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B.N. COOPERSTEIN

A FINITE FLAG-TRANSITIVE GEOMETRY OF EXTENDED G_2-TYPE

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A finite Flag-Transitive Geometry of Extended G_2-Type *)

by

Bruce N. Cooperstein **)

ABSTRACT

The purpose of this note is to give an explicit construction of a finite flag-transitive GAB with an extended G_2-diagram having the group $G_2(3)$ as automorphism group. In our example it will be apparent that the intersection property is satisfied.

KEY WORDS & PHRASES: *Buekenhout-Tits geometries, geometries of extended* G_2 *type, Chevalley group* $G_2(3)$

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*) This report will be submitted for publication elsewhere.
**) University of California, Santa Cruz, California, 95064

0. INTRODUCTION

Geometries that are almost buildings (GABs) were introduced by TITS in [11]. They are BUEKENHOUT-TITS geometries [1] in which all rank two residual geometries are generalized polygons, except they need not satisfy the intersection property. Tits has shown that they exist in great number, including finite ones. In [7] KANTOR remarks that the situation for finite GABs with large automorphism groups, other than those arising from buildings appear to be rare. In [7] KANTOR briefly describes four finite GABs having flag-transitive automorphism groups. The only other known finite flag-transitive GAB was constructed by RONAN and SMITH [9] from the Suzuki sporadic group. The purpose of this note is to give an explicit construction of a finite flag-transitive GAB with an extended G_2-diagram having the group $G_2(3)$ as automorphism group. In our example it will be apparent that the intersection property is satisfied.

1. GEOMETRIES OF EXTENDED G_2-TYPE

We will be concerned here with incidence structures $\mathcal{I} = (P,L,\Pi;I)$ with three types of objects:

P, whose elements are called *points*;

L, whose elements are called *lines*; and

Π, whose elements are called *planes*, together with a symmetric relation I on $\mathcal{E} = P \cup L \cup \Pi$. We set $\mathcal{O} = \{P,L,\Pi\}$.

Suppose $\{X,Y,Z\} = \mathcal{O}$, $x \in X$. Set

$$Y_x = \{y \in Y | x \; I \; y\}, \quad Z_x = \{z \in Z \mid x \; I z\},$$

and

$$I_x = I\Big|_{Y_x \cup Z_x}.$$

$(Y_x, Z_x; I_x)$ is called the *residue* of $\mathcal{I}$ at x, $\mathcal{I}_x$.

We say $\mathcal{I}$ is of *extended* G_2*-type* or *belongs to the diagram* •—o≡o if the following are satisfied

(i) If $X \in \mathcal{O}$, $x \neq y \in X$, then x and y are not incident;

(ii) If $x \in P$, I_x is a generalized hexagon;

(iii) If $\ell \in L$, I_ℓ is a complete bipartite graph; and

(iv) If $\pi \in \Pi$, I_π is a projective plane.

In our construction all residues will have order two.

2. THE GEOMETRY OF NON-ISOTROPIC POINTS IN THREE DIMENSIONAL UNITARY SPACE

Let $K = \mathbb{F}_{q^2}$, $\langle\tau\rangle = \mathrm{Gal}(K/k)$, $k = \mathbb{F}_q$. Denote images under τ by $\overline{}$. Let V be a three-dimensional vector space over K and $h: V\times V \longrightarrow K$ a non-degenerate hermitian form, that is, h should satisfy

(i) For $v \in V$, $w \longrightarrow h_v(w) = h(w,v)$ is linear

(ii) For $v,w \in V$, $h(w,v) = \overline{h(v,w)}$

(iii) $h_v \equiv 0$ if and only if $v = 0$.

(V,h) is a *unitary space over K.*

Let $N = \{\langle v\rangle \mid v \in V,\ h(v,v) \neq 0\}$. N is the set of *non-isotropic* points of (V,h). Define a graph on N as follows: for $p \neq q \in N$, $p \sim q$ if and only if $h(p,q) = 0$. Let Λ be the collection of maximal cliques in $(N,\sim)$. Clearly these are triples. We collect some facts about the partial linear space (N,Λ).

(2.1) $|N| = q^2(q^2-q+1)$

This is easy: $|PG(V)| = q^4 + q^2 + 1$. The number of isotropic or absolute points ($\langle x\rangle$ is absolute if $h(x,x) = 0$), is $q^3 + 1$.

For $x \in N$, set $\Gamma(x) = \{y \in N: x \sim y\}$. We let $d(,) = d_\Gamma(,)$ be the usual metric associated with (N,Γ) and $\Gamma_t(x)$ be the points at distance t from x.

(2.2) $|\Gamma(x)| = q^2-q.$

This is again easy: $x^\perp = \langle v \in V: h(x,v) = 0\rangle$ is a non-degenerate two space, so $|PG(x^\perp)| = q^2 + 1$. The number of absolute points in $x^\perp$ is $q + 1$.

Set $\Lambda_x = \{\lambda \in \Lambda : x \in \lambda\}$. Then clearly from (2.2) we have

(2.3) $|\Lambda_2| = (q^2-q)/2$

Next suppose $x \not\perp y$. There are clearly two possibilities:

(i) $\langle x,y\rangle$ is non-degenerate; and

(ii) Rad $\langle x,y\rangle = \langle x,y\rangle^\perp \cap \langle x,y\rangle$ is a (isotropic) point.

In case (i) we see that $x^\perp \cap y^\perp$ is a single point in N, while in (ii) $\langle x,y\rangle^\perp$ = Rad $\langle x,y\rangle$. Thus in case (i), $d_\Gamma(x,y) = 2$ and in case (ii) $d_\Gamma(x,y) \geq 3$. A simple count yields

(2.4) $|\Gamma_2(x)| = (q^2-q)(q^2-q-2)$.

Now it is not difficult to see that $O(V,h) = \{T \in GL(V): h(Tv,Tw) = h(v,w)\}$ acts transitively on pairs $\{x,y\} \subseteq N$ with $d(x,y) = 2$, and on pairs with Rad $\langle x,y\rangle \neq 0$. It therefore follows that

(2.5) (N,Γ) is distance transitive with diameter 3.

We now turn our attention to pairs $\{x,y\}$ with Rad $\langle x,y\rangle \neq 0$. Let $x = \langle v\rangle$, $y = \langle w\rangle$ where $h(v,v) = h(w,w) = 1$. Suppose $\lambda = \{x = x_1, x_2, x_3\}$ is a line on x. Let $v = v_1$, $x_i = \langle v_i\rangle$ where $h(v_i,v_i) = 1$, $i = 2,3$. Now set R = Rad $\langle x,y\rangle$. Then $w = a\,v + r$ where $r \in R$. Since $h(w,w) = h(v,v) = 1$, without loss of generality we may assume $a = 1$. Since $h(v,r) = 0$ and $v^\perp = x^\perp = \langle x_2,x_3\rangle$, there are $b,c \in K$ so $r = b\,v_2 + c\,v_3$. Since $h(r,r) = 0$ we must have

(2.6) $b\bar{b} + c\bar{c} = 0$.

We will determine conditions for $d(y,x_i) = 2$ for $i = 2,3$. Now $d(y,x_i) = 2$ if and only if $y^\perp \cap x_i^\perp \in N$. $x_i^\perp = \langle x_1,x_j\rangle$ where $\{i,j\} = \{2,3\}$. Note if $\alpha\, v_1 + \beta\, v_j \in y^\perp$, then $\alpha\,\beta \neq 0$, so we may take $\beta = 1$. If $h(\alpha v_1 + v_j\,,\, v_1 + bv_2 + cv_3) = 0$ then

$$\alpha = \begin{cases} -\bar{b} & j = 2. \\ -\bar{c} & j = 3 \end{cases} \qquad \text{Set } \alpha_j = \begin{cases} -\bar{b} & j = 2 \\ -\bar{c} & J = 3 \end{cases}$$

Now $h(\alpha_j v_1+v_j, \alpha_j v_1+v_j) = \alpha_j\overline{\alpha}_j + 1$. Now if we assume char (k) $\neq 2$, then by (2.6) we cannot have $\alpha_2\,\overline{\alpha}_2 + 1 = 0 = \alpha_3\,\overline{\alpha}_3 + 1$. We have therefore shown

(2.7) If $d(x,y) = 2$, $\lambda \in \Lambda_x$, then $\lambda \cap \Gamma_2(y) \neq \emptyset$.

Now if $q > 3$ we can easily see that there are pairs x,y with $d(x,y) = 3$ and lines λ on x such that $\lambda - \{x\} \subseteq \Gamma_2(y)$. However, from (2.6) we see that

(2.8) If $q = 3$, $y \in \Gamma_3(x)$, $\lambda \in \Lambda_x$, then $|\lambda \cap \Gamma_2(y)| = 1$.

We have thus demonstrated all we need for

(2.9) THEOREM. *The geometry* (N,Λ) *is a generalized hexagon if, and only if,* $q = 3$.

(2.10) REMARK. When $q = 3$ the generalized hexagon (N,Λ) is the dual of the usual (2,2)-generalized hexagon associated with $G_2(2)$. [The usual $G_2(2^n)$ hexagon is the one embeddable in $PG(5,2^n)$].

This follows from the fact that if $Q = O_2(G_x)$, $x \in N$, then Q', the commutator subgroup of Q, has order two.

3. THE OCTAVES AND THE G_2-GENERALIZED HEXAGON

Let k be a commutative field, $\mathbb{O}(k) = \mathbb{O}$ the split octaves over k . $\mathbb{O}$ is a *composition algebra,* that is $\mathbb{O}$ is an algebra with identity and admits a non-degenerate quadratic form Q such that $Q(x.y) = Q(x)\,Q(y)$. We can find an othonormal base $1 = e_0, e_1, \ldots, e_7$ for $\mathbb{O}$, with e_0 the identity element such that multiplication in $\mathbb{O}$ is determined by

(3.1) $e_i^2 = -1$, $1 \leq i \leq 7$

and

(3.2) $e_i e_j = -e_j e_i = e_k$ whenever (ijk) is one of the three cycles (1+r,2+r,4+r) where i,j,k,r run through the integers modulo seven and take their values in $\{1,2,\ldots,7\}$.

Let $W = e_0^{\perp} = \langle e_1, \ldots, e_7\rangle$, so that $Q|W$ is a non-degenerate quadratic form (with maximal Witt index). Let $G = \mathrm{Aut}(\mathbb{O})$. Clearly G leaves e_0 and W invariant.

It is well-known (cf.[5])that G is the Chevalley group $G_2(k)$. Moreover, it is well known that there is, up to isomorphism, only one such algebra $\mathbb{O}$(cf.(3.1) in [5]). Consequently we have

(3.3) If $f_1,\dots,f_7$ satisfy (3.1),(3.2) then there exist $\sigma \in Aut(\mathbb{O})$ with $\sigma f_i = e_i$, $1 \leq i \leq 7$.

Of interest to us will be the set $\Phi = \{\{e_i,-e_i\}, 1 \leq i \leq 7\}$, and its full stabilizer, G_Φ , in G. This group is also well-known (see [3] and [4]):

(3.4) G_Φ is a non-split extension of an elementary abelian group E of order 8 by $PSL_3(2)$.

For the sequel we assume that -1 is not a square in k. Then $K = \langle e_0,e_1\rangle = k(e_1) \cong k[t]/_{(t^2+1)}$ is a quadratic extension of k.

Set $V = \langle e_2,\dots,e_7\rangle = W \cap e_1^{\perp}$. Note that V becomes a three dimensional vector space over K by restriction of the multiplication of $\mathbb{O}$ to $K\times V$.

For $u = ae_0 + be_1 \in K$, set $\overline{u} = ae_0 - be_1$. Then $\overline{}$ generates the Galois group of K over k. Next define $h: V\times V \longrightarrow K$ to be $p \circ \mu\,|V\times V$, where μ is the multiplication of $\mathbb{O}$ and p is the projection of $\mathbb{O}$ onto K. Then

(3.5) h is a non-degenerate hermitian form on V with associated automorphism $\overline{}$. Moreover, $\sigma \in G_{\langle e_1\rangle}$ if and only if $\sigma|V$ is a unitary transformation, i.e. preserves h.

4. THE CONSTRUCTION OF THE EXTENDED G_2 - GEOMETRY OF ORDER 2

We retain the notation of the previous sections. Further we set

$$P = \langle e_1\rangle^G = \{\langle w\rangle : w \in W,\ Q(w) = 1\}$$

$$L = \ell^G, \text{ where } \ell = \{\langle e_1\rangle,\langle e_2\rangle,\langle e_4\rangle\} \text{ ; and}$$

$$\Pi = \pi^G \text{ where } \pi = \{\langle e_i\rangle : 1 \leq i \leq 7\}.$$

Let I be symmeterized inclusion of subsets of P restricted to $\mathcal{E} = P \cup L \cup \Pi$. We will prove

<u>(4.1) THEOREM:</u> *$\mathcal{I}$ = (P,L,Π;I) is a geometry of extended G_2-type if and only if $k = \mathbb{F}_3$.*

We proceed to prove this in a series of steps. Note that $G = \mathrm{Aut}(\Phi)$ is flag-transitive by (3.3). Because of this it suffices to check the residues $I_{<e_1>}$, I_ℓ, I_π.

(4.2). I_π is a projective plane of order 2.

pf: $\{<e_i>, <e_j>, <e_k>\} \in L_\pi$ for all triples (ijk) as in (3.2). If these are all the lines in L_π, then the result is clear. Now $C = G_\Phi \leq G_\pi$ and is two transitive on π. Moreover, for any $i \neq j$ $C_{<e_i>,<e_j>}$ fixes $<e_k>$ where (ijk) is as in (3.2) and is transitive on $\pi - \{<e_i>,<e_j>,<e_k>\}$. It therefore suffices to prove $\{<e_1>,<e_2>,<e_3>\} \notin L$. But this is obvious since G tra L and elements of G preserve multiplication.

(4.3) $I_{<e_1>}$ is a generalized hexagon if and only if $k = \mathbb{F}_3$.

pf: Let $\ell \in L_{<e_1>}$. Define $\Theta(\ell) = \ell \cap e_1^\perp$. Now $\ell \in L_{<e_1>}$ if and only if the subalgebra generated by e_0 and ℓ is $k e_0 + k\ \ell$, and this occurs if and only if $\Theta(\ell)$ is a one dimensional non-isotropic subspace of the unitary K-space $V = W \cap e_1^\perp$. Thus $\Theta(L_{<e_1>}) = N(V,h)$, the non-isotropic points. Because of the flag transitivity we see that $\ell_1, \ell_2 \in L_{<e_1>}$ lie in a common plane if and only if the points $\Theta(\ell_1)$, $\Theta(\ell_2)$ of V are orthogonal. Thus $I_{<e_1>}$ is isomorphic to the geometry (N,Λ) of section two. By (2.10) $I_{<e_1>}$ is a generalized hexagon if and only if $k = \mathbb{F}_3$, and in this case $I_{<e_1>}$ is the dual of the usual (2,2)-hexagon.

(4.4) If $k = \mathbb{F}_3$ then I_ℓ is the complete bipartite graph $K_{3,3}$.

pf: Clearly I_ℓ is a complete bipartite graph and as $|\ell| = 3$ it suffices to prove that there are three planes containing ℓ. However, if $\pi' \in \Pi_\ell$, then $\pi' \in \Pi_{<e_1>}$ and π' is a line of $I_{<e_1>}$ containing ℓ. Since $I_{<e_1>}$ has order (2,2) it follows that there are precisely three planes containing ℓ.

This completes the proof of the theorem.

5. CONCLUDING REMARKS

Following KANTOR [4] we define an apartment to be a subset Δ of P such that there are automorphisms r,s,t of I leaving Δ invariant such that r^Δ, s^Δ, t^Δ satisfy the relations •——•≡• while $<r,s,t>^\Delta$ acts flag-transitively. We have

(5.1) PROPOSITION. *Apartments do not exist.*

PROOF. Suppose that Δ, r, s, t exist. Then there is a point $P_0 \in \Delta$ fixed by s and t and points $P_1, \dots, P_6$ in Δ all collinear with P_0 such that P_i is collinear with P_{i+1} (modulo 6). Moreover triangles in Δ are not lines of I. However, $\langle s^\Delta, t^\Delta \rangle$ contains an elementary group of order 4 which cannot act regularly on $\{P_1, \dots, P_6\}$, therefore there is an involution τ in $\langle s,t \rangle$ fixing at least two of $\{P_1, \dots, P_6\}$. But then since τ fixes P_0, $|Fix_\Delta(\tau)| \geq 3$. However, for any involution in G, $Fix_\Delta(\tau)$ is a line. This contradicts the fact that Δ contains no full lines.

(5.2) REMARK. In [6] Goldschmidt studied groups G generated by a pair of subgroups P_1, P_2 where P_1, P_2 contain a common 2-Sylow S of G and $|P_i:S| = 3$, $i = 1,2$, and determined all such groups. In analogy with the groups of Lie type we might call such subgroups P_i parabolics. This theory has been extended to the case $|P_1:S| = q + 1$, $P_i/O_2(P_i) \cong L_2(q)$. CHERMAK [2], TIMMESFELD [10] and others have considered the problem of determining groups $G = \langle P_1, \dots, P_n \rangle$ with $n \geq 3$ such that all P_i contain a common 2-Sylow S, $|P_i:S| = q + 1$ and $P_i/O_2(P_i) \cong PSL_2(q)$, q a power of two, especially the case where for any $i \neq j$, $\langle P_i, P_j \rangle / O_2(\langle P_i, P_j \rangle) \cong L_2(q) \times L_2(q)$ or $L_3(q)$ or $SL_3(q)$. In general the resulting group is a Chevalley group of type A_n, D_n, E_n over a field of characteristic two. Our construction provides an example of a group G generated by three "parabolics" P_1, P_2, P_3 containing a common two Sylow S with $|P_i:S| = 3$ and such that for any $i \neq j$, $\langle P_i, P_j \rangle / O_2(\langle P_i, P_j \rangle)$ is one of $PSL_3(2)$, $PSL_2(2) \times PSL_2(2)$, $G_2(2)$. This suggests that a general classification will prove extremely difficult.

(5.3) FINAL REMARK. In some sense the existence of such a geometry for $G_2(3)$ should not be surprising: A result of MASON'S [8] classifies the groups G of characteristic 2,3-type under very mild hypotheses. The groups are $PSP_4(3)$, $U_4(3)$ and $G_2(3)$. Of course $PSP_4(3)$ is not particularly exceptional because of the isomorphism $PSP_4(3) \cong \Omega_6^-(2)$. In [7] KANTOR constructs a GAB for $U_4(3)$, and our construction is for $G_2(3)$. Thus these groups act flag-transitively on $\mathbb{F}_3$-buildings and as flag-transitive groups on $\mathbb{F}_2$-'near' buildings.

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