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Approximating the Projective Model

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An approximation principle for the projective model is given which makes it possible to prove assertions in this model by proving them in an infinite sequence of certain finite process algebras. Motivated from this principle a new model for process algebras is defined and its relationship to the projective model is studied.

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1. INTRODUCTION

In the formal analog of Milner's **Calculus of Communicating Systems** (see [M]), as this is described by Bergstra and Klop in [BK], one builds large systems of processes by assembling together **atomic processes** (or **actions**) chosen from a **finite** set A of such atomic processes (see [H]). These systems of processes satisfy a set of equational laws, called the axioms of the **theory of the algebras of communicating processes** (or **theory of process algebras**). The models of this theory are called **process algebras**. Its axioms are described in a signature that includes: $+$ (alternative composition or sum), $\cdot$ (sequential composition or product), $\|$ (parallel composition or merge), $\lfloor\!\lfloor$ (left merge), $|$ (communication merge), ∂_H (encapsulation, for H a subset of A), δ (deadlock or failure) and the atom a (for each $a \in A$) (the atomic processes). In the table below the equational laws for process algebras are given; the list (A) consists of the basic axioms, (C) the axioms of communication, (CM) the axioms of merge, and (D) the axioms for the encapsulation. The communication function $| : A_\delta \times A_\delta \to A_\delta$ (where A_δ consists of the atoms in A including δ) is initially given on atomic processes. In the absence of communication, axiom (CM1) should be replaced with $x \| y = y \lfloor\!\lfloor x + x \lfloor\!\lfloor y$. The theory (in the signature $+, ., \|, \lfloor\!\lfloor$, a (with $a \in A$)) consisting of the first five axioms in (A) plus the first four axioms in (CM) is known as (basic) process algebra and is abbreviated by PA; ACP (algebra of communicating processes) consists of the axioms in (A), (C), (CM) and (D). As usual, the multiplication sign as well as the universal quantifiers, which quantify the variables x, y, z, will be omitted. The letters a, b range over A_δ.

(A):
$$x + y = y + x$$
$$x + (y + z) = (x + y) + z$$
$$x + x = x$$
$$(x + y)z = xz + yz$$
$$(xy)z = x(yz)$$
$$x + \delta = \delta$$
$$\delta x = \delta$$

(D):
$$\partial_H(a) = a \text{ if } a \in H$$
$$\partial_H(a) = \delta \text{ if } a \notin H$$
$$\partial_H(x + y) = \partial_H(x) + \partial_H(y)$$
$$\partial_H(xy) = \partial_H(x)\, \partial_H(y)$$

(C):
$$a | b = b | a$$
$$(a | b) | c = a | (b | c)$$
$$\delta | a = \delta$$

(CM):
$$x \| y = y \lfloor\!\lfloor x + x \lfloor\!\lfloor y + x | y$$
$$a \lfloor\!\lfloor x = ax$$
$$(ax) \lfloor\!\lfloor y = a(x \| y)$$
$$(x + y) \lfloor\!\lfloor z = x \lfloor\!\lfloor z + y \lfloor\!\lfloor z$$
$$(ax) | b = (a | b)x$$
$$a | (bx) = (a | b)x$$
$$(ax) | (by) = (a | b)(x \| y)$$
$$(x + y) | z = x | z + y | z$$
$$x | (y + z) = x | y + x | z$$

The underlying signature for all the results of the present paper will be any subset of the signature of the above theory, i.e. $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, δ, a (with $a \in A$).

In this axiomatic framework the term (or initial) model A_ω is defined as the set of all processes built-up from the atomic processes $a \in A_\delta$, via the operations in the given signature. If one thinks of the elements of A_ω as finite trees with edges labeled by the atoms in A_δ then one can also consider the model A_n consisting of those trees which have height at most n (see [BK]). More formally, following [BK], for each $n > 0$ let the projection function $(.)_n$ be defined on A_ω as follows:

$$(a)_n = a,$$
$$(at)_1 = a,$$
$$(at)_n = a(t)_{n-1}, \text{ for } n > 1,$$
$$(t + t')_n = (t)_n + (t')_n \text{ for } n > 0.$$

The finite process algebras A_n are defined by $A_n = \{(t)_n : t \in A_\omega\}$. Also for any binary (respectively unary) operation $*$ (respectively π) in the given signature define an operation $*_n$ (respectively π_n) as follows:

$$x *_n y = (x * y)_n,$$
$$\pi_n (x) = (\pi (x))_n.$$

It is now easy to show (see [BK]) that (for finite A) each A_n is a finite model of the theory of process algebras. The **projective (or standard) model**, denoted by A^∞, consists of all infinite sequences $\langle p_1,p_2,\dots,p_n,\dots\rangle$ such that $p_n \in A_n$ and $(p_{n+1})_n = p_n$, for all $n > 0$.

Throughout the present paper $T(x_1,\dots,x_n)$, $S(x_1,\dots,x_n)$ (with or without subscripts) will always denote **(polynomial) operators**, i.e. terms built-up from the variables $x_1,\dots,x_n$, the atoms a (with $a \in A_\delta$) and the operations of the given signature.

The class **P** of **positive formulas** is the smallest class of well founded formulas in the signature $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, δ, a (with $a \in A$), which satisfies the following properties:

(i) For all polynomial operators T, S and any variables $v_1,\dots,v_n$,

$T(v_1,\dots,v_n) = S(v_1,\dots,v_n) \in \mathbf{P}$.

(ii) $\Phi, \Psi \in \mathbf{P} \Rightarrow \Phi \vee \Psi \in \mathbf{P}$.

(iii) For any countable $\Delta \subseteq \mathbf{P}$, the conjuction of the formulas in Δ belongs to $\mathbf{P}$.

(iv) If $\Phi(v_1,\dots,v_n,\dots) \in \mathbf{P}$ and $\{u_1,\dots,u_k,\dots\} \subseteq \{v_1,\dots,v_n,\dots\}$ then both formulas $(\exists u_1 \dots \exists u_k \dots)\, \Phi(v_1,\dots,v_n,\dots)$, $(\forall u_1 \dots \forall u_k \dots)\, \Phi(v_1,\dots,v_n,\dots) \in \mathbf{P}$.

The class $\mathbf{P_0}$ of **finite positive formulas** is the smallest class of well founded formulas in the signature $+$, $.$, $\|$, $\|\!\_$, $|$, ∂_H, δ, a (with $a \in A$), which satisfies the following properties:

(i) For all polynomial operators T, S and any variables $v_1,\dots,v_n$,

$T(v_1,\dots,v_n) = S(v_1,\dots,v_n) \in \mathbf{P_0}$.

(ii) $\Phi, \Psi \in \mathbf{P_0} \Rightarrow \Phi \vee \Psi, \Phi \wedge \Psi \in \mathbf{P_0}$.

(iii) If $\Phi(v_1,\dots,v_n) \in \mathbf{P_0}$ and $\{u_1,\dots,u_k\} \subseteq \{v_1,\dots,v_n\}$ then both formulas $(\exists u_1 \dots \exists u_k)\, \Phi(v_1,\dots,v_n)$, $(\forall u_1 \dots \forall u_k)\, \Phi(v_1,\dots,v_n) \in \mathbf{P}$.

Most of section 3 will be dedicated to a proof of the following theorem.

Theorem 1.1 [Approximation Theorem]
Any formula $\Phi(v_1,\dots,v_n,\dots) \in \mathbf{P}$ satisfies the following approximation principle: for any convergent sequences $\{x_{1,n}\},\dots,\{x_{k,n}\},\dots$ such that $x_{k,n} \in A_n$ for all k, n, if the set $\{n > 0 : A_n \models \Phi(x_{1,n},\dots,x_{k,n},\dots)\}$ is infinite then it is true that $A^\infty \models \Phi(\lim_{n\to\infty} x_{1,n},\dots, \lim_{n\to\infty} x_{k,n},\dots)$.

Such formulas Φ occur when one wants to prove that a system of fixed point equations has a solution, e.g. consider the infinite system $\Sigma = \{x_k = T_k(x_1,\dots,x_{n(k)}) : k > 0\}$ of fixed point equations, where each T_k is a polynomial operator in the variables indicated. The assertion "Σ has a solution in $(A^\infty)^\omega$" can be expressed by the formula $(\exists x_1 \dots \exists x_k \dots)\Psi$, where Ψ is the countable conjunction of all the formulas $x_k = T_k(x_1,\dots,x_{n(k)})$, for $k > 0$. Theorem 1.1 states that in order to prove that Σ has a solution in A^∞, it is enough to show that Σ has a solution in in infinitely many A_n's.

It is also possible to prove a partial converse of the approximation principle. This is stated in the theorem below.

Theorem 1.2 [Converse of the Approximation Principle]
For any positive formula $\Phi(v_1,\dots,v_k,\dots) \in \mathbf{P}$ and any $p_1,\dots,p_k,\dots \in A^\infty$ the following statements are equivalent:

(i) $A^\infty \models \Phi(p_1,\dots,p_k,\dots)$.

(ii) $\{n > 0 : A_n \models \Phi((p_1)_n,\dots,(p_k)_n,\dots)\}$ is infinite.

(iii) $\forall n > 0[A_n \models \Phi((p_1)_n,\dots,(p_k)_n,\dots)]$.

Motivated from the approximation theorem one can define a new process algebra, which is an extension of the projective algebra A^∞. To state the next theorem the notion of ultrafilter on the set N of positive integers will be required. Call D a **(nonprincipal) ultrafilter** on N if D is a nonempty set of subsets of N satisfying the following properties for all X, $Y \subseteq N$: (i) $\emptyset \notin D$ (ii) $X \in D$ and $X \subseteq Y \Rightarrow Y \in D$ (iii) X, $Y \in D \Rightarrow X \cap Y \in D$ (iv) $X \in D$ or $N - X \in D$ (v) $X \in D \Rightarrow$ is infinite. Notice that the existence of such ultrafilters requires the axiom of choice (see [E] or [CK]).

The main theorem of section 4 is the following:

Theorem 1.3
For any ultrafilter D on the set N of positive integers there exists a process algebra A^D, which is a proper extension of the projective algebra A^∞. Moreover, for any finite, positive formula $\Phi(v_1,\dots,v_k) \in \mathbf{P_0}$ and any $p_1,\dots,p_k \in A^\infty$ the following statements are equivalent:

(i) $A^D \models \Phi(p_1,\dots,p_k)$.

(ii) $A^\infty \models \Phi(p_1,\dots,p_k)$.

(iii) $\{n > 0 : A_n \models \Phi((p_1)_n,\dots,(p_k)_n)\} \in D$.

(iv) $\{n > 0 : A_n \models \Phi((p_1)_n,\dots,(p_k)_n)\}$ is infinite.

(v) $\forall n > 0[A_n \models \Phi((p_1)_n,\dots,(p_k)_n)]$.

2. TOPOLOGY OF THE PROJECTIVE MODEL

As explained before the projective model A^∞ consists of all infinite sequences $\langle p_1,p_2,\dots,p_n,\dots\rangle$ such that $p_n \in A_n$ and $(p_{n+1})_n = p_n$, for all $n > 0$. The term model A_ω can be embedded in a natural way in the projective model; because of this, it is considered a subset of the projective model (see [BK]). For any such $p = \langle p_1,p_2,\dots,p_n,\dots\rangle \in A^\infty$ put $(p)_n = p_n$. For any $p,q \in A^\infty$ such that $p \neq q$ let $k(p,q)$ = the least $n > 0$ such that $(p)_n \neq (q)_n$. The set A^∞ can be endowed with a **metric space structure** by defining a **distance function** d as follows: $d(p,q) = 2^{-k(p,q)}$ if $p \neq q$, and $d(p,q) = 0$ otherwise.

This metric was used by Arnold and Nivat (see [AN]) in the context of denotational semantics of concurrency. An equivalent metric was also used by de Bakker and Zucker (see [dBZ]). For additional information the interested reader is advised to consult [L] and [Ro].

The following result summarizes all the basic properties of the metric space (A^∞,d) and will be used frequently in the sequel.

Theorem 2.1 [In the signature $+$, $.$, $\|$, $\mathbin{⫡}$, $|$, ∂_H, δ, a (with $a \in A$)]

(i) (A^∞, d) is an ultrametric space, i.e. it satisfies the following properties:

(a) $d(p,q) = 0 \Leftrightarrow p = q$.

(b) $d(p,q) = d(q,p)$.

(c) $d(p,q) \leq \max\{d(p,r), d(r,q)\}$.

(ii) $p^{(r)} \to p \Leftrightarrow \forall n \exists m \forall k \geq m\ (p^{(k)})_n = (p)_n$.

(iii) (A^∞, d) is the metric completion of the metric space (A_ω, d'), where d' is the restriction of d on A_ω.

(iv) For all $p \in A^\infty$, $n > 0$, $d(p,(p)_n) \leq 2^{-n}$. Hence, $\lim_{n\to\infty}(p)_n = p$.

(v) The operations $(.)_n : A^\infty \to A_n$ are continuous.

(vi) Any operator $T(x_1,\dots,x_n)$ is continuous in the variables $x_1,\dots,x_n$. In fact, for any $p_1,\dots,p_n, q_1,\dots,q_n \in A^\infty$,

$$d(T(p_1,\dots,p_n),T(q_1,\dots,q_n)) \leq \max\{d(p_1,q_1),\dots,d(p_n,q_n)\}.$$

Proof: The proof is omitted. For more details of the proof the reader can consult [K], [L] and [AN].

Theorem 2.2 [In the signature $+$, $.$, $\|$, $\mathbin{\|\!\_}$, $|$, ∂_H, δ, a (with $a \in A$)]
A is finite $\Leftrightarrow$ (A^∞, d) is compact.

Proof: ($\Leftarrow$) Assume on the contrary that A is infinite and let $a_1,\ldots,a_n,\ldots$ be an infinite list of pairwise distinct atoms in A. Then the sequence $\{a_n\}$ cannot have any convergent subsequence since $d(a_n, a_m) = 1/2$, for $n \neq m$. Clearly, this is a contradiction.
($\Rightarrow$) This is immediate from [Du], page 429.

In view of the previous theorem from now on and for the rest of the paper it will always be assumed that A is finite. This will guarantee that A^∞ is compact.

3. THE APPROXIMATION PRINCIPLE

Intuitively, the approximation principle enables one to verify assertions in the projective model by proving that the same assertion is valid in infinitely many A_n. To be more precise a formula Φ $(v_1,\ldots,v_n,\ldots)$ satisfies the **approximation priniple**, and this will be abbreviated by $\mathfrak{a}(\Phi)$, if the following property holds: for any convergent sequences $\{x_{1,n}\},\ldots, \{x_{k,n}\},\ldots$ such that $x_{k,n} \in A_n$ for all k, n, if the set $\{n > 0 : A_n \models \Phi\ (x_{1,n},\ldots,x_{k,n},\ldots)\}$ is infinite then it is true that $A^\infty \models \Phi\ (\lim_{n\to\infty} x_{1,n},\ldots, \lim_{n\to\infty} x_{k,n},\ldots)$.

Now it is possible to prove theorem 1.1

Proof of theorem 1.1: It is enough to show that for any formula $\Phi \in \mathbf{P}$, $\mathfrak{a}(\Phi)$ holds. The proof is by induction on the construction of the formula Φ.

Case 1: $\Phi \equiv T(v_1,\ldots,v_m) = S(v_1,\ldots,v_m)$, where T, S are polynomial operators.

For any operator T $(v_1,\ldots,v_m)$ let T^n $(v_1,\ldots,v_m)$ denote the interpretation of T in the model A_n. Using induction on the construction of T and the definitions of the operations in the process algebras A_n (see section 1) it is easy to show that

Lemma 3.1 For all $x_1,\ldots,x_m \in A_n$, $T^n(x_1,\ldots,x_m) = (T(x_1,\ldots,x_m))_n$.

Now, it is required to show that the formula $T(v_1,\ldots,v_m) = S(v_1,\ldots,v_m)$ satisfies the appriximation principle $\mathfrak{a}$. Indeed, let $\{x_{1,n}\},\ldots, \{x_{k,n}\},\ldots$ be any convergent sequences such that $x_{k,n} \in A_n$ for all k, n and the set

$$J = \{n > 0 : A_n \models T(x_{1,n},\ldots,x_{m,n}) = S(x_{1,n},\ldots,x_{m,n})\}$$

is infinite. It is enough to show that

$$A^\infty \models T(\lim_{n\to\infty} x_{1,n},\ldots,\lim_{n\to\infty} x_{m,n}) = S(\lim_{n\to\infty} x_{1,n},\ldots,\lim_{n\to\infty} x_{m,n}).$$

For $k = 1,\ldots,m$ put $x_k = \lim_{n\to\infty} x_{k,n}$. It is clear that for all $n \in J$,

$$T^n(x_{1,n},\ldots,x_{m,n}) = S^n(x_{1,n},\ldots,x_{m,n}).$$

Using this last equation and lemma 3.1 it is easy to show that for all $n \in J$,

$$(T(x_{1,n},\ldots,x_{m,n}))_n = (S(x_{1,n},\ldots,x_{m,n}))_n.$$

However, the following result is an easy consequence of the definition of convergence in A^∞ via the metric d:

Lemma 3.2 For any sequence $\{u_n\}$ of terms in A_ω if $u_n \to u$ then $(u_n)_n \to u$.

Using the continuity of the operators T, S (see theorem 2.1) it follows that

$$T(x_{1,n},\ldots,x_{m,n}) \to T(x_1,\ldots,x_m),$$
$$S(x_{1,n},\ldots,x_{m,n}) \to S(x_1,\ldots,x_m).$$

Hence, using lemma 3.2 and the last equation it follows that

$$A^\infty \models T(x_1,\ldots,x_m) = S(x_1,\ldots,x_m),$$

which completes the proof in case 1.

Case 2: $\Phi \equiv X \vee \Psi$.

Let $\{x_{1,n}\},\ldots, \{x_{k,n}\},\ldots$ be convergent sequences such that $x_{k,n} \in A_n$ for all k, n, and the set $J = \{n > 0 : A_n \models \Phi(x_{1,n},\ldots,x_{k,n},\ldots)\}$ is infinite. To show that

$$A^\infty \models \Phi(\lim_{n\to\infty} x_{1,n},\ldots,\lim_{n\to\infty} x_{k,n},\ldots).$$

Put

$$K = \{n > 0 : A_n \models X (x_{1,n},\dots,x_{k,n},\dots)\},\ L = \{n > 0 : A_n \models \Psi (x_{1,n},\dots,x_{k,n},\dots)\}.$$

Since, $J = K \cup L$ it is clear that at least one of the sets K, L (say K) must be infinite. It follows from the induction hypothesis that

$$A^{\infty} \models X (\lim_{n\to\infty} x_{1,n},\dots,\lim_{n\to\infty} x_{k,n},\dots),$$

and hence also

$$A^{\infty} \models \Phi (\lim_{n\to\infty} x_{1,n},\dots,\lim_{n\to\infty} x_{k,n},\dots),$$

which completes the proof in case 2.

Case 3: $\Phi \equiv \bigwedge \{\Phi_i : i > 0\}$.

Let $\{x_{1,n}\},\dots, \{x_{k,n}\},\dots$ be convergent sequences such that $x_{k,n} \in A_n$ for all k, n, and the set $J = \{n > 0 : A_n \models \Phi (x_{1,n},\dots,x_{k,n},\dots)\}$ is infinite. To show that

$$A^{\infty} \models \Phi (\lim_{n\to\infty} x_{1,n},\dots,\lim_{n\to\infty} x_{k,n},\dots).$$

For each $i > 0$ put

$$J_i = \{n > 0 : A_n \models \Phi_i (x_{1,n},\dots,x_{k,n},\dots)\}.$$

Clearly, each J_i is infinite and hence the induction hypothesis implies that for all $i > 0$,

$$A^{\infty} \models \Phi_i (\lim_{n\to\infty} x_{1,n},\dots,\lim_{n\to\infty} x_{k,n},\dots).$$

This completes the proof in case 3.

Case 4: $\Phi \equiv (\exists u_1 \dots \exists u_k \dots) \Psi (u_1,\dots,u_k,\dots,v_1,\dots,v_n,\dots)$.

Actually, this is the only part of the proof which requires the compactness of A^{∞}. Let $\{x_{1,n}\},\dots, \{x_{k,n}\},\dots$ be convergent sequences such that $x_{k,n} \in A_n$ for all k, n, and the set $J = \{n > 0 : A_n \models \Phi (x_{1,n},\dots,x_{k,n},\dots)\}$ is infinite. To show that

$$A^{\infty} \models \Phi (\lim_{n\to\infty} x_{1,n},\dots,\lim_{n\to\infty} x_{k,n},\dots).$$

By assumption, for each $n \in J$ there exist elements $y_{k,n} \in A_n$ such that

$$A_n \models \Psi (y_{1,n},\dots,y_{k,n},\dots,x_{1,n},\dots,x_{k,n},\dots).$$

The sequences $(\{y_{k,n}\} : k > 0)$ need not be convergent. However, using the compactness of the metric space A^{∞} (in fact one can define a compact metric on the cartesian product of countably many copies of A^{∞}) there exists an infinite subset L of J such that each of the sequences $\{y_{k,n}\}_{n \in L}$

is convergent. Let $y_k = \lim_{n\in L, n\to\infty} y_{k,n}$, for $k > 0$. Now apply the induction hypothesis to the sequences $\{x_{k,n}\}_{n\in L}$, $\{y_{k,n}\}_{n\in L}$, for $k > 0$, and the formula Ψ to obtain that

$$A^\infty \models \Psi\,(y_1,\ldots,y_k,\ldots,\ x_1,\ldots,x_k,\ldots).$$

This completes the proof of case 4.

Case 5: $\Phi \equiv (\forall u_1 \ldots \forall u_k \ldots)\ \Psi\,(u_1,\ldots,u_k,\ldots,v_1,\ldots,v_n,\ldots)$.

This is similar to the cases above and it is left to the reader. Now the proof of theorem 1.1 is complete.

The two examples given below show that theorem 1.1 would not be true if the class **P** of formulas defined above were assumed to be closed either under negations or under countable disjunctions.

Example 3.3 The approximation principle cannot be valid for formulas involving infinitary disjunctions. To see this consider the formula

$$\Phi\,(x) \equiv \bigvee \{\, x = a^n : n > 0\},$$

i.e. Φ is the countable disjunction of the formulas $x = a^n$, for $n > 0$. It is clear that $A^\infty \not\models \Phi\,(a^\omega)$. Moreover, consider the sequence $x_n = a^n$. It is then true that for all $n > 0$, $A_n \models \Phi\,(x_n)$. Hence, Φ does not satisfy the approximation principle $\mathfrak{A}$, $x_n \to a^\omega$. Another example (communicated to the author by H. Mulder), which provides a similar formula but with no free variables is the following:

$$\Psi \equiv \bigvee \{\, \exists x\,(xa = a^n) : n > 0\}.$$

It is clear that $A^\infty \not\models \Psi$ and for all $n > 0$, $A_n \models \Psi$. Hence, Ψ does not satisfy the approximation principle $\mathfrak{A}$.

Example 3.4 The approximation principle cannot be valid for formulas involving negations. To see this consider the formula

$$\Psi\,(x,y) \equiv x = a^n \wedge y = a^{n-1} \wedge x \neq y.$$

Clearly, for all $n > 0$, $A_n \models \Psi\,(a^n, a^{n-1})$. But $A^\infty \not\models \Psi\,(a^\omega, a^\omega)$. Hence, Ψ does not satisfy the approximation principle $\mathfrak{A}$.

Proof of theorem 1.2: In view of theorem 1.1 it is enough to prove that (i) implies (iii). In fact, it is enough to show by induction on positive formulas $\Phi(v_1,\ldots,v_k,\ldots)$ that for all $p_1,\ldots,p_k,\ldots \in A^\infty$,

$$A^{\infty} \models \Phi(p_1,\ldots,p_k,\ldots) \Rightarrow \forall n > 0[A_n \models \Phi((p_1)_n,\ldots,(p_k)_n,\ldots)].$$

The initial step of the proof is for formulas of the form $\Phi \equiv T(v_1,\ldots,v_k) = S(v_1,\ldots,v_k)$, where T, S are polynomial operators. Suppose that $A^{\infty} \models T(p_1,\ldots,p_k) = S(p_1,\ldots,p_k)$. However, if $p = q$ then $(p)_n = (q)_n$, for all $n > 0$. Hence,

$$(T^n((p_1)_n,\ldots,(p_k)_n))_n = (T(p_1,\ldots,p_k))_n =$$
$$(S(p_1,\ldots,p_k))_n = (S^n((p_1)_n,\ldots,(p_k)_n))_n,$$

for all $n > 0$. It follows that $\forall n > 0[A_n \models \Phi((p_1)_n,\ldots,(p_k)_n)]$. This completes the proof in the case where Φ is an equality between two polynomial operators. The proof of the other cases in the inductive proof create no particular difficulties and are left as an exercise to the reader. Moreover, the compactness of A^{∞} is not necessary for this proof. This completes the proof of the theorem.

4. THE ULTRAPRODUCT MODEL

Perhaps the most natural way to interpret the approximation principle is via the ultraproduct model. According to [CK], the **ultraproduct construction** (to be outlined below) was originally discovered by Skolem and used to construct nonstandard models of Peano arithmetic. It was later further extended and used by Łoś to prove his seminal fundamental theorem for ultraproducts (see [E] or [EK]); its definition, although algebraic in nature, arose from mathematical logic.

Given an ultrafilter D on the set N of positive integers, define an equivalence relation $\equiv_D$ on the product set $\Pi\{A_n : n > 0\}$ (= the set of all functions $f : N \to \cup\{A_n : n > 0\}$ such that for all $n > 0$, $f(n) \in A_n$) as follows:

$$f \equiv_D g \Leftrightarrow \{n > 0 : f(n) = g(n)\} \in D.$$

Clearly, $\equiv_D$ is an equivalence relation on the set $\Pi\{A_n : n > 0\}$. For each $f \in \Pi\{A_n : n > 0\}$, let $[f]_D$ denote the equivalence class of f modulo $\equiv_D$ and let A^D be the set of all such equivalence classes. For each binary (respectively unary) operation $*$ (respectively π) define an operation $*_D$ (respectively π_D) on A^D as follows:

$$[f]_D *_D [g]_D = [\langle (f(n) *_n g(n))_n : n > 0\rangle]_D,$$

$$\pi_D([f]_D) = [\langle \pi_n(f(n)) : n > 0\rangle]_D.$$

These operations are well defined on A^D (see [E]) moreover the following fundamental theorem holds (see [E]):

Theorem 4.1 [Fundamental Theorem for Ultraproducts, Łoś]
For any functions $f_1,\dots,f_k$ and any well founded, finite formula $\Phi(v_1,\dots,v_k)$ in the given signature the following result holds:

$$A^D \models \Phi([f_1]_D,\dots,[f_k]_D) \Leftrightarrow \{n > 0 : A_n \models \Phi(f_1(n),\dots,f_k(n))\} \in D.$$

As an immediate consequence of the fundamental theorem, one obtains that A^D is a model of the theory of process algebras. In fact , lemma 4.2 implies that it is an extension of the projective model A^∞.

Lemma 4.2 The mapping $F : A^\infty \to A^D : p \to [\langle (p)_n : n > 0\rangle]_D$ is an embedding of A^∞ into A^D.

Proof: It is easy to show that F is a homomorphism (one merely has to go through the definitions of the operations in the projective model as given in [BK]). To show that it is injective notice that for any $p, q \in A^\infty$,

$$\begin{aligned} F(p) = F(q) &\Leftrightarrow [\langle (p)_n : n > 0\rangle]_D = [\langle (q)_n : n > 0\rangle]_D \\ &\Leftrightarrow \{n > 0 : (p)_n = (q)_n\} \in D \\ &\Leftrightarrow \{n > 0 : (p)_n = (q)_n\} \text{ is infinite} \\ &\Leftrightarrow p = q. \end{aligned}$$

This completes the proof of the lemma.

In view of lemma 4.2, the elements of A^∞ will be identified with their corresponding images in A^D via the embedding F. Moreover, A^∞ will be considered a subset of A^D.

Now it is possible to prove the main result of this section.

Proof of theorem 1.3: Let $\Phi(v_1,\dots,v_k)$ be a finite, positive formula in the given signature. The equivalence of (i), (iii) is an immediate consequence of the fundamental theorem for ultraproducts (see theorem 4.1). Since the ultrafilter D is nonprincipal (i.e. all its elements are infinite) the implications (v) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are also immediate. The implication (iv) $\Rightarrow$ (ii) is consequence of the approximation theorem (see theorem 1.1) and the fact that $\lim_{n \to \infty} (p_j)_n = p_j$, for $j = 1,\ldots,k$ (see theorem 2.1.iv). It remains to prove that (ii) $\Rightarrow$ (v). But this is a special case of theorem 1.2. This completes the proof of the theorem.

The result of theorem 1.3 is best possible. It cannot be extended to well founded formulas with negation.

Example 4.3 Consider the formula

$$\sigma \equiv (\exists x, y)[\, x = ax \wedge y^2 = ay^2 \wedge x \neq y \,].$$

It can be shown that

$$A^D \models \sigma \text{ and } A^\infty \not\models \sigma.$$

To show that $A^D \models \sigma$ it is enough to show that $\{n > 0 : A_n \models \sigma\} \in D$ (see theorem 4.1). Indeed, in A_n, equation $x = ax$ has a unique solution, namely $x = a^n$. But, $y^2 = ay^2$ has more than one solution, in A_n, (if $n > 1$), namely $y = a^k, a^{k+1},\ldots,a^n$, where k is the least integer greater than or equal to $(n - 1)/2$. Since D is a nonprincipal ultrafilter it follows that $\{n > 0 : A_n \models \sigma\} \in D$. This shows that $A^D \models \sigma$. In order to show that $A^\infty \not\models \sigma$ it is enough to prove that the only solution (x, y) of the system $x = ax \wedge y^2 = ay^2$ in A^∞ must satisfy $x = y = a^\omega$. Clearly, $x = a^\omega$ is the unique solution of $x = ax$. Let $p \in A^\infty$ be any solution of $y^2 = ay^2$ in A^∞. Then $p^2 = ap^2$ and hence $p^2 = a^\omega$. An easy induction on $n > 0$ shows that for all $n > 0$, $(p)_n = a^n$. Hence, $p = a^\omega$ and the unique solution of $y^2 = ay^2$ in A^∞ is a^ω.

The (ultraproduct) process algebra A^D makes it possible to define nonstandard processes, as the example below shows.

Example 4.4 Let $a \in A$ be a fixed atom. For any function $\sigma : N \to N$ such that for all $n > 0$, $\sigma(n) \leq n$, define the following element $a^\sigma \in A^D$:

$$a^\sigma = [\langle a^{\sigma(n)} : n > 0 \rangle]_D.$$

It can be shown that for any such function σ, the following result holds:

Lemma 4.5

$$a^\sigma \in A^\infty \Leftrightarrow (\exists X \in D)[\, \sigma \text{ is either constant or the identity on } X].$$

Proof of the lemma: The proof of ($\Leftarrow$) is trivial. To prove ($\Rightarrow$) assume that for some $p \in A^\infty$, $a^\sigma = [\langle (p)_n : n > 0 \rangle]_D$. Clearly, the set $J = \{n > 0 :$

$a^{\sigma(n)} = (p)_n\} \in D$. Let $n < m$, with $n, m \in J$. It is then true that

$$a^{\sigma(n)} = (p)_n \text{ and } a^{\sigma(m)} = (p)_m.$$

It follows that

$$a^{\sigma(n)} = (a^{\sigma(m)})_n = a^{\min\{n, \sigma(m)\}},$$

and consequently, $\sigma(n) = \min\{n, \sigma(m)\}$. Write the elements of J in an increasing sequence: $n_1 < n_2 < ... < n_k < ....$ If $\sigma(n_k) < n_k$, for some $k > 0$, then $\sigma(n_{k+i}) = \sigma(n_k)$, for all $i > 0$. Otherwise, $\sigma(n_k) = n_k$, for all $k > 0$. This completes the proof of the lemma.

It can also be shown that for different size finite sets A the models A^∞ need not satisfy exactly the same well founded formulas.

Example 4.6 Let a, b be two distinct atoms and consider the formula $\tau \equiv (\exists x, y)[\, x = x^2 \wedge y = y^2 \wedge x \neq y\,]$. Then it is not hard to see that

$$\{a, b\}^\infty \models \tau \text{ and } \{a\}^\infty \not\models \tau.$$

(This is because in $\{a\}^\infty$, **with δ not occurring in the signature**, the only solution of $x = x^2$ is a^ω.)

5. DISCUSSION AND OPEN PROBLEMS

The proof of the approximation principle (theorem 1.1) requires the compactness of the topological space A^∞. This not only forces the set A of atoms to be finite, (see theorem 2.2) but it also excludes the possibility of using τ (silent or internal action) (see [K], as well as [BK] for the definition of this last concept). It is not known, however, if the approximation principle could be proved for the same class of positive formulas without these restrictions.

The ultraproduct construction is quite general and it seems it would be interesting to study the ultraproduct obtained when one takes countably many copies of the finite term model A_ω (which, by the way, is not any longer an extension of the projective model), as well as its relation to the so called graph models (see [BK]). It might also be possible to use the ultraproduct construction in order to prove that certain concepts in process algebras are undefinable in a given signature. (Such a use of ultraproducts is well-known in model theory, e.g. see [E], corollary 3.4).

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