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AFDELING TOEGEPASTE WISKUNDE TW 248/83 DECEMBER
(DEPARTMENT OF APPLIED MATHEMATICS)

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LINEAR VOLTERRA CONVOLUTION EQUATIONS: SEMIGROUPS, SMALL SOLUTIONS AND CONVERGENCE OF PROJECTION OPERATORS

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Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 45D05,47D05

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by

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ABSTRACT

In this paper we consider the initial function semigroup and the forcing function semigroup generated by linear Volterra integral equations of convolution type. We prove that the two types are adjoints of each other in the sense that the adjoint of the one type is the other type semigroup corresponding to the equation with transposed kernel. Moreover the semigroups are equivalent. We prove that the absence of small solutions is equivalent to the injectivity of a structural operator F which maps initial functions into forcing functions. We show the convergence of the spectral projection operators corresponding to the (purely) point spectrum of the infinitesimal generators on a dense subset of the state space for a special class of equations.

KEY WORDS & PHRASES: *Volterra integral equation, semigroup, adjoint semigroup, structural operator, decomposition according to the spectrum of the infinitesimal generator, convergence of projection operators, small solution.*

1. INTRODUCTION

We discuss two types of semigroups for the Volterra convolution equation

$$x(t) = \int_{-\infty}^{t} \zeta(t-\tau)x(\tau)d\tau, \qquad t \in \mathbb{R}_+ .$$

Here x takes values in $\mathbb{R}^n$, $\mathbb{R}_+ = [0,\infty)$, and we assume that ζ is a $n \times n$ matrix with elements in $L_1[0,b]$, $0 < b < \infty$, which vanishes for $t \geq b$. Therefore we can rewrite this equation as

$$(1.1) \qquad x(t) = \int_0^b \zeta(\tau)x(t-\tau)d\tau, \qquad t \in \mathbb{R}_+ ,$$

which we provide with initial condition

$$(1.2) \qquad x(t) = \phi(t), \qquad -b \leq t < 0,$$

where

$$\phi \in L_p[-b,0], \qquad 1 \leq p \leq \infty.$$

The first semigroup associated by (1.1)-(1.2) acts on initial functions and is defined by translation along the solution. One solves (1.1)-(1.2) (see section 2) and defines $(T(s)\phi)(t) = x(t+s)$, $s \in \mathbb{R}_+$, $-b \leq t \leq 0$.

Related with equation

$$(1.3) \qquad x(t) = \int_0^t \zeta(\tau)x(t-\tau)d\tau + f(t),$$

where $f(t) \in \tilde{L}_p[0,b] = \{g \in L_p(\mathbb{R}_+) \mid g(t) = 0 \text{ for } t \geq b\}$, $1 \leq p \leq \infty$, is the semigroup which is defined by tracing the *forcing* of the translated equation. $(S(s)f)(t) = x(t+s) - \int_0^t \zeta(\tau)x(s+t-\tau)d\tau$. See Diekmann [10], Miller [22], Miller & Sell [23]. It was shown for the first time by Burns & Herdman [5] in the case of a Volterra integro-differential equation with infinite delay that these two semigroups are related by duality provided that one replaces in one of the equations ζ by its conjugated transpose ζ^T. In [11] Diekmann pointed out that this is a quite general property of delay equations. For neutral differential equations this has been worked out in detail by Salamon [26]. See also Staffans [27] for a general functional equation.

The use of the two semigroups makes the bilinear form (see for instance [14,16]), redundant. The semigroups are intertwined by so called structural operators F and G. F maps the space of initial functions into the space of forcing functions, whereas G does the opposite. The intertwining relations are

$$T(s)G = GS(s), \qquad FT(s) = S(s)F.$$

Quite similar operators have been introduced by Bernier and Manitius [3,21], but there the distinction between forcing functions and initial functions is less explicit.

One of the aims of this paper is to line up the results obtained by Delfour and Manitius (see [9,21]), making a systematic use of the two semigroup approaches. Here this is done for Volterra integral equations. See Verduyn Lunel [31] for corresponding results for functional differential equations of retarded type.

We also study some properties of solutions of the Volterra equation, which are closely related to properties of the semigroups. First we show that the absence of "small solutions", these are solutions of (1.1) which vanish after finite time, is equivalent to F being injective. This extends a result of Manitius [21]. See also Verduyn Lunel [31].

In the second place we prove that the conjecture, see Salamon [26, pag.136], that the state space $L_p[-b,0]$ can be decomposed as $\overline{R(T(nb))} \oplus N(T(nb))$ cannot be true in general (compare also Hale [14, pag.64]). We propose another conjecture

$$(1.4) \qquad L_p[-b,0] = \overline{\overline{R(T(nb))} \oplus N(T(nb))}\ ?$$

Note that as a consequence of Henry's result [15, Corollary 2] $\overline{R(T(s))}$ and $N(T(s))$ are constant for $s \geq nb$.

In his famous article "On the integral equation of renewal theory" Feller [13] already remarked that the series expansion of the solution of

(1.1) in terms of the generalized eigenfunctions of the infinitesimal generator of $\{T(s)\}$ does not have to converge. In [2] Bellman & Cooke have extensively studied such expansions for some scalar differential difference equations. Their results were extended to systems of equations by Banks & Manitius [1]. Here we prove corresponding results for a class of Volterra equations, which include the results of [1] and [2].

The paper is organised as follows. In section 2 we introduce the semigroups $\{T(s)\}$ and $\{S(s)\}$ and the structural operators F and G. In section 3 we study the small solutions and state space decompositions. The main results of that section are contained in Theorem 3.10 and 3.12. The last section is devoted to convergence results.

Notation

$\mathbb{R}^n$	real n-dimensional Euclidian space
$\mathbb{C}^n$	complex n-dimensional space
$\tilde{L}_p[0,b]$	$\{f \in L_p(\mathbb{R}_+) \mid f(t) = 0 \quad \text{for } t \geq b\}$ $(1 \leq p \leq \infty)$
$\tilde{W}^{1,p}[0,b]$	the Sobolev space of absolute continuous functions on $\mathbb{R}_+$ which vanish for $t \geq b$ and with derivative in $L_p(\mathbb{R}_+)$
$L(X;Y)$	the set of bounded linear operators of the normed space X into the normed space Y.
$f*g$	$f*g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$
f_s	$f_s(t) = f(t+s)$
supp	the support of an L_p function is meant in the sense of distributions.

2. THE TWO SEMIGROUPS

We will define x to be a solution of (1.1)-(1.2) on the interval $[-b,\omega)$, $0 < \omega \leq \infty$, if $x \in L_p^{loc}[-b,\omega)$ satisfies (1.2) on the interval $[-b,0)$ and (1.1) on $[0,\omega)$. We will see in a moment that we can take $\omega = \infty$.

We can rewrite equation (1.1)-(1.2) as the renewal equation (1.3) where the forcing function f equals $f(t) = \int_t^b \zeta(\tau)\phi(t-\tau)d\tau$.

The Resolvent R

Equation (1.3) can be solved explicitly in terms of the so-called resolvent. More precisely

$$(2.1) \qquad x(t) = f(t) - R*f(t), \qquad t \in \mathbb{R}_+ ,$$

where R satisfies the matrix equation

$$(2.2) \qquad R(t) = \zeta*R(t) - \zeta(t), \qquad t \in \mathbb{R}_+ .$$

THEOREM 2.1. *Equation* (2.2) *has a unique matrix-valued solution* $R \in L_1^{loc}(\mathbb{R}_+)$. *This solution, which is called the resolvent, has the following properties:*

(i) *for any* $f \in L_p^{loc}(\mathbb{R}_+)$ *the equation* (1.3) *has a unique solution* $x = x(t;f) \in L_p^{loc}(\mathbb{R}_+)$ *given explicitly by* (2.1);

(ii) R *commutes with* ζ *in the convolution algebra, i.e.,* $R*\zeta = \zeta*R$;

(iii) *there exists a real number* λ_o *such that the mapping* $t \mapsto R(t)e^{-\lambda t}$ *belongs to* $L_1(\mathbb{R}_+)$ *for* $\mathrm{Re}\ \lambda > \lambda_o$.

Part (iii) of this theorem, which is the essential part, is based on the theorem of Wiener & Levy and we refer to Paley and Wiener [24, section 18], Miller [22, section IV.5 and appendix I.4] or Cordoneanu [8, section I.3].

As a consequence, (1.1)-(1.2) has a unique solution on $[-b,\infty)$ which we denote by $x(t;\phi)$.

The semigroup T(s)

Define for $s \in \mathbb{R}_+$, $\phi \in L_p[-b,0]$ and $-b \le t \le 0$:

$$(2.3) \qquad (T(s)\phi)(t) = x(s+t;\phi).$$

THEOREM 2.2. T(s) *is a strongly continuous semigroup in the space* $L_p[-b,0]$, *i.e.:*

(i) $T(s)T(\sigma) = T(s+\sigma), \qquad s,\sigma \in \mathbb{R}_+$,

(ii) $T(0) = Id$

(iii) $\lim_{h\downarrow 0} \|T(h)\phi-\phi\|_{L_p[-b,0]} = 0, \qquad \forall\phi \in L_p[-b,0]$, *Moreover, this semigroup satisfies:*

(iv) *the explicit representation of* T(s) *in terms of the kernel and the resolvent reads:*

$$(T(s)\phi)(t) = \begin{cases} \phi(t+s) & -b \le t < \max\{-s,-b\} \\ \int_0^b Q(\tau,t+s)\phi(-\tau)\,d\tau & \max\{-s,-b\} \le t \le 0, \end{cases}$$

where by definition

$$Q(t,s) = \zeta_t(s) - R*\zeta_t(s),$$

(v) *for* $s > 0$, $T(s)$ *is the sum of the nilpotent bounded linear operator* $U(s)$

$$(U(s)\phi)(t) = \begin{cases} \phi(t+s) & -b \le t \le \max\{-s,-b\} \\ 0 & \max\{-s,-b\} \le t \le 0, \end{cases}$$

and the compact linear operator $V(s)$

$$(V(s)\phi)(t) = \begin{cases} 0 & -b \le t < \max\{-s,-b\} \\ \int_0^b Q(\tau,t+s)\phi(-\tau)\,d\tau. & \max\{-s,-b\} \le t \le 0, \end{cases}$$

(vi) $T(s)$ *is compact for* $s \ge b$.

PROOF. (i) holds because equation (1.1) is autonomous and submitted to initial condition (1.2) admits a unique solution on $[-b,\infty)$. From the definition of $T(s)$ (ii) is clear. (iii) follows from the fact that translation is continuous in $L_p(\mathbb{R})$, $1 \le p < \infty$. Rewriting (1.1) - (1.2) as the renewal equation (1.3), the forcing function is given by $f(t) = \int_t^b \zeta(\tau)\phi(t-\tau)\,d\tau = \int_0^b \zeta_t(\tau)\phi(-\tau)\,d\tau$. Now (iv) follows by applying (2.1). The first statement in (v) is trivial. The second one follows from the observation that for a given element $a \in L_1[0,b]$ the mapping $f \mapsto a*f$ from $L_p[0,b]$ into itself is compact. This follows easily from the compactness criterium in L_p spaces which is due to Riesz (see e.g. [19,Thm 2.13.1). As U(s) vanishes for $s \geq b$ (vi) is a consequence of (v). □

REMARK. If $p = \infty$ then T(s) is a semigroup which is however in general not strongly continuous. The representation given in Theorem 2.iv equally well holds. The restriction of T(s) to the closed subspace M of C[-b,0], where $M = \{\phi \in C[-b,0] \mid \phi(0) = \int_0^b \zeta(\tau)\phi(-\tau)d\tau\}$, is a strongly continuous semigroup.

In the next theorem we characterize the infinitesimal generator A of the semigroup T(s), $1 \leq p < \infty$.

THEOREM 2.3.

(i) $\mathcal{D}(A) = \{\phi \in W^{1,p}[-b,0] \mid \phi(0) = \int_0^b \zeta(\tau)\phi(-\tau)d\tau\}$

(ii) $A\phi = \phi'$

(iii) *The resolvent* $(A-\lambda I)^{-1}$ *is given explicitly by*

$$(A-\lambda I)^{-1}\psi = \int_0^t e^{\lambda(t-\tau)}\psi(\tau)d\tau - \Delta(\lambda)^{-1}\int_0^b e^{\lambda(t-s)}\left(\int_s^b \zeta(\tau)\psi(s-\tau)d\tau\right)ds.$$

Consequently A *has compact resolvent and*

$$\sigma(A) = P_\sigma(A) = \{\lambda \mid \det \Delta(\lambda) = 0\},$$

where

$$\Delta(\lambda) = I - \int_0^b e^{-\lambda\tau}\zeta(\tau)d\tau.$$

PROOF. Suppose that $\phi \in \mathcal{D}(A)$. As T(s) is a translation semigroup we know (see for instance [6, Proposition 1.3.12]) that $\phi \in W^{1,p}[-b,0]$ and $A\phi = \phi'$. Also the solution $x(t;\phi) \in W^{1,p}[0,T]$ for all T positive. Therefore

$$0 = \lim_{h\downarrow 0} \left\| \frac{T(h)\phi-\phi}{h} \right\|_{L_p[-h,0]} = \lim_{h\downarrow 0} \int_{-h}^{0} \left| \frac{x(t+h)-x(t)}{h} \right|^p dt =$$

$$\lim_{h\downarrow 0} \int_{-h}^{0} \left| \frac{\int_0^{t+h}\dot{x}(\tau)d\tau + \int_t^0 \dot{x}(\tau)d\tau + x(0) - x(0-)}{h} \right|^p dt.$$

As the first two terms in this formula go to zero as h goes to zero we conclude that

$$x(0) = \int_0^b \zeta(\tau)\phi(-\tau)d\tau = \phi(0) = x(0-).$$

Conversely assume that $\phi \in W^{1,p}[-b,0]$ and $\phi(0) = \int_0^b \zeta(\tau)\phi(-\tau)d\tau$. First we note that $x \in W^{1,p}$ on $[-b,0)$ and $x \in W^{1,p}$ on $[0,T]$ for all $T > 0$. As $x(0) = x(0-)$ we conclude that $x \in W^{1,p}$ on $[-b,T]$ for all $T > 0$. By standard arguments it follows that

$$\lim_{h\downarrow 0} \left\| \frac{x(t+h;\phi)-x(t;\phi)}{t} - \dot{x}(t;\phi) \right\|_{L_p[-b,0]} = 0.$$

This proves (i) and (ii). To prove (iii) first consider the eigenvalue problem

$$A\phi = \lambda\phi,$$

or equivalently

$$\phi' = \lambda\phi \ \& \int_0^b \zeta(\tau)\phi(-\tau)d\tau = \phi(0).$$

If $\det \Delta(\lambda) = 0$ then there exists a nontrivial element $\phi(0) \in \mathbb{C}^n$ such that $\Delta(\lambda)\phi(0) = 0$. The mapping $t \mapsto e^{\lambda t}\phi(0)$, $-b \le t \le 0$, solves the eigenvalue problem. On the other hand suppose now that $\det \Delta(\lambda) \ne 0$. With the abstract problem $(A-\lambda I)\phi = \psi$ corresponds the differential equation

$$\phi' - \lambda\phi = \psi$$

which has the solution

$$\phi(t) = e^{\lambda t}\phi(0) + \int_0^t e^{\lambda(t-\tau)}\psi(\tau)d\tau.$$

We can achieve that $\phi \in \mathcal{D}(A)$ by choosing

$$\phi(0) = -\Delta(\lambda)^{-1} \int_0^b e^{-\lambda s} \int_s^b \zeta(\tau)\psi(s-\tau)d\tau ds.$$

So $\phi = (A-\lambda I)^{-1}\psi$ is as stated is the theorem. From this explicit expression the correctness of the theorem follows (see for instance Kufner et al [19, Theorem 2.13.1] for the appropriate compactness theorem). $\square$

We conclude this subsection with a description of the generalized null-space and range of the operator $A-\lambda I$. A proof of this theorem is straightforward. Compare for instance [14, section 7.3; 9,II, appendix; 10, appendix]. We need to introduce some notations:

$$(2.4)\qquad C_\lambda : L_p[-b,0] \to L_p[-b,0], \qquad (C_\lambda\phi)(t) = -\int_0^t e^{\lambda(t-\tau)}\phi(\tau)\,d\tau,$$

$$(2.5)\qquad \text{for } i \in \mathbb{N} \cup \{0\}: P_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\Delta(\lambda).$$

We introduce matrices A_k of dimension $kn \times kn$ and column-vectors Φ_k and Ψ_k as follows:

$$(2.6)\qquad A_k = \begin{pmatrix} P_0 & 0 & 0 & \cdots & 0 \\ P_1 & P_0 & 0 & \cdots & 0 \\ P_2 & P_1 & P_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ P_{k-1} & P_{k-2} & P_{k-3} & \cdots & P_0 \end{pmatrix}$$

$$\Psi_k = \mathrm{col}(\Psi_k^1, \Psi_k^2, \ldots, \Psi_k^k),$$

$$\Psi_k^j = (-1)^j \int_0^b \zeta(t)\,(C_\lambda^m\psi)(-t)\,dt.$$

THEOREM 2.4.
(i) $N(A-\lambda I)^k$ *consists of functions* ϕ *of the form*

$$\phi(t) = e^{\lambda t}\left\{\sum_{m=0}^{k-1} \frac{t^m}{m!}\, e_{k-m}\right\}$$

where $E = \mathrm{col}(e_1,\ldots,e_k)$ *satisfies* $A_k E = 0$,
(ii) $\psi \in R(A-\lambda I)^k$ iff $C_k\Psi_k = 0$ *for all row-vectors* C_k *such that* $C_kA_k = 0$.

The semigroup S(s)

Define for $s \in \mathbb{R}_+$, $f \in \tilde{L}_p[0,b]$ and $t \in \mathbb{R}_+$:

$$(2.7) \qquad (S(s)f)(t) = x_s(t;f) - \zeta * x_s(\cdot;f)(t).$$

Recall that $x(t;f)$ is the solution of (1.3). The motivation to choose this particular state space emanates from the fact that if the forcing is the effect of an initial function in $L_p[-b,0]$, it will belong to $\tilde{L}_p[0,b]$. The next three theorems are the counterparts of the foregoing theorems. The proofs are quite similar and for the details, which are given in the case $p = 1$, we refer to Diekmann [10].

THEOREM 2.5. *Let* $1 \le p < \infty$. $S(s)$ *is a strongly continuous semigroup in the space* $\tilde{L}_p[0,b]$. *Furthermore the following properties hold:*

(i) *for* $s > 0$, $S(s)$ *is the sum of the nilpotent bounded linear operator* $U(s)$ *and the compact operator* $V(s)$, *where* $(U(s)f)(t) = f(t+s)$ $(V(s)f)(t) = (\zeta_t - \zeta_t * R) * f(s)$.

(ii) $S(s)$ *is compact for* $s \ge b$.

THEOREM 2.6. *The infinitesimal generator* B *is characterized by*

(i) $\mathcal{D}(B) = \{f \in \check{W}^{1,1}[0,b] \mid f' + \zeta(\cdot)f(0) \in \tilde{L}_p[0,b]\}$

(ii) $Bf(t) = f'(t) + \zeta(t)f(0)$.

(iii) B *has compact resolvent, and* $\sigma(B) = P_\sigma(B) = P_\sigma(A)$,

$$(B-\lambda I)^{-1}g = -\int_t^b e^{\lambda(t-\tau)}f(\tau)\,d\tau - \int_t^b e^{\lambda(t-\tau)}\zeta(\tau)\,d\tau\cdot\Delta(\lambda)^{-1}\cdot\int_0^b e^{-\lambda\tau}f(\tau)\,d\tau.$$

Notation:

$$(2.8) \qquad J_\lambda\colon \tilde{L}_p[0,b] \to \tilde{L}_p[0,b], \quad (J_\lambda f)(t) = \int_t^b e^{\lambda(t-\tau)}f(\tau)\,d\tau.$$

$$(2.9) \qquad L_\lambda\colon \tilde{L}_p[0,b] \to \mathbb{C}^n, \quad L_\lambda f = \int_0^b e^{-\lambda s}f(s)\,ds.$$

$$G_k = \mathrm{col}(G_k^1, G_k^2, \ldots, G_k^k)$$

$$G_k^j = -\frac{1}{(j-1)!}\frac{d^{j-1}}{d\lambda^{j-1}}L_\lambda g.$$

THEOREM 2.7.
(i) $\mathcal{N}(B-\lambda I)^k$ *consists of functions* f *of the form*

$$f(t) = \sum_{m=0}^{k-1} (-1)^m J_\lambda^{m+1}(\zeta(\cdot)e_{k-m})(t)$$

where

$$E_k = \mathrm{col}(e_1,\ldots,e_k) \quad \textit{satisfies} \quad A_k E_k = 0,$$

(ii) $g \in \mathcal{R}(B-\lambda I)^k$ *iff* $C_k G_k = 0$ *for all row-vectors* C_k *such that* $C_k A_k = 0$.

The structural operators

To describe the relation between the semigroups T(s) and S(s) we need the notion of two so called structural operators acting between initial functions and forcing functions.
Define for $1 \le p \le \infty$

$$\mathcal{F}: L_p[-b,0] \to \tilde{L}_p[0,b]$$

by

$$(2.10) \qquad (\mathcal{F}\phi)(t) = \int_t^b \zeta(\tau)\phi(t-\tau)d\tau,$$

and

$$\mathcal{G}: \tilde{L}_p[0,b] \to L_p[-b,0]$$

by

$$(2.11) \qquad (\mathcal{G}f)(t) = x(t+b;f).$$

THEOREM 2.8. $\mathcal{G}$ *is a bounded invertible operator, the inverse is explicitly given by*

$$(\mathcal{G}^{-1}\phi)(t) = \phi_{-b}(t) - \zeta * \phi_{-b}(t) \qquad 0 \le t \le b.$$

PROOF. The first statement is a direct consequence of the explicit formula for G^{-1}. □

THEOREM 2.9.

(i) $T(s)G = GS(s), \quad s \in \mathbb{R}_+$

(ii) $FT(s) = S(s)F, \quad s \in \mathbb{R}_+$

(iii) $\forall\phi \in \mathcal{D}(A): \; F\phi \in \mathcal{D}(B) \;\; \& \;\; FA\phi = BF\phi$

(iv) $\forall f \in \mathcal{D}(B): \; Gf \in \mathcal{D}(A) \;\; \& \;\; AGf = GBf$

(v) $GF = T(b)$

(vi) $FG = S(b)$

(vii) *Providing* $\mathcal{D}(A)$ *and* $\mathcal{D}(B)$ *with their graph-norms,* G *is a bounded invertible operator of* $\mathcal{D}(A)$ *onto* $\mathcal{D}(B)$.

(viii) $G(N(B-\lambda I)^k) = N(A-\lambda I)^k, \quad K \in \mathbb{N}$

(ix) $F(N(A-\lambda I)^k) = N(B-\lambda I)^k, \quad K \in \mathbb{N}$.

PROOF. (i) and (ii) follow directly from the definitions. To prove (iii) let $\phi \in \mathcal{D}(A)$. Then $F\phi \in \tilde{W}^{1,1}[0,b]$ and $(F\phi)'(t) = \int_t^b \zeta(\tau)\phi'(t-\tau)d\tau-\zeta(t)\phi(0)$. Hence $(F\phi)' + \zeta(\cdot)\phi(0) \in \tilde{L}_p(\mathbb{R}_+)$. This shows that $F\phi \in \mathcal{D}(B)$. Now (iii) follows from $B(F\phi)(t) = \frac{d}{dt}(F\phi)(t) + \zeta(t)(F\phi)(0) = \int_t^b \zeta(\tau)\phi'(t-\tau)d\tau = F(A\phi)(t)$. To prove (iv) let $f \in \mathcal{D}(B)$ then $\frac{d}{dt}(Gf)(t) = (f'-R*f')(t+b)-R(t+b)f(0) =$ $= f'(t+b)+\zeta(t+b)f(0) - R*\{f'+\zeta(\cdot)f(0)\}(t+b)$ which implies that $(Gf)' \in L_p[-b,0]$. The compatibility condition is satisfied because $x(b) = \int_0^b \zeta(\tau)x(b-\tau)d\tau$. The identity $AGf = GBf$ follows as above. (v): for $t \in \mathbb{R}_+$ $x(t) = (F\phi)(t)-R*F\phi(t) \Rightarrow T(b+t)\phi = F\phi(t+b) - R*F\phi(t+b) = G(F\phi)(t)$. (vi): using the identity $a*b(s+t) = a_s*b(t)+a*b_t(s)$ we derive

$$FG(f) = -\zeta_b*f_b(t-b) + \zeta_b*(R*f)_b(t-b)$$

$$= \zeta_t*f(b)-\zeta_t*R*f(b) = S(b)f.$$

From the proof of (iv) we see that $\|Gf\|_A \leq C\|f\|_{L_p[0,b]} \|f'+\zeta(\cdot)f(0)\|_{L_p[0,b]} \leq C\|f\|_B$, provided that $f \in \mathcal{D}(A)$. Furthermore, G^{-1} maps $\mathcal{D}(A)$ into $\mathcal{D}(B)$. This proves (vii). The proof of the last two statements goes by induction, employing the linear algebra which is needed to prove Theorem 2.4, 2.7. We omit the details. □

Duality relations

We introduce notation for the semigroups and the structural operators corresponding to the transpose of the kernel ζ. For any initial row-vector $\phi \in L_p[-b,0]$ and any forcing row-vector $f \in \tilde{L}_p[0,b]$ we define

$$(T^+(s)\phi)(t) = y(s+t;\phi) \qquad s \in \mathbb{R}_+, \quad -b \le t \le 0$$

$$(S^+(s)f)(t) = y_s(t;f)-y_s(\cdot;f)*\zeta(t), \qquad s,t \in \mathbb{R}_+,$$

where $y(\cdot;\phi)$ satisfies

$$(2.12) \qquad \begin{cases} y(t) = \int_0^b y(t-\tau)\zeta(\tau)d\tau, & t \in \mathbb{R}_+, \\ y(t) = \phi(t), & -b \le t < 0, \end{cases}$$

and $y(\cdot;f)$ satisfies

$$(2.13) \qquad y(t) = y*\zeta(t) + f(t), \qquad t \in \mathbb{R}_+.$$

$$(2.14) \qquad (F^+\phi)(t) = \int_t^b \phi(t-\tau)\zeta(\tau)d\tau, t \in \mathbb{R}_+.$$

$$(2.15) \qquad (G^+f)(t) = y(t+b;f), \qquad -b \le t \le 0.$$

As a realization of the dual space of $L_p[-b,0]$ $(\tilde{L}_p[0,b])$ $1 \le p < \infty$, we choose $\tilde{L}_q[0,b]$, $(L_q[-b,0])$, $\frac{1}{p}+\frac{1}{q} = 1$, respectively together with the pairing

$$\langle\phi,f\rangle = \int_0^b \phi(-\tau)f(\tau)d\tau.$$

As a realization of the dual space of $C[-b,0]$ we choose $\widetilde{NBV}[0,b]$, which consists of all bounded variation functions on $[0,\infty)$ such that (i) $f(0) = 0$; (ii) f is continuous from the right on $(0,\infty)$; (iii) f is constant for $t \ge b$, the pairing being given by

$$\langle\phi,f\rangle = \int_0^b \phi(-\tau)df(t).$$

The dual space of

$$\tilde{C}[0,b] = \{f \in C(\mathbb{R}_+; \mathbb{R}^n) \mid f(t) = 0 \quad \text{for } t \geq b\}$$

will be NBV[-b,0] which consists of bounded variation functions on [-b,0] which are continuous from the right, and vanishing at zero. The pairing is given by

$$\langle \phi, f \rangle = \int_0^b d\phi(-t) f(t).$$

DEFINITION. If the pair $(T(s),X)$ consists of the (not necessarily strongly continuous) semigroup, $T(s)$ of bounded linear operators on the Banach space X, satisfying $\sup_{0\leq s\leq 1} \|T(s)\| < \infty$, we will denote by

$$\{T(s),X\}$$

the pair $(T_o(s),X_o)$ where X_o is the largest subspace of X (which is closed!) on which $T(s)$ is strongly continuous and $T_o(s)$ is the restriction of $T(s)$ to X_o.

DEFINITION. If $T(s)$, $T_o(s)$, X,X_o are as above then

$$\{T(s),X\}^* = \{T_o(s)^*, X_o^*\}.$$

THEOREM 2.10.
(i) $F_p^* = F_q^+$, $\quad 1 \leq p < \infty$,
(ii) $G_p^* = G_q^+$, $\quad 1 \leq p < \infty$,
(iii) $\{T(s),L_p[-b,0]\}^* = \{S^+(s), \tilde{L}_q[0,b]\}$, $\quad 1 \leq p \leq \infty$,
(iv) $\{S(s), \tilde{L}_p[0,b]\}^* = \{T^+(s),L_q[-b,0]\}$, $\quad 1 \leq p \leq \infty$.
Here, the subindices in the first two statements indicate on which L_p*-space the operators act. In the last two statements the assertion for* $p \in \{1,\infty\}$ *holds after identification of* AC *in the* NBV*-norm with* L_1.

PROOF.

(i) $<\psi,F\phi> = \int_0^b \psi(-t)(F\phi)(t)dt = -\int_0^b \psi(-t)\zeta_b*\phi(t-b)dt =$

$= \int_0^{-b}\psi(t)\zeta_b*\phi(-b-t)dt = \psi*\zeta_b*\phi(-b) =$

$\int_0^b(\int_t^b \psi(\tau)\zeta(t-\tau)d\tau)\phi(-t)dt = <F^+\psi,\phi>$

(ii) $<h,Gf> = \int_0^b h(t)(Gf)(-t)dt = \int_0^b h(t)x(-t+b)dt =$

$\int_0^b h(t)(f(-t+b) - \int_0^{-t+b} R(-t+b-\sigma)f(\sigma)d\sigma)dt =$

$\int_0^b h(b-t)f(t)dt - \int_0^b (\int_0^{b-\sigma} h(t)R(-t+b-\sigma)dt)f(\sigma)d\sigma =$

$\int_0^b (h(b-t) - \int_0^{b-t} h(\tau)R(b-t-\tau)d\tau)f(t)dt = <G^+h,f>.$

(iii) Case 1: $1 < p < \infty$. For any $\phi \in \mathcal{D}(A)$ and $f \in \mathcal{D}(B^+)$:

$<f,A\phi> = \int_0^b f(t)\phi'(-t)dt = \int_0^b f'(t)\phi(-t)dt + f(0)\phi(0) =$

$\int_0^b (f'(t) + f(0)\zeta(t))\phi(-t)dt = <B^+f,\phi>.$

This shows that $\mathcal{D}(A^*) \supset \mathcal{D}(B^+)$ and that $A^*|_{\mathcal{D}(B^+)} = B^+$. As L_p is reflexive for these values of p A^* generates strongly continuous semigroup and by standard arguments [6] one shows that $\mathcal{D}(A^*) = \mathcal{D}(B^+)$ and hence $\{T^*(s),L_p[-b,0]\}^* = \{S^+(s),\tilde{L}_q[0,b]\}$. Note that we do not have to take restrictions in this case!

Case 2: $p = \infty$. We have to restric T(s) to the space

$$X = \{\phi \in C([-b,0];\ \mathbb{R}^n) \mid \int_0^b \zeta(\tau)\phi(-\tau)d\tau = \phi(0)\}$$

as a representation of X^* we choose the Banach space which consists of the restriction of functionals on C[-b,0], i.e. $\widetilde{NBV}[0,b]$, to the space X. The duality pairing is as above. For $f \in \widetilde{NBV}[0,b]$ and $\phi \in X$ we have

$$<f,T(s)\phi> = \int_0^b df(t)T(s)\phi(-t) =$$

$$\int_s^b df(t)\phi(-t+s) + \int_0^s df(t)\Big(\int_0^b Q(\tau,-t+s)\phi(-\tau)d\tau\Big) =$$

$$\int_0^{b-s} df(t+s)\phi(-t) + \int_0^b\int_0^s df(\tau)Q(t,-\tau+s)\phi(-t)dt =$$

$$\int_0^{b-s} df(t+s)\phi(-t) + \int_0^b d_t \int_0^s df(\tau) \int_0^t Q(\xi,-\tau+s)d\xi\ \phi(-t).$$

Therefore

$$(2.16)\qquad T^{*}(s)f(t) = f(t+s)-f(s) + \int_0^s df(\tau)\int_0^t Q(\xi,s-\tau)d\xi .$$

At this point we use Lemma 5.1 of the appendix to rewrite this expression as:

$$(2.17)\qquad T^{*}(s)f(t) = f(t+s)-f(s) + f*(\zeta_t-R*\zeta_t)(s) + f*R(s) - f*R(s)\int_0^t \zeta(\xi)d\xi + f(s)\int_0^t \zeta(\xi)d\xi .$$

As $\operatorname*{Var}_{[0,b]}\ (t \mapsto \int_0^s df(\tau)\int_0^t Q(\xi,s-\tau)d\xi) \le \operatorname*{Var}_{[0,s]}(f)\cdot \|\zeta\|_{L_1[0,b]}\ (1+\|R\|_{L_1[0,s]})$,

we conclude from the general theory on translation semigroups [6, Theorem 1.4.9] that $T^{*}(s)$ is strongly continuous on $AC(\mathbb{R}_+)\cap \widetilde{NBV}[0,b]$. From formula (2.17) we conclude that if f is in this closed subspace of NBV[0,b] then

$$\frac{d}{dt}\,(T^{*}(s)f)(t) = f'(t+s) + f'*(\zeta_t-R*\zeta_t)(s) = S^{+}(s)f'(t).$$

Case 3: $p = 1$. If $f \in \tilde{L}_\infty[0,b]$ and $\phi \in L_1[-b,0]$ then from the identity $\langle f,T(s)\phi\rangle = \langle T^{*}(s)f,\phi\rangle$ we derive that

$$T^{*}(s)f(t) = f(t+s) + f*(\zeta_t-R*\zeta_t)(s).$$

So $\qquad \{T^{*}(s),L_\infty[0,b]\} = (S^{+}(s),\tilde{C}[0,b]) = \{S^{+}(s),\tilde{L}_\infty[0,b]\}.$

(iv) Case 1: $1 < p < \infty$. This follows combining (ii) and (iii). The case $p = 1$ is as easy as above, so we concentrate on $p = \infty$. We restrict S(s) to $\tilde{C}[0,b]$. If $\phi \in \widetilde{NBV}[-b,0]$ and $f \in \tilde{C}[0,b]$ then from $\langle \phi,S(s)f\rangle = \langle S^{*}(s)\phi,f\rangle$ we derive that

$$(2.18)\qquad S^{*}(s)\phi(t) = \begin{cases} \int_0^b d\phi(-\tau)\int_0^{-t} Q^{+}(\tau,s-\xi)d\xi & \max\{-s,-b\} \le t \le 0 \\ \phi(t+s) + \int_0^b d\phi(-\tau)\int_0^s Q^{+}(\tau,s-\xi)d\xi & -b \le t \le \max\{-s,-b\}. \end{cases}$$

Where by definition

$$(2.19) \qquad Q^+(t,s) = \zeta_t(s) - \zeta_t * R(s).$$

Using Lemma 5.2 of the appendix we rewrite this expression as

$$(2.20) \qquad S^*(s)\phi(t) = \begin{cases} \int_0^b \phi(-\tau) \; \{Q^+(\tau,s+t) - Q^+(\tau,s) + \zeta(\tau) \int_s^{t+s} R(\sigma)d\sigma\} \\ \qquad\qquad \max\{-s,-b\} \le t \le 0 \\ \phi(t+s) + \int_0^b \phi(-\tau)\{Q^+(\tau,0) - Q^+(\tau,s) + \zeta(\tau) \int_s^0 R(\sigma)d\sigma\} \\ \qquad\qquad -b \le t \le \max\{-s,-b\}. \end{cases}$$

Already from formula (2.18) it is clear that the subspace on which $S^*(s)$ is strongly continuous consists of the absolutely continuous functions on $[-b,0]$ vanishing at zero. If ϕ is in this set then we derive from (2.20) that

$$\frac{d}{dt}(S^*(s)\phi)(t) = \begin{cases} \int_0^b \phi'(-\tau)Q^+(\tau,t+s) & \max\{-s,-b\} \le t \le 0 \\ \phi'(t+s) & -b \le t \le \max\{-s,-b\}. \end{cases}$$

Therefore $\frac{d}{dt}(S^*(s)\phi)(t) = T^+(s)\phi'(t)$. $\square$

We can interpret (2.16), (2.20) in the following way:

$$(2.21) \qquad T^*(s)f(t) = z_s(t) - z_s(0) - (z_s(\cdot)-z_s(0))*\zeta(t),$$

where $f \in \widetilde{NBV}[0,b]$ and z satisfies

$$(2.22) \qquad z(t) = z*\zeta(t) + f(t), \quad t \ge 0.$$

$$(2.23) \qquad S^*(s)\phi(t) = y_s(t) - y_s(0),$$

where $\phi \in NBV[-b,0]$ and y satisfies

$$(2.24)\qquad \begin{cases} \frac{d}{dt}\{y(t) - \int_0^b y(t-\tau)\zeta(\tau)d\tau\} = 0 & t > 0 \\ y(t) = \phi(t) & -b \le t \le 0. \end{cases}$$

Decomposition according to the spectrum

We formulate the results only for the semigroup $T(s)$. By the equivalence relation $T(s) = GS(s)G^{-1}$ the corresponding results are valid for $S(s)$. From Theorem 2.3 we know that $R(\lambda,A)$ is compact. We apply the general theory for such operators (see for instance [28, Theorem 10.1], [17, section 5.14]).

THEOREM 2.12. *Let* λ *be a pole of* $R(\lambda,A)$ *of order* r, *then the state space* $L_p[-b,0]$ *can be decomposed as the direct sum of the closed subspaces*: $L_p[-b,0] = N(A-\lambda I)^r \oplus R(A-\lambda I)^r$.

We will denote the corresponding spectral projection operator with range $N(A-\lambda I)^r$ by P_λ^A. Recall that $P_\lambda^A = \frac{1}{2\pi i}\int_\Gamma R(w,A)dw$, where Γ is a circle around λ, λ being the only possible singularity of $R(\cdot,A)$ within the closed disk.

Notation $\quad M_\lambda^A = N(A-\lambda I)^r, \quad M^A = \cup_{\lambda\in\sigma} M_\lambda^A \quad$ (by $\cup$ we mean the span of the union)

$$N_\lambda^A = R(A-\lambda I)^r, \quad N^A = \cap_{\lambda\in\sigma} N_\lambda^A.$$

From abstract theory we know that $P_\sigma(T(t)) \subset e^{tP_\sigma(A)} \cup \{0\}$, see [17,Thm. 16.7.2, p.467]. $T(s)$ is compact for $s \ge b$ and hence $T(s)$ has only point spectrum for those values of s. This leads to the next theorem (compare Hale [14, Thm.4.1]).

THEOREM 2.13. *For any real number* β *let* $\Lambda = \Lambda(\beta) = \{\lambda \mid \lambda \in \sigma(A)$ *and* $\mathrm{Re}\,\lambda \ge \beta\}$. *Then*

$$L_p[-b,0] = \bigcup_{\lambda\in\Lambda} M_\lambda^A \oplus \bigcap_{\lambda\in\Lambda} N_\lambda^A,$$

which we will write as

$$M_\Lambda^A \oplus N_\Lambda^A.$$

M_Λ^A *is finite dimensional, and there exist positive numbers* K *and* γ *such that:*

$$\forall\phi \in M_\Lambda^A,\ t \in \mathbb{R}_- : \|T(t)\phi\| \le Ke^{(\beta-\gamma)t}\|\phi\|,$$

$$\forall\phi \in N_\Lambda^A,\ t \in \mathbb{R}_+ : \|T(t)\phi\| \le Ke^{(\beta-\gamma)t}\|\phi\|.$$

3. SMALL SOLUTIONS AND COMPLETENESS OF EIGENFUNCTIONS

DEFINITION. The solution of (1.1) - (1.2) is called a small solution if the mapping $\lambda \mapsto \hat{x}(\lambda) = \int_0^\infty e^{-\lambda t}\, x(t-b)dt$ defines an entire function of $\mathbb{C}$ into $\mathbb{C}^n$. It turns out that small solutions must vanish after finite time. Before we state this precisely as a theorem we need one more definition.

DEFINITION. An exponential function $f(\lambda)$ is of exponential type τ if $\limsup_{r\to\infty} r^{-1}\log M(r) = \tau$, where $M(r) = \max_{|\lambda|=r} |f(\lambda)|$.

THEOREM 3.1. (Henry [15]). *Let* x *be any small solution of* (1.1)-(1.2). *Then* $x(t) = 0$ *for* $t \ge (n-1)b-\tau$, τ *being the exponential type of* $\lambda \mapsto \det\Delta(\lambda)$.

PROOF. Here we briefly indicate the proof. For the details in the case of a retarded functional differential equation see [15]. Define $\hat{x}(\lambda) = \int_0^\infty e^{-\lambda t}x(t-b)dt$. Then $\hat{x}(\lambda)$ satisfies

$$\Delta(\lambda)\hat{x}(\lambda) = g(\lambda), \tag{3.1}$$

where

$$g(\lambda) = \int_0^b e^{-\lambda t}\phi(t-b)\,dt - \int_0^b e^{-\lambda t}\left\{\int_0^t \zeta(\tau)\phi(t-b-\tau)\,d\tau\right\}dt. \tag{3.2}$$

As $\hat{x}(\lambda)$ is entire we infer from this identity that $\hat{x}(\lambda)$ has finite exponential type and $\hat{x}(\lambda) = o(1)$ on the imaginary axis. Define $h(\lambda) = \dfrac{\hat{x}(\lambda)-\hat{x}(0)}{\lambda}$, than h has the same exponential type and is $O(|\frac{1}{\lambda}|)$ on the imaginary axis. Therefore due to a theorem of Paley and Wiener [4,§ 6.8.1]

$$h(\lambda) = \int_0^\sigma e^{-\lambda\tau}\phi(\tau)\,d\tau, \qquad 0 \le \sigma < \infty, \quad \phi \in L_2(\mathbb{R}_+).$$

Combining this with the identity $h(\lambda) = \int_0^\infty e^{-\lambda\tau}(-\int_\tau^\infty x(\mu-b)d\mu)d\tau$ yields that $x(t) = 0$ for $t \geq \sigma-b$.
Multiplying equation (3.1) on both sides by $\mathrm{Adj}\Delta(\lambda)$, the matrix consisting of the cofactors of $\Delta(\lambda)$, we obtain

$$\det\Delta(\lambda)\hat{x}(\lambda) = \mathrm{Adj}\Delta(\lambda).g(\lambda).$$

Since the exponential type of the right hand side is bounded above by nb we conclude that $\sigma \leq nb-\tau$. □

This result motivates the following

DEFINITION. Let α denote the ascent of $T(t)$ i.e.:

$$\alpha = \inf\{t \in \mathbb{R}_+ \mid \forall\varepsilon > 0: N(T(t+\varepsilon)) = N(T(t))\}. \tag{3.3}$$

Similarly δ denotes the ascent of $T^*(t)$:

$$\delta = \inf\{t \in \mathbb{R}_+ \mid \forall\varepsilon > 0: N(T^*(t+\varepsilon)) = N(T(t))\}. \tag{3.4}$$

COROLLARY 3.2. $\alpha \leq nb-\tau$ *and* $\delta \leq nb-\tau$.

From the equivalence of the semigroups $S(s)$ and $T(s)$ it follows that α equals the ascent of $S(t)$ and δ equals the ascent of $S^*(t)$. The inequalities obtained above are not sharp. This we demonstrate by the following example

$$\begin{cases} x_1(t) = \int_0^1 x_2(t-\tau)d\tau \\ x_2(t) = \int_0^2 x_1(t-\tau)d\tau \\ x_3(t) = \int_0^1 x_2(t-\tau)d\tau. \end{cases} \tag{3.5}$$

Here

$$\Delta(\lambda) = \begin{pmatrix} 1 & \frac{e^{-\lambda}-1}{\lambda} & 0 \\ \frac{e^{-2\lambda}-1}{\lambda} & 1 & 0 \\ 0 & \frac{e^{-\lambda}-1}{\lambda} & 1 \end{pmatrix}$$

and type $\det\Delta(\lambda) = 3$, so $nb-\tau = 3.2-3 = 3$.
First restrict to the subsystem

$$(3.6)\qquad \begin{cases} x_1(t) = \int_0^1 x_2(t-\tau)d\tau \\ x_2(t) = \int_0^2 x_1(t-\tau)d\tau. \end{cases}$$

The set of small solutions of (3.6) in $L_p[-b,\infty)$, $1 \le p \le \infty$, is given by

$$\{x \in L_p[-2,\infty) \mid x_1(t) = 0, \quad t \ge -2;\ x_2(t) = 0, \quad t \ge -1\}.$$

Therefore the set of small solutions of (3.5) equals

$$\{x \in L_p[-2,\infty) \mid x_1(t) = 0,\ t \ge -2;\ x_2(t) = 0,\ t \ge -1;\ x_3(t) = 0,\ t \ge 0\}.$$

We conclude that the ascent corresponding to (3.5) equals $nb-\tau-1$.

The set of small solutions of the adjoint system

$$(3.7)\qquad \begin{cases} y_1(t) = \int_0^2 y_2(t-\tau)d\tau \\ y_2(t) = \int_0^1 y_1(t-\tau)d\tau + \int_0^1 y_3(t-\tau)d\tau \\ y_3(t) = 0 \end{cases}$$

is

$$\{y \in L_p[-2,\infty) \mid y_1(t) = 0,\ t \ge 0;\ y_2(t) = 0,\ t \ge -2;$$
$$y_3(t) = -y_1(t),\ -1 \le t \le 0;\ y_3(t) = 0,\ t \ge 0\}.$$

Therefore $\delta = nb-\tau-1$.

In this example $\alpha = \delta$. It is an important open question whether this equality holds in general. We will come back to this question later on.

In the next theorem we give several characterizations of $\mathcal{N}(T(\alpha))$. By the equivalence, the corresponding statements for $\mathcal{N}(S(\alpha))$ are also valid. We omit them.

THEOREM 3.3.

(i) $\mathcal{N}(T(\alpha)) = \{\phi \in L_p[-b,0] \mid$ *the mapping* $\lambda \mapsto \Delta(\lambda)^{-1}L_\lambda(\mathcal{F}\phi)$ *of* $\mathbb{C}$ *into* $\mathbb{C}^n$ *is entire*$\}$

(ii) $\mathcal{N}(T(\alpha)) = \{\phi \in L_p[-b,0] \mid$ *the mapping* $\lambda \mapsto R(\lambda,A)\phi$ *of* $\mathbb{C}$ *into* $L_p[-b,0]$ *is entire*$\}$

(iii) $\mathcal{N}(T(\alpha)) = \bigcap_{\lambda\in\sigma} \mathcal{N}(P_\lambda^A)$.

PROOF. Define $\bar{x}(\lambda) = \int_0^\infty e^{-\lambda t}x(t)dt$, then $\bar{x}(\lambda)$ satisfies

$$\Delta(\lambda)\bar{x}(\lambda) = L_\lambda(\mathcal{F}\phi).$$

Recall $\hat{x}(\lambda)$ defined in Theorem 3.1. The mapping $\lambda \mapsto \bar{x}(\lambda)$ is entire iff the mapping $\lambda \mapsto \hat{x}(\lambda)$ is enitre. This proves (i). (ii) follows from the explicit formula for $R(\lambda,A)$ given in Theorem 2.3. The Laurant series of $R(\lambda,A)$ around a pole λ_0 of order m is given by

$$R(\lambda,A) = \sum_{n=1}^{m} (\lambda-\lambda_0)^{-n}(A-\lambda_0 I)^{n-1} P_{\lambda_0}^A + H(\lambda,A),$$

H being holomorphic in a neighbourhood of λ_0 (see [28, section V.10]). This proves (iii). □

As a consequence of Henry's theorem we state

THEOREM 3.4. $\forall t \geq \delta$: $\overline{\mathcal{M}^A} = \overline{\mathcal{R}(T(t))}$.

PROOF. Let $t \geq \delta$. $\overline{\mathcal{M}^A} = \overline{\bigcup_{\lambda\in\sigma} \mathcal{R}(P_\lambda^A)} = \overline{\bigcup_{\lambda\in\sigma} \mathcal{N}(P_{\bar{\lambda}}^{A^*})^\perp} =$

$$\overline{\bigcup_{\lambda\in\sigma} \mathcal{N}(P_\lambda^{B^+})^\perp} = (\bigcap_{\lambda\in\sigma} \mathcal{N}(P_\lambda^{B^+}))^\perp = \mathcal{N}(S^+(\delta))^\perp =$$

$$\mathcal{N}(S^+(t))^\perp = \overline{\mathcal{R}(T(t))}.$$

Here we have used the identity $N(S^{+}(\delta)) = \cap_{\lambda\in\sigma} N(P_{\lambda}^{B+})$, which is the counterpart of Theorem 3.3-(iii) in the "S-language". □

Completeness of Eigenfunctions

We will call the eigenfunctions of A complete iff $\bar{M}^{A} = L_p[-b,0]$. It follows from the previous theorem that this is the case iff $\delta = 0$. By the semigroup property this will be the case iff $N(T^{+}(b)) = 0$. As $T^{+}(b) = G^{+}F^{+}$ this is equivalent with the identity $N(F^{+}) = \{0\}$. It has been an open question for quite a long time whether the equivalence $N(F) = 0 \iff N(F^{+}) = \{0\}$ holds, see for instance Delfour & Manitius [9]. Or in other words $\alpha = 0 \iff \delta = 0$? A positive answer to this question is given in the next theorem. The corresponding result for functional differential equations has been obtained by Verduyn Lunel [31].

Before we state and prove the theorem we first introduce some notation and recall some facts from linear algebra (see [18,§15]).

By $\det{*}\zeta$ we indicate the element of $L_1(\mathbb{R}_+)$ that is obtained from the expression for $\det \zeta$ by replacing the product in $\mathbb{R}$ by the convolution product. For any square matrix $C = (c_{ij})$ we denote by Adj C the square matrix which consist of all cofactors cofc_{ij} of C. By definition $\mathrm{cofc}_{ij} = (-1)^{i+j} c_{ij} \det C_{ij}^{*}$, where C_{ij}^{*} is obtained from C by leaving out the i-th row and the j-th column. We also use Adj*C. The well known identity $(\mathrm{Adj}C)^{T}.\ C = \det C.\ I$ transfers in the convolution algebra to $(\mathrm{Adj}{*}\zeta)^{T}{*}\zeta(t) = \det{*}\zeta(t).\ I$

Let C be a n×n matrix such that det C = 0 and suppose that the equation Cx = b has a solution. Suppose that the rank of C is r and that $\det \hat{C} \neq 0$, $\hat{C}$ being the r×r submatrix consisting of the elements c_{ij} $1 \le i \le r$, $1 \le j \le r$. All solutions of the equation Cx = b may be obtained by solving the reduced equation

$$c_{11}x_1+\ldots + c_{1r}x_r = b_1 - c_{1r+1}x_{r+1} - \ldots - c_{1n}x_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_{r1}x_1+\ldots + c_{rr}x_r = b_r - c_{rr+1}x_{r+1} - \ldots - c_{rn}x_n.$$

If we choose for $x_{r+1},\ldots,x_n$ the arbitrary constants $d_{r+1},\ldots,d_n$, then the solution set of $Cx = b$ is obtained by Cramers Rule:

$$x_i = \frac{1}{\det \hat{C}} \sum_{j=1}^{r} (b_j - \sum_{k=r+1}^{n} c_{jk} d_k) \operatorname{cof} \hat{c}_{ij}.$$

Finally we state a lemma which we need in the proof of the next theorem

LEMMA 3.5. *Let* $n \in \mathbb{N}$ *and* $a_i \in L_1[0,b]$, $i \in \{1,\dots,n\}$, *be given. For all* $t \in [-b,0]$: $a_1 * \dots * a_n(nb+t) = (-1)^{n-1}(a_1)_b * \dots * (a_n)_b(t)$.

PROOF. The statement trivially holds if $n = 1$, so assume that it also holds if $n = m-1 > 1$. Then the identities:

$$a_1 * \dots\dots * a_m(mb+t) = \int_0^{mb+t} (a_1 * \dots * a_{m-1})(mb+t-\tau) a_m(\tau) d\tau =$$

$$\int_{b+t}^{b} a_1 * \dots * a_{m-1}(mb+t-\tau) a_m(\tau) d\tau =$$

$$-\int_0^t a_1 * \dots\dots * a_{m-1}((m-1)b+t-\tau)(a_m)_b(\tau) d\tau =$$

$$(-1)^{m-1} \int_0^t (a_1)_b * \dots * (a_{m-1})_b(t-\tau)(a_m)_b(\tau) d\tau =$$

$$(-1)^{m-1}(a_1)_b * \dots * (a_m)_b(t),$$

show that the statement is true for $n = m$. □

THEOREM 3.6. *The following assertions are equivalent:*

(i) $N(F) = \{0\}$,

(ii) $\sup \operatorname{supp}(\det * \zeta) = nb$

(iii) type $\det\Delta(\lambda) = nb$.

PROOF.

(ii) ⇒ (i). Suppose $0 \neq \phi \in N(F)$. Then for $t \in [-b,0]$: $\zeta_b * \phi(t) = 0$. Multiplying by $\operatorname{Adj} * \zeta_b$ we obtain that $\det * \zeta_b * \phi$ vanishes identically on the interval $[-b,0]$. From the previous lemma we derive that $\det * \zeta_b(t) = (-1)^{n-1} \cdot \det * \zeta(nb+t)$ for all $t \in [-b,0]$. Our assumption implies that there exists ε positive such that $\det * \zeta_b(t) \neq 0$ [a.e] on the interval $[-\varepsilon,0]$. But then ϕ

must vanish identically on the interval $[-b,0]$ as a consequence of the Theorem of Titchmarsh [29, page 327]. This proves the first part of the theorem.

(i) $\Rightarrow$ (ii). Let us suppose that $nb > c = \sup\operatorname{supp}\det{*}\zeta$. There exist a natural number r, $1 < r < n$, a $r\times r$ submatrix $\hat{\zeta}$ of ζ and a positive number ε such that

(i) $\det{*}\hat{\zeta}_b \neq 0$ [a.e.] on the interval $[-\varepsilon,0]$,

(ii) for any square submatrix with size larger then r the det* vanishes on $[-\varepsilon,0]$.

Without losing generality we assume that $\hat{\zeta} = (\zeta_{ij})$ $1 \le i,\ j \le r$. We construct a nontrivial element in the nullspace of F by using the linear algebra above. We take into account that in general there is no inverse of $\det{*}\hat{\zeta}_b$ in the convolution algebra by letting $\det{*}\hat{\zeta}_b$ be a factor in the elements which we can choose arbitrary. Let ψ by any element in $L_p[-b,0]$ such that $\operatorname{supp}(\psi) \subset [-b,-b+\varepsilon]$.

Define

$$\text{(3.8)}\qquad \begin{aligned} \phi_{r+1} &= \det{*}\hat{\zeta}_b{*}\psi;\ \phi_{r+2} \equiv \ldots \equiv \phi_n \equiv 0.\\ \phi_i &= -\sum_{j=1}^{n} (\zeta_b)_{j\ r+1} {*}\psi{*}\operatorname{cof}(\hat{\zeta}_b)_{ji}, \qquad i \in \{1,\ldots,r\}. \end{aligned}$$

Then ϕ satisfies $\zeta_b{*}\phi = 0$ on $[-b,0]$ and ϕ does not vanish identically. This proves (i) $\Rightarrow$ (ii). The equivalence of (ii) and (iii) is trivial. Even a stronger assertion is true: if $\sup\operatorname{supp}\det{*}\zeta \in ((n-1)b,nb]$ then it is equal to type $\det(\Delta(\lambda))$. $\square$

As an immediate consequence we mention that $\alpha = 0$ iff $\delta = 0$ and

<u>THEOREM 3.7</u>. *The following assertions are equivalent*

(i) $\overline{M^A} = L_p[-b,0]$

(ii) $\alpha = 0$.

By having a closer look at the proof of Theorem 3.5 we can prove the equality $\alpha = nb-\tau$ in the following case

THEOREM 3.8. *Suppose that* $\rho = \sup \operatorname{supp} \det{*}\zeta \in ((n-1)b,nb)$, *and there exist a natural number* r, $1 < r < n$, *a* $r\times r$ *submatrix* $\hat{\zeta}$ *of* ζ *and a positive number* ε *such that*

(i) $\det{*}\hat{\zeta}_b \neq 0$ [a.e] *on the interval* $[-\varepsilon,0]$,

(ii) *for any submatrix of* ζ_b *with size larger than* r *the* det* *vanishes on the interval* $[\rho-nb,0]$ [a.e], *then* $\alpha = \delta = nb-\rho$.

PROOF. All small solutions vanish for $t \geq (n-1)b-\rho$. Let ψ in (3.8) be such that $\psi(t) = 1$, $-b \leq t \leq (n-1)b-\rho$ and $\psi(t) = 0$, $(n-1)b-\rho \leq t \leq 0$. Then ϕ defined in (3.8) satisfies $F\phi = 0$ and $\sup \operatorname{supp} \phi = (n-1)b-\rho$. Conditions (i) and (ii) as well as the construction of ϕ remain valid if we change from ζ to ζ^T. This proves the theorem. □

COROLLARY 3.9. *Let* $n = 2$. *Then* $\alpha < b \Rightarrow \alpha = \delta$.

PROOF. Let $\rho = \sup \operatorname{supp} \det{*}\zeta$. If $\rho = 2b$ then $\alpha = \delta = 0$. If $\rho \in (b,2b)$ then we can apply the previous theorem. If $\rho \leq b$ then type $\det\Delta(\lambda) \leq b$ and hence $\alpha \geq b$. □

F-completeness

DEFINITION. A solution of (1.1) - (1.2) is called a "trivial small solution" if it vanishes for $t \geq 0$.

The notion of trivial small solutions is closely related to the concept of F-completeness, which was introduced by Manitius [21]. The idea behind the concept is to study the eigenspaces M_λ^B in the closure of the range of F.

DEFINITION. The system (1.1) is F complete iff $\overline{FM^A} = \overline{R(F)}$.

THEOREM 3.10. *The following assertions are equivalent:*

(i) *system* (1.1) *is* F*-complete*

(ii) $\overline{M^B} = \overline{R(F)}$

(iii) $\cap_{\lambda\in\sigma} P_\lambda^{A^+} = N(F^+)$,

(iv) *the transposed equation has only trivial small solutions,*

(v) $N(F^+G^+) \cap R(F^+) = \{0\}$,

(vi) $F^+G^+F^+ = 0 \Rightarrow F^+ = 0$.

PROOF. As $\mathcal{F}M_\lambda^A = M_\lambda^B$ (i) and (ii) are equivalent.

$$\overline{\mathcal{F}M^A} = \overline{\mathcal{R}(\mathcal{F})} \iff \overline{M^B}^{\perp} = \overline{\mathcal{R}(\mathcal{F})}^{\perp} \iff \bigcap_{\lambda\in\sigma} \mathcal{N}(P_\lambda^{A+}) = \mathcal{N}(\mathcal{F}^+),$$

which proves the equivalence of (ii) and (iii). From Theorem 3.3 the equivalence of (iii) and (iv) follows. The transposed equation has only trivial small solutions iff $\mathcal{N}(S^+(h)) \cap \mathcal{R}(\mathcal{F}^+) = \{0\}$. As $S^+(h) = \mathcal{F}^+G^+$ (v) is a restatement of (iv). Another restatement reads $T^+(2h)\phi = 0 \Rightarrow T^+(h)\phi = 0$. Multiplying on both sides with $(G^+)^{-1}$ yields (vi). □

EXAMPLE. Consider the system

$$(3.9)\qquad \begin{cases} x_1(t) = \int_0^1 x_1(t-\tau)\,d\tau \\ x_2(t) = \int_0^2 x_2(t-\tau)\,d\tau. \end{cases}$$

The set of all small solutions in $L_p[-2,\infty)$ is given by $\{x \in L_p[-2,\infty) \mid x_1(t) = 0,\ t \geq -1;\ x_2(t) = 0,\ t \geq -2\}$. Due to Theorem 3.8 (iv) the system is $\mathcal{F}$-complete, it is however not complete. In the next example neither completeness nor $\mathcal{F}$ completeness holds.

$$(3.10)\qquad \begin{cases} x_1(t) = \int_0^2 x_2(t-\tau)\,d\tau \\ x_2(t) = \int_0^1 x_1(t-\tau)\,d\tau + \int_0^2 x_3(t-\tau)\,d\tau \\ x_3(t) = 0. \end{cases}$$

The set of small solutions of the transposed system is

$$\{x \in L_p[-2,\infty) \mid x_1(t) = 0,\ t \geq -2;\ x_2(t) = 0\ t \geq -1;$$
$$x_3(t) = \int_0^2 x_2(t-\tau)\,d\tau,\ 0 \leq t \leq 1,\ x_3(t) = 0\quad t \geq 1\}.$$

Decomposition of the state space

In a first naive guess one would like to prove that the state space can be decomposed into the closure of the span of the generalized eigenfunctions and the initial states of the small solutions.

$$L_p[-b,0] = \overline{M^A} \oplus N(T(nb)) \quad ?$$

$$\tilde{L}_p[0,b] = \overline{M^B} \oplus N(S(nb)) \quad ?$$

However, the next example shows that this cannot be true in general. Allthough the example is artificial, it definitely shows what happens in systems of equations where several delays are involved.

$$(3.11) \qquad \begin{cases} x(t) = \int_0^1 x(t-\tau)\,d\tau, \\ \\ x(t) = \phi(t); \qquad \phi \in L_2[-2,0]. \end{cases}$$

Let E be the characteristic function of the interval [0,1]. Then equivalently we consider

$$(3.12) \qquad x(t) = E * x(t) + f(t);$$

where $f \in X = \{g \in L_2(\mathbb{R}_+) \mid \mathrm{supp}(g) \subset [0,2]\}$. From Theorem 3.4 we derive that $\overline{M^B} = \overline{R(S(1))}$. As $N(T(1)) = \{\phi \in L_2[-2,0] \mid \mathrm{supp}(\phi) \subset [-2,-1]\}$ it follows that

$$\overline{R(s(1))} = \{f \in L_2(\mathbb{R}_+) \mid \mathrm{supp}(f) \subset [0,1]\}.$$

Furthermore

$$N(S(1)) = \{f = x - E*x \mid x \in L_2[0,1]\}.$$

Therefore $X \supsetneq N(S(1)) \oplus \overline{R(S(1))}$, because each element of $N(S(1)) \oplus \overline{R(S(1))}$ is absolutely continuous on the interval [1,2]. Note that in this example

$$(3.13) \qquad X = \overline{N(S(1))\oplus\overline{R(S(1))}}.$$

We did not find any counterexample to this last identity.

We conclude this subsection with some equivalent formulations of this identity in the special case that $\alpha,\delta \le b$.

THEOREM 3.11. *Assume* $\alpha \le b$, $\delta \le b$. *Then the following statements are equivalent*:

(i) $L_p[-b,0] = \overline{\overline{R(T(b))} \oplus N(T(b))}$,

(ii) $\tilde{L}_p(\mathbb{R}_+) = \overline{\overline{R(S(b))} \oplus N(S(b))}$,

(iii) $\{0\} = N(F^+G^+) \cap {}^{\perp}N(F)$,

(iv) $\{0\} = {}^{\perp}N(FG) \cap N(F^+)$,

(v) $T^*(b)|_{{}^{\perp}N(F)}$ *is one-to-one*,

(vi) $S^*(b)|_{{}^{\perp}N(FG)}$ *is one-to-one*.

PROOF. The equivalence of $T(s)$ and $S(s)$ implies that (i) $\Leftrightarrow$ (ii). Applying the identities: ${}^{\perp}(L_1\oplus L_2) = {}^{\perp}L_1 \cap {}^{\perp}L_2$ and ${}^{\perp}({}^{\perp}L_1\cap{}^{\perp}L_2) = \overline{L_1\oplus L_2}$, which hold if L_1 and L_2 are linear subspaces of the normed linear space X, to (i) and (ii) yields (iii) and (iv). Recall that $FG = S(b)$ and $GF = S(b)$. The last two statements are straightforward reformulations. □

4. ON THE CONVERGENCE OF THE PROJECTION OPERATORS

One cannot expect that the sum of the projection operators P_λ^A converges to the identity on the whole state space. For instance if ϕ is a small solution then $P_\lambda^A\phi = 0$, $\forall\lambda \in \sigma$. There are some convergence results in cases where small solutions are absent, see [1,2,20]. Here we give the corresponding results for equation (1.1). Our assumptions on the kernel ζ are in such a way as to include the results of [1,2]. In fact we combine arguments employed by Verblunsky [30] with those used by Bellman & Cooke [2] and Banks & Manitius [1]. Therefore our proof is sketchy in order not to repeat almost literally the argumentation in [1,2,30].

THEOREM 4.1. *Let* $\zeta \in \tilde{L}_p(\mathbb{R}_+)$ *be of bounded variation such that*
(i) $\lim_{t\to b} \zeta(t) = \zeta(b)$
(ii) $\det \zeta(b) \neq 0$.

If $\phi \in L_{p'}[-b,0]$, $p' > p$, *then for* $t \geq b$

$$\lim_{r\to\infty} \| T(t)\phi - \sum_{\substack{\lambda\in\sigma \\ |\lambda|\leq r}} P_\lambda(T(t)\phi)\|_{L_p} = 0.$$

If $\phi \in \mathcal{D}(A)$ *and* $\phi' \in L_{p'}[-b,0]$, $p' > p$, *then the same convergence hold for* $t \geq 0$.

Sketch of the Proof. Consider the scalar case $n = 1$. If $n > 1$, modifications like the ones in [1] have to be made. We rewrite the characteristic function as

$$\Delta(\lambda) = \frac{1}{\lambda e^{\lambda b}} g(\lambda) \tag{4.1}$$

where

$$g(\lambda) = \lambda e^{\lambda b} + \zeta(b) - \zeta(0)e^{\lambda b} - \int_0^b e^{\lambda(b-\tau)} d\zeta(\tau). \tag{4.2}$$

Let for $c > 0$

$$V_c = \{\lambda \in C \mid |\mathrm{Re}(\lambda + \tfrac{1}{b}\log\lambda)| \leq c\}.$$

Then for c and r large enough all zeros of g with modulus larger than r are contained in V_c (compare [2, Theorem 12.9]).
This suggests the transformation

$$z = \lambda + \frac{1}{b}\log\lambda. \tag{4.3}$$

Let $g'(z) = g(\lambda)$ etc. All zero's of g' with large modulus are contained in

$$V_c = \{z \in C \mid |\mathrm{Re}\ z| \leq c\}.$$

The proofs of Lemmas 1-2 in [30,I] carry over to the entire function $g'(z)$. We draw two conclusions. In the first place, the zeros of g' are uniformly bounded away from each other. Let us say that $|z'-z''| \geq K > 0$ if z', z'' are different zeros of g'. In the second place, let each zero of g' be the center of a disk of radius ρ, then there exists a positive η depending on ρ such that $|g'(z)| \geq \eta$ if z is not an element of one of the disks.

Let $\bar{x}(\lambda)$ be the Laplace transform of x defined for Re λ sufficiently large. Then $\bar{x}(\lambda)$ satisfies

$$\Delta(\lambda)\bar{x}(\lambda) = p(\lambda,\phi), \tag{4.4}$$

where

$$p(\lambda,\phi) = L_\lambda(F\phi). \tag{4.5}$$

Therefore

$$x(t) = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} e^{\lambda t}\,\Delta(\lambda)^{-1}\,p(\lambda,\phi)\,d\lambda,$$

where for any $\lambda \in \sigma(A)$: Re $\lambda < \mu$.

By the above arguments there exists a unbounded increasing sequence of positive numbers r_p and a small positive number ρ (the radius of the disks) such that the circle C_p: $|z| = r_p$ has no points in common with the disks. Let us, for p large enough, denote by Γ_p the positively oriented curve that consists of the part of C_p which lies to the left of $L = \{z \in \mathbb{C} \mid \text{Re } z = \mu\}$. The points of intersection of C_p and L we will call $\mu \pm ia_p$.

$$\int_{\mu-ia_p}^{\mu+ia_p} e^{\lambda t}\,\Delta(\lambda)^{-1}p(\lambda,\phi)\,d\lambda = \sum_{\lambda_\nu\in\sigma\cap B_{r_p}} \text{Res}\{e^{\lambda t}\,\Delta(\lambda)^{-1}p(\lambda,\phi)\}_{\lambda=\lambda_\nu} - \int_{\Gamma_p} e^{\lambda t}\Delta(\lambda)^{-1}p(\lambda,\phi)\,d\lambda. \tag{4.6}$$

We concentrate on the integral part of the right hand side. Following the arguments in [2, chapter 6] we find after some lengthy but straightforward calculations that

(4.7) $$\lim_{p\to\infty} \Big\| \int_{\Gamma_p} \frac{1}{\lambda^a} \Delta(\lambda)^{-1} e^{\lambda\cdot} \Big\|_{L_p[-b,0]} = 0, \qquad a > \frac{2p-1}{p}.$$

We proof the existence of positive constants C and ε such that $\| p(\lambda,\phi) \| \leq$ $\leq C\lambda^{(1-2p)/p-\varepsilon}$.

$$\begin{aligned} p(\lambda,\phi) &= \int_0^b e^{-\lambda\tau}\zeta(\tau)\Big(\int_{-\tau}^0 e^{-\lambda\tau}\phi(t)\,d\tau\Big)d\tau \\ &= \frac{1}{\lambda}\Big\{\int_0^b e^{-\lambda\tau}d\Big(\zeta(\tau)\int_{-\tau}^0 e^{-\lambda t}\phi(t)\,dt\Big) - e^{-\lambda b}\zeta(b)\int_{-b}^0 e^{-\lambda t}\phi(t)\,dt\Big\} \\ &= \frac{1}{\lambda}\Big\{\int_0^b e^{-\lambda\tau}\Big(\int_{-\tau}^0 e^{-\lambda t}\phi(t)\,dt\Big)d\zeta(\tau) + \int_0^b \zeta(\tau)\phi(-\tau)\,d\tau + \\ &\qquad - e^{-\lambda b}\zeta(b)\int_{-b}^0 e^{-\lambda t}\phi(t)\,dt\Big\}. \end{aligned}$$

Using Hölders inequality

$$\Big| \int_{-b}^0 e^{-\lambda t}\phi(-t)\,dt \Big| \leq C(p')\Bigg(\frac{1}{|\lambda|^{\frac{p'-1}{p'}}+1}\Bigg), \qquad \mathrm{Re}\ \lambda \leq 0,$$

we conclude that if $p' > p$ we can factor out the desired power of λ in the first and the third term. To estimate the integral involving the second term in $p(\lambda,\phi)$ we use the inequality, which is obtained in the same way as the one above

(4.8) $$\lim_{p\to\infty} \Big\| \int_{\Gamma_p} \frac{1}{\lambda^a} \Delta(\lambda)^{-1} e^{\lambda(b+\cdot)} \Big\|_{L_p[-b,0]} = 0, \qquad a > \frac{p-1}{p}.$$

Thus we obtain that

$$\lim_{p\to\infty} \Big\| \int_{\Gamma_p} e^{\lambda(b+\cdot)}\Delta(\lambda)^{-1}p(\lambda,\phi)\,d\lambda \Big\|_{L_p[-b,0]} = 0,$$

and the first statement of the theorem follows. The last statement of the theorem makes use of the straightforward identity which holds for all ϕ in the domain of the generator

$$(4.9)\qquad \forall\phi \in \mathcal{D}(A): \quad p(\lambda,\phi) = \frac{1}{\lambda}\{p(\lambda,\phi')+\Delta(\lambda)\phi(0)\}.$$

The factor λ^{-1} so obtained does the job. □

REMARK. The condition $\phi \in L_{p'}$, $p' > p$, can be weakened to a condition on the interval $[-\varepsilon,0]$: $\phi \in L_p[-b,0] \cap L_{p'}[-\varepsilon,0]$, ε small but positive, and resembles a condition involving backward continuation. Compare for instance [1, Corollary 4.2].

While finishing this paper Verduyn Lunel has obtained the answer to one of the questions posed in section 3, and he will publish in [31] the result: $\alpha = \delta$.

5. APPENDIX

LEMMA 5.1. *Recall that* $Q(t,s) = \zeta_t(s) - R*\zeta_t(s)$. $\int_0^t Q(\xi,s-\tau)d\xi$ *is absolutely continuous as a function of* τ *with derivative* $Q(0,s-\tau)-Q(t,s-\tau)+R(s-\tau)\int_0^t\zeta(\xi)d\xi$.

PROOF.

$$\int_0^t Q(\xi,s-\tau)d\xi = \int_0^t (\zeta_\xi(s-\tau)-R(s-\tau)*\zeta_\xi(\tau)d\xi =$$

$$\int_0^t \zeta(\xi+s-\tau)d\xi - \int_0^t\int_0^{s-\tau} R(\sigma)\zeta(\xi+s-\tau-\sigma)d\sigma d\xi =$$

$$\int_{s-\tau}^{t+s-\tau} \zeta(\xi)d\xi - \int_0^{s-\tau} R(\sigma)d\sigma\left(\int_{s-\tau}^{t+s-\tau} \zeta(\xi-\sigma)d\xi\right).$$

Therefore $\int_0^t Q(\xi,s-\tau)d\xi$ is absolutely continuous as a function of τ with derivative

$$\frac{\partial}{\partial\tau}\int_0^t Q(\xi,s-\tau)d\xi = -\zeta(t+s-\tau)+\zeta(s-\tau)+R(s-\tau)\int_{s-\tau}^{t+s-\tau} \zeta(\xi-s+\tau)d\xi +$$

$$-\int_0^{s-\tau} R(\sigma)(\zeta(s-\tau-\sigma)-\zeta(t+s-\tau-\sigma))d\sigma =$$

$$-\zeta_t(s-\tau)+\zeta(s-\tau)+R*\zeta_t(s-\tau)-R*\zeta(s-\tau)+R(s-\tau)\cdot\int_s^{t+s} \zeta(\xi-s)d\xi =$$

$$Q(0,s-\tau)-Q(t,s-\tau)+R(s-\tau)\int_0^t \zeta(\xi)d\xi. \quad \square$$

LEMMA 5.2. *Recall that* $Q^+(t,s) = \zeta_t(s)-\zeta_t*R(s)$. $\int_0^t Q^+(\tau,s+\xi)d\xi$ *is absolutely continuous as a function of* τ *with derivative* $Q^+(\tau,s+t)-Q^+(\tau,s) + \zeta(\tau)\int_s^{t+s} R(\sigma)d\sigma$.

PROOF.

$$\int_0^t Q^+(\tau,s+\xi)\,d\xi = \int_0^t \zeta(\tau+s+\xi)\,d\xi - \int_0^t \int_0^{s+\xi} \zeta(\tau+s+\xi-\sigma)R(\sigma)\,d\sigma d\xi =$$

$$\int_{\tau+s}^{\tau+s+t} \zeta(\tau)\,d\tau - \int_0^s \int_0^t \zeta(\tau+s+\xi-\sigma)\,d\xi R(\sigma)\,d\sigma +$$

$$- \int_s^{s+t} \int_{\sigma-s}^{t} \zeta(\tau+s+\xi-\sigma)\,d\xi R(\sigma)\,d\sigma =$$

$$\int_{\tau+s}^{\tau+s+t} \zeta(\sigma)\,d\sigma - \int_0^s \int_\tau^{t+\tau} \zeta\,(s+\xi-\sigma)\,d\xi R(\sigma)\,d\sigma +$$

$$- \int_s^{s+t} \int_{\sigma+\tau-s}^{t+\tau} \zeta(s+\xi-\sigma)\,d\xi R(\sigma)\,d\sigma.$$

Therefore $\int_0^t Q^+(\tau,s+\xi)d\xi$ is absolutely continuous as a function of τ with derivative

$$\frac{\partial}{\partial\tau}\int_0^t Q^+(\tau,s+\xi)\,d\xi = \zeta(\tau+s+t) - \zeta(\tau+s)$$

$$- \int_0^s \zeta(\tau+s+t-\sigma)R(\sigma)\,d\sigma + \int_0^s \zeta(\tau+s-\sigma)R(\sigma)\,d\sigma +$$

$$- \int_s^{s+t} \zeta(\tau+s+t-\sigma)R(\sigma)\,d\sigma + \int_s^{s+t} \zeta(\tau)R(\sigma)\,d\sigma =$$

$$\zeta_\tau(s+t)-\zeta_\tau * R(s+t)-\zeta_\tau(s)+\zeta_\tau * R(s)+\zeta(\tau)\int_s^{s+t} R(\zeta)\,d\zeta =$$

$$Q^+(\tau,s+t)-Q^+(\tau,s)-\zeta(\tau)\int_s^{t+s} R(\sigma)\,d\sigma. \quad \square$$

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