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A Note on the Behaviour of the Odd-Even Hopscotch Scheme for a Convection-Diffusion Problem with Discontinuous Initial Data

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In this note we study the behaviour of the odd-even hopscotch scheme, when applied to a convection-diffusion problem with discontinuous initial data. It turns out that the odd-even hopscotch solution is dependent upon the implementation of the scheme.

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1. Introduction

The odd-even hopscotch (OEH) scheme is an integration technique for time-dependent partial differential equations [2,3]. It is a combination of the explicit and implicit Euler rule. When combined with a suitable finite difference space discretization, the scheme is explicit in the sense that no sets of equations have to be solved. In this note we consider the OEH scheme when applied to a convection-diffusion problem with discontinuous initial data. The ideas behind the analysis are borrowed from [1], which deals with the related diffusion problem.

Many time-integration techniques do not simulate the behaviour of the solution in the neighbourhood of the discontinuity in a proper way. For example, the Crank-Nicolson scheme exhibits an oscillatory behaviour near the discontinuity, which even increases in amplitude with decreasing mesh size (with fixed time step) [4]. The OEH scheme behaves in a peculiar way in the neighbourhood of the discontinuity, i.e. the form of the solution obtained depends on the implementation of the OEH scheme. We can distinguish two cases. When the initial approximation at the point adjacent to the discontinuity is implicit (case 1), then the subsequent solution appears smooth. If however, the initial approximation in that point is explicit (case 2), then a wave-like disturbance can propagate through the domain, which even increases in amplitude with decreasing mesh size (with fixed time step). This disturbance will die away for increasing time, but for small time-values it will destroy the solution.

The purpose of this paper is to demonstrate this phenomenon for the linear, 1-dimensional convection-diffusion problem. The contents of the paper is the following. In section 2 we introduce the OEH scheme for the convection-diffusion problem. The solution of the scheme is analyzed in Section 3 and 4 for respectively case 2 and case 1.

2. Problem specification and solution method

Consider the following convection-diffusion problem

$$u_t + qu_x = \epsilon u_{xx},\ 0<x<1,\ t>0 \tag{2.1a}$$

$$u(x,0)=0,\ 0<x\leqslant 1 \tag{2.1b}$$

$$u(0,t) = 1,\ u(1,t)=0,\ t\geqslant 0, \tag{2.1c}$$

where $u(x,t)$ is the dependent variable, q the (constant) convection velocity and $\epsilon>0$ is the diffusion parameter. We assume that $q>0$. Note that the initial function $u(x,0)$ is discontinuous at $x=0$. An

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analysis of the OEH scheme for (2.1) in case $q=0$ (diffusion equation) is given in [1].

The interval [0,1] is covered with a uniform space mesh $\{x_i\}(i=0(1)N+1)$ with mesh size $h=1/(N+1)$. We suppose that N is odd, though this is not essential. For space discretization we use standard central differences. Let U_i^n denote an approximation to $u(ih,n\tau)$ where τ is the time step. Then the OEH scheme for (2.1a) reads

$$U_i^{n+1} = U_i^n - c\theta_i^n(U_{i+1}^n - U_{i-1}^n) + d\theta_i^n(U_{i+1}^n - 2U_i^n + U_{i-1}^n) - c\theta_i^{n+1}(U_{i+1}^{n+1} - U_{i-1}^{n+1}) + d\theta_i^{n+1}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}), \quad i=1(1)N, \quad n=0,1,2,\cdots. \tag{2.2}$$

In (2.2) $c=\tau q/2h$, $d=\tau\epsilon/h^2(c,\ d>0)$ and θ_i^n is the odd-even function defined by

$$\theta_i^n = \begin{cases} 1 \text{ if } n+i \text{ is odd (odd points)} \\ 0 \text{ if } n+i \text{ is even (even points)} \end{cases} \tag{2.3}$$

The initial and boundary conditions become

$$U_i^0=0, \quad i=1(1)N \tag{2.4a}$$

$$U_0^n=1, \quad U_{N+1}^n=0, \quad n=0,1,2,\cdots. \tag{2.4b}$$

Note that the OEH scheme is just the forward Euler rule at the odd points and the backward Euler rule at the even points. Also note that the initial approximation U_1^1 immediately adjacent to the discontinuity is explicit.

In order to investigate the behaviour of the solution of the finite difference equations, the OEH scheme is considered as a 2-stage explicit process. Writing down two successive steps of (2.2) yields, after some algebraic manipulation, the following results (see also [1])

$$(1+2d)U_i^{2m+2} = (1-2d+4d^2-4c^2)U_i^{2m} + 2(-c+d)(1-2d)U_{i+1}^{2m} + 2(c+d)(1-2d)U_{i-1}^{2m} + 2(-c+d)^2U_{i+2}^{2m} + 2(c+d)^2U_{i-2}^{2m}, \quad m=0,1,2,..., \tag{2.5a}$$

for the even points $i=2(2)N-1$, and

$$(1+2d)^2U_i^{2m+2} = (1-2d)(1+2d+4(d^2-c^2))U_i^{2m} + 2(-c+d)(1+3d^2-3c^2)U_{i+1}^{2m} + 2(c+d)(1+3d^2-3c^2)U_{i-1}^{2m} + 2(-c+d)^2(1-2d)U_{i+2}^{2m} + 2(c+d)^2(1-2d)U_{i-2}^{2m} + 2(-c+d)^3U_{i+3}^{2m} + 2(c+d)^3U_{i-3}^{2m}, \quad m=0,1,2,..., \tag{2.5b}$$

for the odd points $i=3(2)N-2$. Of the two remaining points $(i=1,N)$ only the solution at $i=1$ is of interest to us. The equation corresponding to (2.5b) for $i=1$ is

$$(1+2d)^2U_1^{2m+2} = (1-2d)(1+2d+2d^2-2c^2)U_1^{2m} + 2(-c+d)(1+2d^2-2c^2)U_2^{2m} + 2(-c+d)^2(1-2d)U_3^{2m} + 2(-c+d)^3U_4^{2m} + 2(c+d)(1+2d+d^2-c^2), \quad m=0,1,2,\cdots. \tag{2.5c}$$

From (2.4) and (2.5) one can easily see that the only non-zero values at $t=2\tau(m=0)$ are:

$$U_1^2 = \frac{2(c+d)(1+2d+d^2-c^2)}{(1+2d)^2} \tag{2.6a}$$

$$U_2^2 = \frac{2(c+d)^2}{1+2d} \tag{2.6b}$$

$$U_3^2 = \frac{2(c+d)^3}{(1+2d)^2}. \tag{2.6c}$$

For arbitrary $m, U_i^{2m}=0$ for $i>2m+1$.

3. Solution of the OEH scheme

Now we examine the behaviour of the leading non-zero values $U_{2m-1}^{2m}, U_{2m}^{2m}$ and U_{2m+1}^{2m} at even number of time steps. Consider equation (2.5b) and (2.5a) for respectively $i=2m+3$ and $i=2m+1$. Taking into account that $U_i^{2m}=0$ for $i>2m+1$, one can easily see that

$$U_{2m+1}^{2m}=\frac{c+d}{1+2d}U_{2m}^{2m}, \quad m=1(1)\frac{N-1}{2}. \tag{3.1}$$

Substitution of this result in (2.5a) for $i=2m+2$ then yields

$$U_{2m+2}^{2m+2}=4\left[\frac{c+d}{1+2d}\right]^2 U_{2m}^{2m}, \quad m=1(1)\frac{N-3}{2}. \tag{3.2}$$

Using (2.6b) as an initial condition, the above recurrence relation for U_{2m}^{2m} has the solution

$$U_{2m}^{2m}=(c+d)\left[\frac{2c+2d}{1+2d}\right]^{2m-1}, \quad m=1(1)\frac{N-1}{2}. \tag{3.3}$$

From (3.1) and (3.3) it follows that

$$U_{2m+1}^{2m}=\frac{1}{2}(c+d)\left[\frac{2c+2d}{1+2d}\right]^{2m}, \quad m=1(1)\frac{N-1}{2}. \tag{3.4}$$

The computation of U_{2m-1}^{2m} is somewhat more tedious. Suppressing the details, it can be shown that U_{2m-1}^{2m} satisfies the recurrence relation

$$(1+2d)^2 U_{2m+1}^{2m+2}=4(c+d)^2 U_{2m-1}^{2m}+(c+d)(1-4c^2)\left[\frac{2c+2d}{1+2d}\right]^{2m}, \quad m=1(1)\frac{N-3}{2}. \tag{3.5}$$

Taking the value in (2.6a) as the initial condition, equation (3.5) has the solution

$$U_{2m-1}^{2m}=\left[\frac{m-1}{2}\frac{1-4c^2}{1+2d}+\frac{(d+1)^2-c^2}{1+2d}\right]\cdot\left[\frac{2c+2d}{1+2d}\right]^{2m-1}, \quad m=1(1)\frac{N-1}{2}. \tag{3.6}$$

If $c<\frac{1}{2}$, which is the stability restriction for the OEH scheme (2.2) [5], then U_{2m-1}^{2m}, U_{2m}^{2m} and U_{2m+1}^{2m} decrease as m increases, however the unrealistic values $U_{2m-1}^{2m}, U_{2m}^{2m}, U_{2m+1}^{2m}>1$ are still possible. For example, suppose that $\epsilon=1$, $q=5$, $h=1/40$ and $\tau=1/400$, then $c=1/4$ (hence the scheme is stable), $d=4$ and $U_{2m}^{2m}=1.4346>1$ for $m=10$. Fig. 1. shows the solution corresponding to (3.3), (3.4) and (3.6) for $m=1,2,3,4$ computed for the following set of parameters: $\epsilon=1$, $q=5$, $h=1/20,1/40$ and $\tau=1/100,1/200$, and hence $c=1/2$ and $d=4,8$. This figure clearly demonstrates that a wave-like disturbance propagates through the computational domain, although the computations are stable. Mesh refinement even gives worse results. Although this disturbance dies away for increasing m (provided $c<\frac{1}{2}$), it spoils the solution for small m-values. If we want to suppress this disturbance, we have to choose τ smaller, much smaller, than needed for stability.

4. An alternative implementation

Consider again the OEH scheme (2.2) and suppose that the odd-even function θ_i^n is redefined as

$$\theta_i^n=\begin{cases}1 \text{ if } n+i \text{ is even (even points)}\\ 0 \text{ if } n+i \text{ is odd (odd points).}\end{cases} \tag{4.1}$$

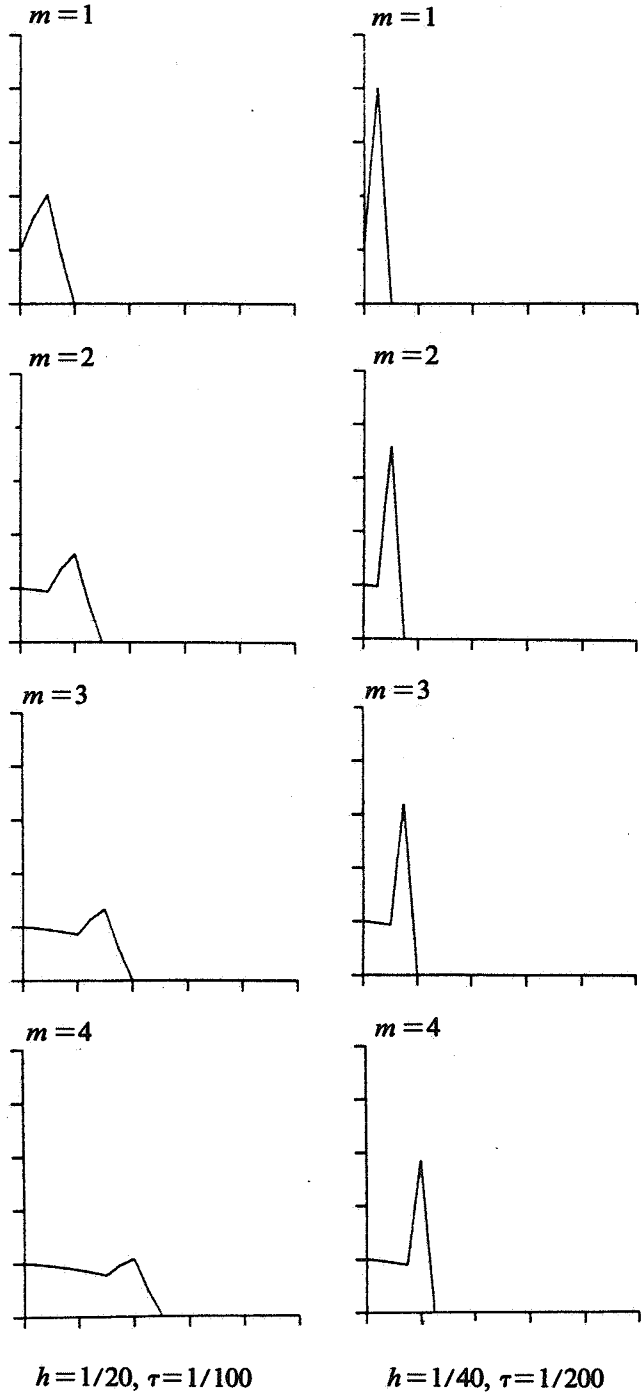

FIGURE 1. Solution (3.3),(3.4),(3.6) for $m=1,2,3,4(\epsilon=1,q=5)$.

Note that the initial approximation U_1^1 immediately adjacent to the discontinuity is now implicit. In this case $U_i^{2m+1}=0$ for $i>2m+1$ and the equations (2.5a)-(2.5c) remain valid if the superscripts $2m+2$ and $2m$ are replaced by respectively $2m+3$ and $2m+1$. Now we consider the behaviour of the leading non-zero terms $U_{2m-1}^{2m+1}, U_{2m}^{2m+1}$ and U_{2m+1}^{2m+1} at odd number of timesteps.

Following a procedure completely analogous to the one outlined in Section 3, one finds the following results for $U_{2m-1}^{2m+1}, U_{2m}^{2m+1}$ and U_{2m+1}^{2m+1} $(m=1(1)\frac{N-1}{2})$ (see also [1])

$$U_{2m-1}^{2m+1} = \frac{1}{2}\left[\frac{2m(1-4c^2)}{(1+2d)^2}+\frac{1+8d+8d^2+4c^2}{(1+2d)^2}\right]\left[\frac{2c+2d}{1+2d}\right]^{2m-1} \tag{4.2a}$$

$$U_{2m}^{2m+1} = \left[\frac{2c+2d}{1+2d}\right]^{2m} \tag{4.2b}$$

$$U_{2m+1}^{2m+1} = \frac{1}{2}\left[\frac{2c+2d}{1+2d}\right]^{2m+1} . \tag{4.2c}$$

Clearly $1>U_{2m}^{2m+1}>U_{2m+1}^{2m+1}$ if $c<\frac{1}{2}$. A little further analysis shows that also $U_{2m+1}^{2m+1}<1$ for $c<\frac{1}{2}$. These results don't give any information about the other non-zero terms, but numerical experiments suggest that $U_i^{2m+1}<1$ for all i values. This means that the numerical solution in this case behaves in a smooth way, and doesn't exhibit the wave-like disturbance mentioned in Section 3.

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