Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

E. Kranakis

Fixed point equations with parameters in the projective model

Computer Science/Department of Algorithmics & Architecture Report CS-R8606 January

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

Copyright © Stichting Mathematisch Centrum, Amsterdam

Fixed Point Equations with Parameters in the Projective Model

E. Kranakis

Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Existence and uniqueness theorems are given for solving infinite and finite systems of fixed point equations with parameters in the projective model. The three main methods discussed depend on the topological properties of the projective model and they include: compactness argument, density argument, and Banach's contraction principle. As a converse to the uniqueness theorem it is also shown that in certain signatures guarded equations are the only ones that have unique fixed points.

1980 Mathematics Subject Classification: 68B05. 69F11, 69F12, 69F32, 69F43
Key Words & Phrases: process algebra, process, fixed point equations, polynomial operator, metric space, continuous, contraction, dense, compact, guard, fixed point.
Note: This report will be submitted for publication elsewhere.

Report CS-R8606
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

1. INTRODUCTION

There has been a lot of effort in the current literature to understand the mathematical behavior of processes. Beginning with Milner's seminal work on **Calculus of Communicating Systems**, as described in [M], an attempt was made to bring the provability of correctness of computer programs under a solid mathematical foundation. In fact, one of Milner's main contributions is to regard the basic concepts of communication and parallelism as algebraic in nature. Motivated from this Bergstra and Klop gave an axiomatic-algebraic framework for studying processes (see [BK] for a survey introduction to their equational laws), which is more easily amenable to formal analysis and mathematical proof verification. In many respects their axiomatization constitutes a formal analog of some basic concepts in Milner's Calculus of Communicating Systems (see [M]).

Starting from a given set of **atomic processes (or actions)** one can assemble together large systems of processes. The atomic processes of such a system may interact with one another, communicate, be executed in parallel or even lead to a deadlock (see [H1]). The experience accumulated from studying the behavior of processes (see [H1]) has led to a set of equational laws (see [BK]). In the list below the axioms of the theory of the algebras of communicating processes are given (the reader is advised to look in [BK] for details and further discussion of the axiom system). The given signature is: $+$ (alternative composition or sum), $\cdot$ (sequential composition or product), $\|$ (parallel composition or merge), $\mathbin{|\!\!\lfloor}$ (left merge), $|$ (communication merge), ∂_H (encapsulation, for H a subset of A), δ (deadlock or failure), τ (silent or internal action) and the atom a (for each $a \in A$) (the atomic processes). The list (A) consists of the basic axioms, (C) the axioms of communication, (CM) the axioms of merge, (T) the axioms of the internal action and (D) the axioms for the encapsulation. The communication function $| : A_\delta \times A_\delta \to A_\delta$ (where A_δ consists of the atoms in A including δ) is initially defined on atomic processes. In the absence of communication, axiom (CM1) should be replaced with $x \| y = y \mathbin{|\!\!\lfloor} x + x \mathbin{|\!\!\lfloor} y$. The theory (in the signature $+$, $.$, $\|$, $\mathbin{|\!\!\lfloor}$, a (with $a \in A$)) consisting of the first five axioms in (A) plus the first four axioms in (CM) is known as (basic) process algebra and is abbreviated by PA; ACP (algebra of communicating processes) consists of the axioms in (A), (C), (CM) and (D); finally, ACP[τ] consists of ACP plus the axioms in (T). As usual, the multiplication sign as well as the universal quantifiers, which quantify the variables x, y, z, will be omitted. The letters a, b range over A.

(A):
$$
\begin{aligned}
&x + y = y + x\\
&x + (y + z) = (x + y) + z\\
&x + x = x\\
&(x + y)z = xz + yz\\
&(xy)z = x(yz)\\
&x + \delta = \delta\\
&\delta x = \delta
\end{aligned}
$$

(C):
$$
\begin{aligned}
&a \mid b = b \mid a\\
&(a \mid b) \mid c = a \mid (b \mid c)\\
&\delta \mid a = \delta
\end{aligned}
$$

(D):
$$
\begin{aligned}
&\partial_H(\tau) = \tau\\
&\partial_H(a) = a \text{ if } a \in H\\
&\partial_H(a) = \delta \text{ if } a \notin H\\
&\partial_H(x + y) = \partial_H(x) + \partial_H(y)\\
&\partial_H(xy) = \partial_H(x)\,\partial_H(y)
\end{aligned}
$$

(CM):
$$
\begin{aligned}
&x \parallel y = y \mathbin{|\!\!\lfloor} x + x \mathbin{|\!\!\lfloor} y + x \mid y\\
&a \mathbin{|\!\!\lfloor} x = ax\\
&(ax) \mathbin{|\!\!\lfloor} y = a(x \parallel y)\\
&(x + y) \mathbin{|\!\!\lfloor} z = x \mathbin{|\!\!\lfloor} z + y \mathbin{|\!\!\lfloor} z\\
&(ax) \mid b = (a \mid b)x\\
&a \mid (bx) = (a \mid b)x\\
&(ax) \mid (by) = (a \mid b)(x \parallel y)\\
&(x + y) \mid z = x \mid z + y \mid z\\
&x \mid (y + z) = x \mid y + x \mid z
\end{aligned}
$$

(T):
$$
\begin{aligned}
&x\tau = x\\
&\tau x + x = \tau x\\
&a(\tau x + y) = a(\tau x + y)\\
&\tau \mathbin{|\!\!\lfloor} x = \tau x\\
&(\tau x) \mathbin{|\!\!\lfloor} y = \tau(x \parallel y)\\
&\tau \mid x = \delta\\
&x \mid \tau = \delta\\
&(\tau x) \mid y = x \mid y\\
&x \mid (\tau y) = x \mid y
\end{aligned}
$$

In this axiomatic framework one can define the so-called **term (or initial) model** A_ω (= the set of all processes built-up from the atomic processes a in the set A_δ, via the operations in the given signature), as well as the models A_n, where $n > 0$. In fact, if one thinks of the elements of A_ω as finite trees with edges labeled by atoms then A_n can be considered as consisting of those trees which have height at most n (see [BK]).

Given any term t in A_ω and any positive integer n let $(t)_n$ be the subtree of t of height at most n obtained from t by deleting all those nodes which are located at height bigger than n. Thus, $(.)_n$ can be considered as projecting the term model A_ω onto the model A_n. Now the **projective (or standard) model,** denoted by A^∞ consists of all infinite sequences

$\langle p_1, p_2, \ldots, p_n, \ldots \rangle$ such that $p_n \in A_n$ and $(p_{n+1})_n = p_n$, for all $n > 0$.

In the study of the theory of concurrent processes one is particularly interested in solving **fixed point equations**, i.e. equations of the form $x=T(x)$, where $T(x)$ is a **(polynomial) operator** built up from the atomic processes, the variable x and the operations of the given signature. Such equations, or even systems of such equations arise naturally in the description of several well known concepts in computer science, like stack, bag, counter, mutual exclusion, etc. (see [BK] for a description of such concepts via process algebra).

In general one is interested in establishing criteria that will guarantee both the existence as well as the uniqueness of solutions in systems of fixed point equations. For finite systems two such theorems are given in [BK] and [BK1] for the above mentioned projective model (in the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H$, a (with $a \in A$)):

Theorem 1.1 [Existence Theorem]
Every finite system $\Sigma = \{x_k = T_k(x_1,\ldots,x_n) : k = 1,\ldots,n\}$ of fixed point equations has a solution in $(A^\infty)^n$.

Theorem 1.1 is stated in [BK] only for the signature $+, \cdot, \|, \lfloor\!\lfloor$, a (with $a \in A$). Also, [BK1] provides a proof of the theorem for the case $k = 1$ and the signature $+, \cdot, \|, \lfloor\!\lfloor$, a (with $a \in A$). The full statement of theorem 1.1, as this stated above, was communicated to the author by J. W. Klop and will appear in a forthcoming revised version of [BK1].

A similar existence theorem for systems of arbitrary size (without parameters) has recently been proved by R. J. van Glabbeek (see [vG]) for the case of the countably branching graph model.

Theorem 1.2 [Uniqueness Theorem]
Every finite system $\Sigma = \{x_k = T_k(x_1,\ldots,x_n) : k = 1,\ldots,n\}$ of guarded fixed point equations has a unique solution in $(A^\infty)^n$.

The present paper generalizes both of the above theorems in two directions: on the one hand it allows the systems to have parameters in the projective model and on the other it permits systems with a countable number of fixed point equations. In the case of finite systems without parameters the finiteness of the set A of atoms is not an issue; one can assume, without loss of generality, that A is a finite set containing all the atoms occurring in all the specifications of the given (finite) system of fixed point equations. However, the situation is different in the case of a system of fixed point specifications with parameters. This is due to the fact that if A is infinite and $a_1,\dots,a_n,\dots$ is an infinite list of mutually distinct atomic processes in A then the process $p = \langle a_1, a_1a_2, \dots, a_1\dots a_n, \dots\rangle \in A^\infty$ can occur as a parameter in a fixed point specification. In particular it is shown that

Theorem 1.3 [Extended Existence Theorem; A is finite]
Every countable system $\Sigma = \{x_k = T_k(x_1,\dots,x_{n(k)},p_1,\dots,p_{m(k)}) : k > 0\}$ of fixed point equations, with parameters $p_1,\dots,p_m,\dots \in A^\infty$, has a solution in $(A^\infty)^\omega$.

Theorem 1.4 [Extended Existence Theorem; A is arbitrary]
Every finite system $\Sigma = \{x_k = T_k(x_1,\dots,x_n,p_1,\dots,p_m) : k = 1,\dots,n\}$ of fixed point equations, with parameters $p_1,\dots,p_m \in A^\infty$, has a solution in $(A^\infty)^\omega$.

As an immediate corollary of theorem 1.4 one obtains, for arbitrary A, an existence theorem for countable, infinite, **diagonal** systems of fixed point equations with parameters.

Moreover, the notion of **guardedness** given in [BK] is generalized to include fixed point equations with parameters. **Guarded** operators T(x) do not always provide equations x = T(x) which have unique solutions in every model of process algebra (pathological counterexamples are in fact easy to give). However, it is one of the many interesting properties of the projective model that in the signature $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, δ, τ, a (with $a \in A$) one can prove the following result: (**notice the omission of the abstraction operator τ_I**).

Theorem 1.5 [Extended Uniqueness Theorem; A is arbitrary]
Every countable system $\Sigma = \{x_k = T_k(x_1,\dots,x_{n(k)},p_1,\dots,p_{m(k)}) : k > 0\}$ of guarded fixed point equations, with parameters $p_1,\dots,p_m,\dots \in A^\infty$, has a

unique solution in $(A^{\infty})^{\omega}$.

In fact, the last theorem is proved for arbitrary (even uncountable) systems of fixed point specifications. However, it appears that it is only the countable case which is applicable in practice.

The converse of the uniqueness theorem appears to be much more intricate. In general, one is interested if the notion of **guardedness** given in the paper fully captures all the specifications which have unique solutions. To be more specific the following partial converse to theorem 1.5 is proved.

Theorem 1.6 [Converse of the Uniqueness Theorem]
Let $T(x)$ be an operator in the signature $+, \cdot, \|, \lfloor\!\lfloor, a$ (with $a \in A$) such that the equation $x = T(x)$ has a unique solution in A^{∞}. If A has an atom which does not occur in $T(x)$ then there exists a guarded operator $S(x)$, without any parameters other than atomic processes in A, such that the equations $x = T(x)$, $x = S(x)$ have exactly the same solutions. In addition, if A has at least two atoms then $T(x)$ itself is guarded.

Remark on Notation:
Throughout the present paper $T(x_1,\ldots,x_n)$ will always denote a (polynomial) operator, i.e. a term built-up from the atomic processes, the variables $x_1,\ldots,x_n$, the atoms a (with $a \in A$) and the operations of the given signature.

2. TOPOLOGY OF THE PROJECTIVE MODEL

Let A_{ω} be the term model; it consists of all terms modulo the equivalence determined by the corresponding theory in the given signature. In addition let the projection function $(.)_n$ be defined as follows on A_{ω}:

$$(a)_n = a,$$
$$(at)_1 = a,$$
$$(at)_n = a(t)_{n-1}, \text{ for } n > 1,$$
$$(t + t')_n = (t)_n + (t')_n \text{ for } n > 0.$$

The atomic process δ (deadlock) will be treated like the atomic processes in A (if δ is in the signature). In the case of τ (internal action) one uses, in addition to the above defining axioms, the following: $(\tau)_n = \tau$ and $(\tau t)_n = \tau(t)_n$. Let $A_n = \{(t)_n : t \in A_\omega\}$. The projective model A^∞ consists of all infinite sequences $\langle p_1, p_2, \ldots, p_n, \ldots\rangle$ such that each p_n belongs to A_n and $(p_{n+1})_n = p_n$, for all $n > 0$. The term model can be embedded in a natural way in the projective model; because of this, it is considered a subset of the projective model (see [BK]). For any such $p = \langle p_1, p_2, \ldots, p_n, \ldots\rangle$ in A^∞ put $(p)_n = p_n$. For any $p,q \in A^\infty$ such that p is not equal to q let $k(p,q)$ = the least $n > 0$ such that $(p)_n$ is not equal to $(q)_n$. The set A^∞ can be endowed with a **metric space structure** by defining a **distance function** d as follows: $d(p,q) = 2^{-k(p,q)}$ if p is different from q, and $d(p,q) = 0$ otherwise.

This metric was used by Arnold and Nivat (see [AN]) in the context of Denotational Semantics of Concurrency. An equivalent metric was also defined by de Bakker and Zucker (see [dBZ]). For additional information the reader is advised to consult [L] and [Ro].

The following result summarizes all the basic properties of the metric space (A^∞, d) and will be used frequently in the sequel.

Theorem 2.1 [In the signature +, ., $\|$, $\lfloor\!\lfloor$, |, ∂_H, δ, τ, a (with $a \in A$)]
(i) (A^∞, d) is an ultrametric space, i.e. it satisfies the following properties:

(a) $d(p,q) = 0 \Leftrightarrow p = q$.
(b) $d(p,q) = d(q,p)$.
(c) $d(p,q) \le \max\{d(p,r), d(r,q)\}$.

(ii) $p^{(r)} \to p \Leftrightarrow \forall n\, \exists m\, \forall k \ge m\, (p^{(k)})_n = (p)_n$.
(iii) (A^∞, d) is the metric completion of the metric space (A_ω, d'), where d' is the restriction of d on A_ω.
(iv) For all $p \in A^\infty$, $n > 0$, $d(p,(p)_n) \le 2^{-n}$. Hence, $\lim_{n\to\infty}(p)_n = p$.
(v) The operations $(.)_n : A^\infty \to A_n$ are continuous.

Proof: The proof is omitted. For details the reader can consult [L] and [AN].

The forthcoming results of the section will require a finer analysis of the algebraic structure of A^∞. The appropriate signature is +, ., $\|$, $\lfloor\!\lfloor$, |, ∂_H, δ,

τ, a (with $a \in A$).

Lemma 2.2 [Bergstra-Klop]
For any $p \in A^\infty$, and any integers n, m, $((p)_n)_m = (p)_{\min\{n,m\}}$.

Proof: It is enough to prove that $((u)_n)_m = (u)_{\min\{n,m\}}$ holds for terms u in A_ω (the lemma will then follow by passing to the limit using the fact that A_ω is dense in A^∞). The proof is by induction on the length of the given term u. Write u as a finite sum $u = \Sigma_i a_i u_i + \Sigma_r \tau v_r + \Sigma_j b_j + \tau$, where a_i, b_j are atomic processes in A and u_i, v_r are terms; from the representation of u above empty sums are set equal to δ and the term τ may or may not be missing (in the presence of τ the next to the last summand is not necessary since $b_j\tau = b_j$). Then it is true that

$$\begin{aligned}((u)_n)_m &= ((\Sigma_i a_i u_i + \Sigma_r \tau v_r + \Sigma_j b_j + \tau)_n)_m\\ &= (\Sigma_j a_i (u_r)_{n-1} + \Sigma_r \tau (v_r)_n)_m + \Sigma_j b_j + \tau\\ &= \Sigma_j a_i ((u_r)_{n-1})_{m-1} + \Sigma_r \tau((v_r)_n)_m + \Sigma_j b_j + \tau\\ &= \Sigma_j a_i (u_r)_{\min\{n-1,m-1\}} + \Sigma_r \tau(v_r)_{\min\{n,m\}} + \Sigma_j b_j + \tau\\ &= (u)_{\min\{n,m\}}.\end{aligned}$$

This completes the proof of the lemma.

Lemma 2.3
Let $*$ (respectively π) denote any of the binary (respectively unary) operations in the signature $+$, $.$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H. Then for any p, q in A^∞ and any integer n the following equalities hold:

$$(p * q)_n = ((p)_n * (q)_n)_n,$$
$$(\pi(p))_n = (\pi((p)_n))_n.$$

Proof: As before it is enough to prove that the lemma holds for terms u, $v \in A_\omega$ (the lemma will then follow by passing to the limit using the fact that A_ω is dense in A^∞). The proof is tedious but straightforward and can be given by induction on the construction of the terms u, v simultaneously for all the operations in the given signature.

As an immediate corollary one obtains that

Lemma 2.4
Let $*$ (respectively π) denote any of the binary (respectively unary) operations in the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H$. Then for any p, p_1, q, q_1 in A^∞,

$$d(p*p_1, q*q_1) \le \max\{d(p, q), d(p_1, q_1)\}$$
$$d(\pi(p), \pi(q)) \le d(p, q).$$

Consequently, for any operator $T(x_1,\ldots,x_n)$ and any $p_1,\ldots,p_n, q_1,\ldots,q_n \in A^\infty$,

$$d(T(p_1,\ldots,p_n), T(q_1,\ldots,q_n)) \le \max\{d(p_1, q_1),\ldots,d(p_n, q_n)\}.$$

Proof: The second part of the lemma concerning operators follows from the first part using induction on the construction of the operator T. To prove the first part let $k = k(p, q)$, $k_1 = k(p_1, q_1)$ and $s = \min\{k, k_1\}$. Then it is clear that for all $i < s$, $(p)_i = (q)_i$ and $(p_1)_i = (q_1)_i$. It follows from lemma 2.3 that $(p*p_1)_{s-1} = (q*q_1)_{s-1}$ and hence, $s \le k(p*p_1, q*q_1)$, which completes the proof of the lemma.

Example 2.5 [J. W. Klop (unpublished)]
In the presence of τ the space A^∞ is not compact. To see this, construct a sequence $\{t_n\}$ of terms such that for all $n \ne m$, $t_n \ne t_m$ and $(t_n)_1 = t_n$; such a sequence cannot have any convergent subsequence since $d(t_n, t_m) = 1/2$, for $n \ne m$. The first five members of the sequence are given by: $t_0 = a$, $t_1 = \tau a$, $t_2 = \tau$, $t_3 = \tau(a + \tau)$, $t_4 = a + \tau a$. For higher indices one defines by induction

$$t_{4k+i} = \tau\, t_{4k+i-1} \qquad \text{if } i = 1, 3,$$
$$t_{4k+i} = t_{4k+i-3} + t_{4k+i-5} \qquad \text{if } i = 0, 2.$$

On the contrary, if τ is not present in the signature then the space A^∞ can be compact as the theorem below shows.

Theorem 2.6 [In the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, \delta, a$ (with $a \in A$)]

(i) A is finite $\Leftrightarrow$ (A^∞, d) is compact.

(ii) In fact, if A is finite then (A^∞, d) must be topologically homeomorphic to the Cantor set.

Proof: (i) ($\Leftarrow$) Assume on the contrary that A is infinite and let $a_1,\ldots,a_n,\ldots$

be an infinite list of pairwise distinct atoms in A. Then the sequence $\{a_n\}$ cannot have any convergent subsequence since $d(a_n, a_m) = 1/2$, for $n \neq m$. Clearly, this is a contradiction.
($\Rightarrow$) Since τ is not in the signature and A is finite each A_n is finite and hence compact. It follows that A^∞ is compact (see [Du], page 429).
(ii) This is immediate from [Ri], page 223. A more direct proof can be given along the following lines. For each $u \in A_\omega$ let $C(u) = \{p \in A^\infty : (p)_n = u$, for some integer $n > 0\}$ and let $n(u)$ = the least n such that $(u)_n = u$.. It can be shown that $\{C(u) : u \in A_\omega\}$ is a family of nonempty subsets of A^∞ such that for all $u, v \in A^\infty$ exactly one of the following three conditions holds: $C(u) \subseteq C(v)$ or $C(v) \subseteq C(u)$ or $C(u)$ and $C(v)$ are disjoint. Moreover, each $C(u)$ is the (finite) disjoint union of those sets $C(v)$ such that $n(v) = n(u) + 1$ and $(v)_{n(u)} = u$. Finally, the homeomorphism between A^∞ and the Cantor set can be constructed as in [Di], page 84. Details are left to the reader.

3. SOLVING EQUATIONS WITH PARAMETERS

Suppose that it is desired to find a solution to an equation of the form $x = T(x)$. If $T(x)$ is contractive (see idea 3.3 below) then for any element $q \in A^\infty$, $\lim_{n\to\infty} T^n(q)$ is the unique solution of the equation $x = T(x)$. However, if $T(x)$ is not contractive then Banach's contraction principle does not apply. Thus, one is faced with the problem of finding solutions to $x = T(x)$, for an arbitrary (not necessarily contractive) operator $T(x)$. Motivated by Banach's contraction principle one is tempted to prove that for any $q \in A_\omega$, $\lim_{n\to\infty} T^n(q)$ is a solution of the equation $x = T(x)$. In fact, this trick works. An outline of the idea of the proof, due to [BK1], is as follows. Let $q \in A_\omega$. One shows by induction on m that the sequence $\{(T^n(q))_m\}$ is constant, for all but a finite number of n. This is done by induction on the construction of T; to handle the operation +, which is also the most difficult case, one needs to use Koenig's infinity lemma (i.e. any infinite, finite branching tree has an infinite branch). For details the reader should consult [BK1].

Now suppose that it is required to solve a fixed point equation $x = T(x,p)$, where $p \in A^{\infty}$ is a parameter and $T(x,y)$ is an operator built-up from the variables x,y in the signature $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, a (with $a \in A$). Depending on the topological properties of the space (A^{∞},d) and the structure of the operator T one of the following three ideas can be used.

Idea 3.1 Compactness Argument:

For each positive integer n consider the fixed point equation $x = T(x,(p)_n)$. Each such equation has a solution in A^{∞} (by the existence theorem 1.1), say x_n, such that $x_n = T(x_n,(p)_n)$. However, if A is compact then the sequence $\{x_n\}$ must have a convergent subsequence, say $\{x_{n(k)}\}$, such that $x_{n(k)}$ converges to the limit point $x \in A^{\infty}$. But, it is clear from the continuity of the operator T that $x = \lim_{k\to\infty} x_{n(k)} = \lim_{k\to\infty} T(x_{n(k)}, (p)_{n(k)}) = T(\lim_{k\to\infty} x_{n(k)}, \lim_{k\to\infty} (p)_{n(k)}) = T(x,p)$. Thus, the limit point x is the desired solution of the given fixed equation.

The main limitation of this method is that it works only in the case where the topological space (A^{∞},d) is compact (this excludes the possibility of an infinite set A of atoms or even using τ in the signature).

Idea 3.2 Density Argument:

For each $t \in A_{\omega}$ let T_t be the operator obtained from T by substituting each occurrence of y in $T(x,y)$ by t. The solution of the equation $x = T(x,t)$ is obtained as the limit of the sequence $\{T_{n,t}(a)\}$, where $a \in A$ is a given fixed atom and $T_{n,t}(a)$ is the n-th iteration of the operator T_t, i.e. it is defined inductively as follows: $T_{1,t}(a) = T_t(a)$ and $T_{n+1,t}(a) = T_t(T_{n,t}(a))$. Let $\sigma_T : A_{\omega} \to A^{\infty}$ be the function defined by $t \to \sigma_T(t) = \lim_{n\to\infty} T_{n,t}(a)$. It can be shown that the function σ_T is uniformly continuous. In fact the claim below states that σ_T is non expansive.

Claim: $d(\sigma_T(u),\sigma_T(u)) \le d(u,v)$, for all $u, v \in A_{\omega}$.

Proof of the claim: Using the continuity of the distance function d one obtains that

$$d(\sigma_T(u),\sigma_T(u)) = \lim_{n\to\infty} d\,(T_{n,u}(a), T_{n,v}(a)) \le d(u,v),$$

which proves the claim.

Thus, σ_T is a uniformly continuous mapping from a dense subset A_ω of A^∞ into the complete metric space A^∞. It follows that σ_T can be extended **by continuity** to a continuous mapping $\omega_T : A^\infty \to A^\infty$ (see [Di]). Moreover, for any $p \in A^\infty$ it is true that $\omega_T(p) = \lim_{n\to\infty} \sigma_T((p)_n)$. Now it is possible to find a fixed point of the original equation. Indeed, $\sigma_T((p)_n) = T(\sigma_T((p)_n), (p)_n)$. Using the continuity of the operator T and passing to the limit as $n\to\infty$ it follows that $\omega_T(p) = T(\omega_T(p), p)$, as desired.

The main advantage of this method is that it allows the set A of atomic actions to be finite or infinite. In fact, one uses only the density of A_ω (in A^∞) as well as the completeness of the metric space (A^∞, d). Its main disadvantage is that one must have a priori a **uniform** way of obtaining solutions of the equation $x = T(x,t)$ (i.e. uniformly in t) as was the case above.

Idea 3.3. Banach's Contraction Principle:

Any operator $T(x,p)$ (with parameter p in A^∞) determines a continuous (in fact nonexpansive) mapping $x \to T(x, p)$ from A^∞ into A^∞ (see lemma 2.4). In case it is a contraction one can find fixed points by iterating the operator. Call the operator T **contractive** if for all $x, y \in A^\infty$, $d(T(x, p), T(y, p)) \le (1/2)d(x, y)$. It follows from Banach's Contraction Principle (see [DG]) that for any $q \in A^\infty$, $\lim_{n\to\infty} T^n(q, p)$ is the unique fixed point of the equation $x = T(x, p)$ (the reader will benefit from the discussion in [L] and [Ro]).

The three ideas considered above will be used extensively in the sequel in order to solve arbitrary systems of fixed point equations with parameters in A^∞.

4. EXISTENCE THEOREMS IN THE PROJECTIVE MODEL

In this section a proof of theorem 1.3 will be given. First the case of countable systems without parameters is handled; the general theorem will then follow by applying a compactness argument as in section 3. If the space A^∞ is compact so is the **Tychonoff product** $(A^\infty)^\omega$ of countably many copies of A^∞. This can be seen by defining a new metric d_1 on $(A^\infty)^\omega$ as follows: $d_1(x, y) = \Sigma_{n\geq 1} 2^{-n} d(x_n, y_n)$, where , $x = \langle x_n : n \geq 1\rangle$, $y = \langle y_n : n \geq 1\rangle$ (see [Di]). To be more specific it can be shown that

Lemma 4.1 [A is finite]
Every countable system $\Sigma = \{x_k = T_k(x_1,\dots,x_{n(k)}) : k > 0\}$ of fixed point equations has a solution in $(A^\infty)^\omega$.

Proof: Let $a \in A$ be an arbitrary but fixed atomic process. For each positive integer m consider the following finite system Σ_m of m fixed point equations:

$$x_1 = T_1(x_1,\dots,x_m,a,\dots,a)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_m = T_m(x_1,\dots,x_m,a,\dots,a),$$

i.e. for each $k = 1,\dots,m$ replace each occurrence of the variables $x_{m+1},\dots,x_{n(k)}$ in T_k by the above atomic process a. Theorem 1.1 implies that each system Σ_m has a solution, say $s_{1,m},\dots,s_{m,m}$, such that for all $k = 1,\dots,m$, $s_{k,m} = T_k(s_{1,m},\dots,s_{m,m},a,\dots,a)$. For each m let s_m denote the infinite sequence $\langle s_{1,m},\dots,s_{m,m},a,\dots,a,\dots\rangle$. Since A^∞ is compact so is the Tychonoff product space $(A^\infty)^\omega$. It follows that the sequence $\{s_m\}$ has a convergent subsequence, say $s_{m(i)} \to u = \langle u_1,\dots,u_m,\dots\rangle$, as $i \to \infty$. By the choice of the sequence s_m it is true that for all integers i and all $k = 1,\dots,m(i)$,

$$s_{k,m(i)} = T_k(s_{1,m(i)},\dots,s_{m(i),m(i)},a,\dots,a).$$

Now fix the integer k. Then there exists an integer i_0 such that for all $i \geq i_0$, $m(i) \geq n(k)$. However, for each $i \geq i_0$, $m(i) \geq n(k)$ and hence the above equation becomes

$$s_{k,m(i)} = T_k(s_{1,m(i)},\ldots,s_{n(k),m(i)}).$$

Using the continuity of T_k and passing to the limit as $i \to \infty$ one easily obtains that

$$u_k = T_k(u_1,\ldots,u_{n(k)}).$$

This completes the proof of the lemma.

Proof of theorem 1.3 Let $\Sigma = \{x_k = T_k(x_1,\ldots,x_{n(k)},p_1,\ldots,p_{m(k)}) : k > 0\}$ be the system of fixed point equations of theorem 1.3, where $p_1,\ldots,p_m,\ldots \in A^\infty$ are the given parameters. For each integer r consider the countably infinite system Σ_r given by the equations below:

$$x_1 = T_1(x_1,\ldots,x_{n(1)},(p_1)_r,\ldots,(p_{m(1)})_r)$$

$$\begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}$$

$$x_k = T_k(x_1,\ldots,x_{n(k)},(p_1)_r,\ldots,(p_{m(k)})_r)$$

$$\begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}$$

Each Σ_r is a countable system of fixed point equations **without parameters**. Hence lemma 4.1 applies to each Σ_r. For each r, let $s_r = \langle s_{1,r},\ldots,s_{k,r},\ldots\rangle$ be a solution of the system Σ_r, i.e. for all integers k, r, it is true that $s_{k,r}=T_k(s_{1,r},\ldots,x_{n(k),r},(p_1)_r,\ldots,(p_{m(k)})_r)$. Using the compactness of the Tychonoff product space $(A^\infty)^\omega$, it follows that the sequence $\{s_r\}$ has a convergent subsequence, say $s_{r(i)} \to u = \langle u_1,\ldots,u_k,\ldots\rangle$, as $i \to \infty$. Let k be fixed. It follows from the choice of the sequence $\{s_r\}$ that for all integers i, $s_{k,r(i)} = T_k(s_{1,r(i)},\ldots,x_{n(k),r(i)},(p_1)_{r(i)},\ldots,(p_{m(k)})_{r(i)})$. Passing to the limit as $i \to \infty$ and using the continuity of the operator T_k one easily obtains that $u_k = T_k(u_1,\ldots,u_{n(k)},p_1,\ldots,p_{m(k)})$. This completes the proof of theorem 1.3.

Proof of theorem 1.4 Let $\Sigma = \{x_k = T_k(x_1,\dots,x_n,p_1,\dots,p_m) : k = 1,\dots n\}$ be the system of fixed point equations of theorem 1.4, where $p_1,\dots,p_m \in A^\infty$ are the given parameters. For each $t_1,\dots,t_m \in A_\omega$ and each positive integer $i = 1,\dots,n$ define the functions $\sigma_i : (A_\omega)^m \to A^\infty : \langle t_1,\dots,t_m \rangle \to \sigma_i(t_1,\dots,t_m) = \lim_{k\to\infty} \sigma_{i,k}(t_1,\dots,t_m)$, where the sequence $\sigma_{i,k}(t_1,\dots,t_m)$ is defined by induction on k as follows (for simplicity put $t = \langle t_1,\dots,t_m \rangle$):

$\sigma_{i,0}(t) = a$, $(k = 1,\dots,n)$,

$\sigma_{1,k+1}(t) = T_{k+1}(\sigma_{1,k}(t),\dots,\sigma_{n,k}(t),t)$,

$\sigma_{i+1,k+1}(t) = T_{k+1}(\sigma_{1,k+1}(t),\dots,\sigma_{i,k+1}(t),\sigma_{i+1,k}(t),\dots,\sigma_{n,k}(t),t)$, $(k \geq 0)$,

where $(n > i \geq 0)$ and $a \in A$ is a fixed atomic process (the existence of the limit $\lim_{k\to\infty} \sigma_{i,k}(t_1,\dots,t_m)$ will appear in a revised version of [BK1]). It is clear that for each i, k there exists an operator $S_{i,k}(u_1,\dots,u_n,v_1,\dots,v_m)$ such that for all t,

$$\sigma_{i,k+1}(t) = S_{1,k}(\sigma_{1,1}(t),\dots,\sigma_{i-1,k}(t),\sigma_{i,k+1}(t),\dots,\sigma_{i,n}(t),t).$$

The proof is by induction on i; for each i one proves the assertion above in succession for i = 1, i = 2,..., i = n. Just like in section 3 (using the last equation) it can be shown that the functions σ_i are uniformly continuous, (in fact they are nonexpansive). Hence, each σ_i can be expanded to a uniformly continuous mapping $\omega_i : (A^\infty)^m \to A^\infty$. The rest of the details are as in 3.2 and are left to the reader. This completes the proof of theorem 1.4.

As an immediate corollary of theorem 1.4 one can also show that

Corollary 4.2 [Existence theorem for diagonal systems; A is arbitrary] Every countable **diagonal** system $\Sigma = \{x_k = T_k(x_1,\dots,x_k,p_1,\dots,p_{m(k)}) : k > 0\}$ of fixed point equations with parameters $p_1,\dots,p_m,\dots \in A^\infty$ has a solution in $(A^\infty)^\omega$.

Proof: By theorem 1.4, $x_1 = T_1(x_1,p_1,\dots,p_{m(1)})$ has a solution, say s_1. Apply theorem 1.4 once again to find a solution of $x_2 = T_k(s_1,x_2,p_1,\dots,p_{m(2)})$, say s_2. Proceed in this fashion to obtain a solution $\langle s_1,s_2,\dots\rangle$ of the system Σ.

5. GUARDED EQUATIONS

It will be proved in the sequel that a sufficient condition for a fixed point equation to have a unique solution is the notion of **guardedness.** This will be made precise later. However, in order to obtain the most general definition of **guardedness** it will be necessary to define the notion of **guard** (see also [H1], page 28).

Definition 5.1 [In the signature $+, ., \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a$ (with $a \in A$)]
Call $g \in A^\infty$ a **guard** if and only if every finite branch of (the tree corresponding to) g has an edge which is labeled with an atomic process other than τ.

The above definition arises from the following observation. To obtain a uniqueness theorem for fixed point equations one wants to consider fixed point equations of the form $x = T(x)$ such that $T(x)$ is a contraction. Clearly, by lemma 2.4, $T(x)$ is not distance increasing (at least in the signature $+, ., \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a$ ($a \in A$)). Since an operator T is built-up from the signature $+, ., \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a$ ($a \in A$)) and the variable x, it is apparent that one must at first search for a distance contraction principle for the nontrivial operators of minimal length; such terms are of the form gv, with g a parameter and v a variable (the operators τv, $g\|v$, $g\lfloor\!\lfloor v$, $g|v$ etc, are also of minimal length but they will not be considered as guarded since they lead to fixed point equations which do not necessarily have unique solutions.) Hence, one is lead to define g to be a guard if and only if for all $x, y \in A^\infty$, $d(gx, gy) \leq (1/2)\, d(x, y)$ (see definition 5.1 and lemma 5.2.i). It follows from Banach's contraction principle and the completeness of the space (A^∞, d) that the fixed point equation $v = gv$ has a unique fixed point, if g is a guard. It is now an immediate consequence of the definition that that the following result holds.

Lemma 5.2 [In the signature $+, ., \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a$ (with $a \in A$)]
(i) g is a guard $\Leftrightarrow d(gx, gy) \leq (1/2)d(x, y)$, for all $x, y \in A^\infty$.
(ii) If $g_1, g_2 \in A^\infty$ are guards then so are $g_1 + g_2$, $g_1 x$ (for any $x \in A^\infty$), $g_1 \| g_2$, $g_1 \lfloor\!\lfloor g_2$, $g_1 | g_2$, $\partial_H(g_1)$.

Proof: (i) ($\Leftarrow$) Assume g is not a guard. Then g has a finite branch all of whose edges are labeled with τ. It follows from the definition of $(.)_n$ and axiom A4 (on left distributivity) that the inequality $d(gx, gy) \leq (1/2)d(x, y)$ cannot be true for all $x, y \in A^\infty$, a contradiction.

($\Rightarrow$) If g has no finite branch then $gx = g$, for all $x \in A^\infty$, and hence $d(gx, gy) = 0$, for all $x, y \in A^\infty$. Assume that g has finite branches. Since in forming the product gx the process x can only be appended on finite branches of g it is clear that $d(gx, gy) \leq (1/2)d(x, y)$, for all $x, y \in A^\infty$.

(ii) This is straightforward by considering the graphs corresponding to the processes $g_1 + g_2$, $g_1 x$ (for any $x \in A^\infty$), $g_1 \| g_2$, $g_1 \mathbin{\|\!\_} g_2$, $g_1 | g_2$, $\partial_H(g_1)$.

It is now possible to define the notion of guarded operators.

Definition 5.3 [In the signature $+, ., \|, \mathbin{\|\!\_}, |, \partial_H, \delta, \tau, a$ (with $a \in A$)]
Let $T(v_1,\dots,v_n, p_1,\dots,p_m)$ be an operator with variables $v_1,\dots,v_n$ and parameters $p_1,\dots,p_m \in A^\infty$. Call T **guarded** if the following conditions hold:

(i) $T \equiv p$, where $p \in A_\omega \cup \{p_1,\dots,p_m\}$ is a guard.

(ii) $T \equiv pv_i$, where $p \in A_\omega \cup \{p_1,\dots,p_m\}$ is a guard.

(iii) $T \equiv T_1 \cdot T_2$, where T_1 is guarded.

(iv) $T \equiv T_1 + T_2$ or $T_1 \| T_2$ or $T_1 \mathbin{\|\!\_} T_2$ or $T_1 | T_2$, where both operators T_1, T_2 are guarded.

(v) $T \equiv \partial_H(T_1)$, where T_1 is guarded.

Remark: As in [BBK] one might be tempted to define $T(v_1,\dots,v_n,p_1,\dots,p_m)$ guarded if for any occurrence of a variable v_i in T, the operator T has a subterm of the form pS, where $p \in A_\omega \cup \{p_1,\dots,p_m\}$ is a guard, and this occurrence of v_i occurs in S. However, it can be shown by induction on the construction of operators that every operator guarded in this sense will also be guarded in the sense of definition 5.3.

The next result will be useful in the sequel.

Lemma 5.4 [In the signature $+, ., \|, \mathbin{\|\!\_}, |, \partial_H, \delta, \tau, a$ (with $a \in A$)]
For any guarded operator $T(v_1,\dots,v_n,p_1,\dots,p_m)$ with variables $v_1,\dots,v_n$ and parameters $p_1,\dots,p_m \in A^\infty$ and any $x_1,\dots,x_n \in A^\infty$, $T(x_1,\dots,x_n, p_1,\dots,p_m)$ is a guard.

Proof: The proof is by induction on the construction of T using part (ii) of lemma 5.2.

6. UNIQUENESS THEOREMS IN THE PROJECTIVE MODEL

The main lemma used in proving the uniqueness of the solutions of a system of guarded fixed point specifications is given below.

Lemma 6.1 [In the signature $+$, $.$, $\|$, $⫛$, $|$, ∂_H, δ, τ, a (with $a \in A$)]
For any guarded operator $T(v_1,\ldots,v_n,p_1,\ldots,p_m)$ with variables $v_1,\ldots,v_n$ and parameters $p_1,\ldots,p_m \in A^\infty$ and any $x_1,\ldots,x_n$, $y_1,\ldots,y_n \in A^\infty$, one can prove that $d(T(x_1,\ldots,x_n, p_1,\ldots,p_m), T(y_1,\ldots,y_n, p_1,\ldots,p_m) \leq (1/2) \max \{d(x_1,y_1),\ldots, d(x_n,y_n)\}$.

Proof: Let x, y, p be abbreviations for the sequences $x_1,\ldots,x_n$, $y_1,\ldots,y_n$, $p_1,\ldots,p_m$ respectively. The proof is by induction on the construction of the operator T. The result is clear if T is one of the forms p or pv_i. If T is of one of the forms $T_1 + T_2$, $T_1 \| T_2$, $T_1 ⫛ T_2$, $T_1 \mid T_2$, then by definition of guardedness both T_1, T_2 must be guarded. Hence it follows by the induction hypothesis that

$$\begin{aligned} d(T(x, p), T(y, p)) &\leq \max \{d(T_1(x, p), T_1(y, p)), d(T_2(x, p), T_2(y, p))\} \\ &\leq (1/2) \max \{d(x_1,y_1),\ldots, d(x_n,y_n)\}. \end{aligned}$$

The case $T \equiv \partial_H (T_1)$ is similar. It remains to consider the case $T \equiv T_1 \cdot T_2$, where T_1 is (but T_2 does not have to be) guarded. Now, it is clear that

$$\begin{aligned} d(T(x, p), T(y, p)) &= d(T_1(x, p)T_2(x, p)), T_1(y, p)T_2(y, p)) \\ &= d(g_1u_1, g_2u_2), \end{aligned}$$

where $g_1 = T_1(x, p)$, $u_1 = T_2(x, p)$, $g_2 = T_1(y, p)$, $u_2 = T_2(y, p)$. However, lemma 5.4 implies that both g_1, g_2 are guards. Hence, the result will follow from the following claim:

Claim: $d(g_1u_1, g_2u_2) \leq \max \{d(g_1, g_2), (1/2)\, d(u_1, u_2)\}$.

Assuming the claim the remaining proof is easy. Indeed, using lemma 2.4 and the induction hypothesis:

$$d(T(x, p), T(y, p)) = d(g_1u_1, g_2u_2)$$
$$\le \max\{d(g_1, g_2), (1/2)\, d(u_1, u_2)\}$$
$$\le \max\{d(T_1(x, p), T_1(y, p)), (1/2) d(T_2(x, p), T_2(y, p))\}$$
$$\le (1/2) \max\{d(x_1,y_1),\dots, d(x_n,y_n)\}.$$

Proof of the claim: Using part (i) of lemma 5.2 and the fact that d is an ultrametric it is easy to show that

$$d(g_1u_1, g_2u_2) \le \max\{d(g_1u_1, g_1u_2), d(g_1u_2, g_2u_2)\}$$
$$\le \max\{(1/2)\, d(u_1, u_2), d(g_1, g_2)\}.$$

This completes the proof of the lemma.

Theorem 6.2 [Uniqueness theorem, in: +, ., $\|$, $\|\!\_$, |, ∂_H, δ, τ, a (with $a \in A$)] Let $\Sigma = \{v_i = T_i(V_i, P_i) : i \in I\}$ be an arbitrary system of guarded fixed point specifications such each V_i (respectively P_i) is a finite set of variables (respectively parameters in A^∞) such that $\{v_i : i \in I\} = \cup\{V_i : i \in I\}$. Then Σ has a unique fixed point in $(A^\infty)^I$.

Proof: Consider the metric space $(E, D) = ((A^\infty)^I, D)$, where the metric D is defined as follows: $D(X,Y) = \sup\{d(x_i, y_i) : i \in I\}$, for $X = \langle x_i : i \in I\rangle$, $Y = \langle y_i : i \in I\rangle$. The proof uses the following

Claim: (E, D) is a complete metric space.

Proof of the claim: Let $\{X_n\}$ be a Cauchy sequence in (E, D). Let $X_n = \langle x_{n,i} : i \in I\rangle$. Given $\varepsilon > 0$, let n_0 be an integer such that for all $n, m \ge n_0$,

$$D(X_n, X_m) = \sup\{d(x_{n,i}, x_{m,i}) : i \in I\} \le \varepsilon.$$

It follows that for each $i \in I$ the sequence $\{x_{n,i}\}$ is a Cauchy sequence in the metric space (A^∞, d). By completeness of this last metric space, for each $i \in I$, the sequence $\{x_{n,i}\}$ has a limit point, say x_i. Put $X = \langle x_i : i \in I\rangle$. It remains to show that the sequence $\{X_n\}$ converges to X in the metric D. Indeed, let $\varepsilon > 0$ be given and let n_0 be such that for all $n, m \ge n_0$,

$$\sup\{d(x_{n,i}, x_{m,i}) : i \in I\} \le \varepsilon.$$

Fix $i \in I$. Then $d(x_{n,i}, x_{m,i}) \le \varepsilon$, for all $n, m \ge n_0$. Next, pass to the limit as

$m \to \infty$ and use the continuity of d to conclude that $d(x_{n,i}, x_i) \leq \varepsilon$, for all $n \geq n_0$. It follows that $D(X_n, X) \leq \varepsilon$, for all $n \geq n_0$. This completes the proof of the claim.

To finish the proof of the theorem notice that the function $T : E \to E$ defined by:

$$X \to T(X) = \langle T_i(X_i, P_i) : i \in I \rangle,$$

where for each $i \in I$, $X_i = \{x_k : v_k \in V_i\}$, is a contraction. Indeed,

$$\begin{aligned} &D(T(X), T(Y)) \leq \\ &\sup \{d(T_i(X_i, P_i), T_i(Y_i, P_i) : i \in I\} \leq \\ &(1/2) \sup \{d(x_i, y_i) : i \in I\} \leq \\ &(1/2) D(X, Y). \end{aligned}$$

Clearly, this is an immediate consequence of lemma 6.1 using the hypothesis $\{v_i : i \in I\} = \cup\{V_i : i \in I\}$. This completes the proof of the theorem.

It is essential to note that the abstraction operator τ_I (see [BK] for the appropriate definition) is not in the signature of the statement of theorem 6.2.

Example 6.3 (C. A. R. Hoare, J. W. Klop)
Equation $x = a\tau_{\{a\}}(x)$ has more than one solution in $\{a, b\}^\infty$, e.g. any x of the form $x = ay$, where y satisfies $y = \tau_{\{a\}}(y)$.

The last part of this section will be dedicated to the converse of the uniqueness theorem.

Proof of theorem 1.6 Without loss of generality it can be assumed that $\|$ does not occur in the operator T (this is because $x \| y = y \mathbin{|\!\!\lfloor} x + x \mathbin{|\!\!\lfloor} y$., by axiom CM1). It is clear that T(x) must be a sum of terms of the following form:

(i) $a\, t_1(x)$,

(ii) $x \mathbin{|\!\!\lfloor} t_2(x)$,

(iii) $x\, t_3(x)$,

(iv) a,

(v) x.

where the atoms a in (i) and (iv) range over a finite subset B of A and $t_1(x)$, $t_2(x)$, $t_3(x)$ are operators (and some category of summands might be missing from the sum above). Without loss of generality it can be assumed that at least one of the terms in (ii), (iii) or (v) occurs as a summand of T(x). It is clear that for all $x \in A^\infty$,

$$(T(x))_1 = \Sigma_{a \in B}\, a + (x)_1.$$

It is now easy to show by induction on $n \geq 1$, that for all $x \in A^\infty$,

$$(T^n(x))_1 = \Sigma_{a \in B}\, a + (x)_1.$$

For any $q \in A_\omega$ let $l(q) = \lim_{n\to\infty} T^n(q)$ be the fixed point of $x = T(x)$ obtained from q by iterating the operator T (see [BK1]). It is clear that for any $q \in A^\infty$,

$$(l(q))_1 = \Sigma_{a \in B}\, a + (q)_1.$$

It follows that in order to obtain two different fixed point of $x = T(x)$ one must find two terms $p, q \in A_\omega$ such that

$$\Sigma_{a \in B}\, a + (p)_1 \neq \Sigma_{a \in B}\, a + (q)_1.$$

If $B \neq \emptyset$ then the above observation would imply that $x = T(x)$ has at least two solutions, namely $p = \Sigma_{a \in B}\, a$ and $q = b$, with $b \in A-B$, which is a contradiction. Hence it can be assumed that $B = \emptyset$. If A had at least two distinct atoms, say a, b, then by the above observation l(a) and l(b) would be two distinct solutions of $x = T(x)$, which is also a contradiction.

Hence, without loss of generality, it can be assumed that A consists of a single atom, say a, and the operator T(x) is atom-free (i.e. the atom a does not occur in T). Hence, the operator T(x) is a finite sum of terms of the form (ii), (iii) and (v). It will be shown that in fact the summand x cannot occur in T(x). Let x be a fixed point of the operator T. Clearly, $(y)_1 = a$, for all $y \in A^\infty$ (since $A = \{a\}$). Moreover, $(x)_2$ equals a sum consisting of summands of the form:

$$(x)_2,\ ((x)_2(t_2(x))_1)_2 = ((x)_2\, a)_2,\ ((x)_2 \mathbin{\|\!\underline{\ }} (t_3(x))_1)_2 = ((x)_2 \mathbin{\|\!\underline{\ }} a)_2.$$

It follows that for $p = a^2$ and $q = a + a^2$ one can prove by induction on n that

$$(T^n(p))_2 = a^2 \text{ and } (T^n(q))_2 = a + a^2.$$

In this way one obtains two distinct fixed points l(p), l(q) of T(x), since

$(1(p))_2 = a^2$, $(1(q))_2 = a + a^2$, which is a contradiction.

It follows that T(x) is a sum of operators of the form (ii) or (iii). In this case it will be shown that the unique solution of $x = T(x)$ must be a^ω. Hence, there is a guarded operator S(x), namely $S(x) = ax$, such that $x = T(x)$, $x = S(x)$ have exactly the same fixed point. Let $x \in A^\infty$ be a fixed point of T. It will be shown by induction on $n > 0$ that $(x)_n = a^n$. Since $A = \{a\}$, the result is clear for $n = 1$. Assume it is true for $n > 0$; to prove it for $n+1$. Indeed, $(x)_{n+1}$ is a sum of terms of the form

$$((x)_{n+1}(t_2(x))_n)_{n+1},\ ((x)_{n+1} \mathbin{|\!\!\lfloor} (t_3(x))_n)_{n+1}.$$

By induction hypothesis $(x)_n = a^n$. However it is easy to show that every tree $t \in \{a\}_\omega$ all of whose branches have length $> n$ must satisfy $(t)_{n+1} = a^{n+1}$. In particular $(x)_{n+1} = a^{n+1}$, and the proof of the claim is complete. This completes the proof of the theorem.

An immediate consequence of the proof of theorem 1.6 is the following

Corollary 6.4 [In the signature $+$, $\cdot$, $\|$, $\mathbin{|\!\!\lfloor}$, a (with $a \in A$)]
If the operator T(x) is atom-free and the equation $x = T(x)$ has a unique solution in $\{a\}^\infty$ then its unique solution must be a^ω.

Example 6.5
Some examples of atom-free polynomial operators with unique solutions in $\{a\}^\infty$ are: $x = x^n$, with $n > 1$, or $x = x \| x$, etc. (If in addition, the atom δ were present in the signature then δ would also be a solution of $x = x^n$.)

7. DISCUSSION AND OPEN PROBLEMS

The proofs of theorems 1.3, 1.4 and 1.5 are signature-free. In fact they depend only on the statements of theorems 1.1 and 1.2 and the topological properties of A^∞ (compactness of A^∞, if A is finite, completeness and the density of A_ω in A^∞). However, the proof of theorem 1.1 is rather combinatorial in nature. It would be useful if one could prove theorem 1.1 using theorem 2.2.ii and an appropriate fixed point theorem for the Cantor set, because the proof would be topological and hence extendible to bigger signatures. Theorem 1.4 does not seem to be the most general result one might hope to prove. For example, it is not known if the theorem is true for infinite systems, with a not necessarily finite alphabet A (as is the case in theorem 1.3).

Theorem 1.6 is only an attempt to justify the fact that guarded equations are the only ones which have unique fixed points. However, it is not known if the result is true for systems of arbitrarily many equations or even in bigger signatures. The proof given here is combinatorial in nature and hence its direct extension to arbitrary systems would most likely be quite complex. It might be possible however to give a proof using results from the theory of metric spaces (see [DG]). For example, it can be shown that if for all n >0, $T^n(x)$ has at most one fixed point (notice that no existence of the fixed point of the operator $T^n(x)$ is asserted) then T(x) must have exactly one fixed point (see [B]). A similar result can also be proved using the deeper result given in [J].

8. ACKNOWLEDGEMENTS

This research was carried out while the author was visiting the Computer Science Department of the University of Amsterdam. Research was partially supported by Esprit under contract no. 432, Meteor.

I would like to thank all the participants of the P.A.M. seminar for their numerous comments. However, I am particularly indebted to J. Baeten, W. Bouma and J. W. Klop, whose valuable criticisms have guided me throughout my research on the present paper.

REFERENCES

[AN] Arnold, A. and Nivat, M., The Metric Space of Infinite Trees: Algebraic and Topological Properties, Fundamenta Informatica, 3, 4(1980), pp. 445 - 476.

[BBK] Baeten, J., Bergstra, J. A. and Klop, J. W., On the Consistency of Koomen's Fair Abstraction Rule, Department of Computer Science Technical Report, CS-R8421, Centre of Mathematics and Computer Science, Amsterdam, November 1984.

[dBZ] de Bakker, J. W. and Zucker, J. I., Denotational Semantics of

Concurrency, Proc. 14th STOC, pp. 153 - 158, 1982.

[dBZ1] **de Bakker, J. W. and Zucker, J. I., Processes and the Denotational Semantics of Concurrency,** Information and Control, Vol. 54, No. 1/2, pp. 70 - 120, 1982.

[BK] **Bergstra, J. A. and Klop, J. W., Algebra of Communicating Processes,** in: Proceedings of the CWI Symposium on Mathematics and Computer Science, J. W. de Bakker, M. Hazewinkel and J. K. Lenstra, eds., 1986.

[BK1] **Bergstra, J. A. and Klop, J. W., Fixed Point Semantics in Process Algebras,** Department of Computer Science Technical Report, IW 206/82, Mathematisch Centrum, Amsterdam, 1982, to appear in revised form.

[B] **Bessaga, C., On the Converse of the Banach Fixed Point Principle,** Colloquium Mathematicum, Vol. VII, pp. 41 - 43. 1959.

[Du] **Dugundji, J., Topology,** Allyn and Bacon, 1966.

[DG] **Dugundji, J. and Granas, A., Fixed Point Theory,** Vol. I, PWN, Warszaw 1982.

[Di] **Dieudonné, J., Elements d' Analyse,** Tom 1, Gauthier-Villars, Paris, 1968.

[H] **Hoare, C. A. R., Communicating Sequential Processes,** Comm. ACM, 21, 8, 1978.

[H1] **Hoare, C. A. R., Communicating Sequential Processes,** Prentice/Hall, 1985.

[vG] **Van Glabbeek, R. J., Oplossingen van Recursieve Specificaties over ACP_τ,** unpublished manuscript, January 1986.

[J] **Janos, L., A Converse of Banach's Contraction Theorem,** Proceedings of American Mathematical Society, Vol. 18. pp. 287 - 289, 1967.

[L] **Lloyd, J., Foundations of Logic Programming,** Springer Verlag, 1984.

[M] **Milner, R., A Calculus for Communicating Systems,** Springer Verlag Lecture Notes in Computer Science, Vol. 92, 1980.

[Ri] **Rinow, W., Topologie,** VEB DVW, Berlin, 1975.

[Ro] **Rounds, W. C., Applications of Topology to Semantics of Communicating Processes,** in: Seminar on Concurrency, Springer Verlag Lecture Notes in Computer Science, Vol. 197, 1985, pp. 360 - 372.