Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

A.L.M. Dekkers, L. de Haan

On a consistent estimate of the index of an extreme value distribution

Department of Mathematical Statistics   Report MS-R8710   September

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

On a Consistent Estimate of the Index of an Extreme Value Distribution

Arnold L.M. Dekkers
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Laurens de Haan
Erasmus University Rotterdam
P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

Abstract: An easy proof is given for the weak consistency of Pickands' estimate of the main parameter of an extreme-value distribution. Moreover further natural conditions are given for strong consistency and for asymptotic normality of the estimate.
AMS 1980 Subject Classifications: Primary 62F12, Secondary 62G30.
Key Words and Phrases: Extreme-value theory, order statistics, strong consistency, asymptotic normality.

1. Introduction.

Suppose one is given a sequence $X_1,X_2,...$ of i.i.d. observations from some distribution function F. Suppose for some constants $a_n>0$ and b_n and some $\gamma\in\mathbb{R}$

$$\lim_{n\to\infty} P\{\frac{\max(X_1,X_2,\ldots,X_n)-b_n}{a_n}\leqslant x\} = G_\gamma(x) \tag{1.1}$$

for all x where $G_\gamma(x)$ is one of the extreme-value distributions

$$G_\gamma(x) = \exp-(1+\gamma x)^{-1/\gamma}. \tag{1.2}$$

Here γ is a real parameter (interpret $(1+\gamma x)^{-1/\gamma}$ as e^{-x} for $\gamma=0$) and x such that $1+\gamma x>0$. The question is how to estimate γ from a finite sample $X_1,X_2,\ldots,X_n$.

A traditional method uses "yearly maxima" i.e. breaks the sample into blocks of equal size and uses maximum likelihood estimation under the assumption that the maximum in each block follows *exactly* distribution G_γ. Consistency has been proved here under certain conditions (J.P. Cohen, 1986). By using this method some information from the sample seems to be lost.

A less traditional method consists of restricting attention to those observations from $X_1,X_2,\cdots,X_n$ that exceed a certain level $M(n)$ and using the method of maximum likelihood under the assumption that these observations follow *exactly* one of the asymptotic residual life-time distributions. Asymptotic results for this procedure have been obtained by R.L. Smith (1985).

An attractive alternative estimate has been proposed by J. Pickands III (1975): Let $m(n)$ be a sequence of integers tending to infinity and let $m(n)/n\to 0$ $(n\to\infty)$. The estimate is

$$\hat{\gamma}_n := (\log 2)^{-1}\cdot\log\frac{X_{(m)}-X_{(2m)}}{X_{(2m)}-X_{(4m)}} \tag{1.3}$$

where $X_{(1)}\geqslant X_{(2)}\geqslant\cdots\geqslant X_{(n)}$, the descending order statistics of $X_1,X_2,\ldots,X_n$ (note that we suppressed the extra index n in the notation). Pickands proved that this estimate is weakly consistent. We shall give a short proof of this result and show that if the sequence $m(n)$ increases suitably rapidly then there is strong consistency. Also we give quite natural and general conditions under which the estimate is asymptotically normal. The analytical work involved in the translation of the condition for the inverse of F into conditions for F is given in a separate section that should be useful in other

Report MS-R8710
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

contexts as well. In this section we assume that the reader is familiar with the theory of Π -variation and Γ -variation (see e.g. J.L. GELUK AND L. DE HAAN 1986)

Knowing the asymptotic distribution of $\hat{\gamma}$ is particularly important: since there is a discontinuity in the shape of the distribution G_γ at $\gamma = 0$, one often wants to test hypotheses of the type $\gamma = 0, \gamma \geqslant 0$ or $\gamma \leqslant 0$.

2. CONSISTENCY AND ASYMPTOTIC NORMALITY

We shall need the following simple result:

LEMMA 2.1: *If* $F(x) = 1 - e^{-x}$ *(standard exponential distribution),* $m(n) \to \infty$ *and* $m = \frac{m(n)}{n} \to 0$ $(n \to \infty)$, *then*

$$\sqrt{2m}\,(X_{(m)} - X_{(2m)} - \log 2)$$

has asymptotically a standard normal distribution.

PROOF: We use the representation for exponential order statistics usually referred to as Rényi's representation: for each n there exist i.i.d. random variables $Z_1, Z_2, \ldots$ with standard exponential distribution such that $\{X_{(m)} - X_{(m+1)}\}_{m=1}^{n-1} \overset{d}{=} \{Z_m / m\}_{m=1}^{n-1}$. This gives $X_{(m)} - X_{(2m)} \overset{d}{=} \sum_{i=m}^{2m-1} Z_i / i$. The rest of the proof is easy (use e.g. B.V. GNEDENKO and A.N. KOLMOGOROV 1954, chapter 5). □

COROLLARY 2.1: $X_{(m)} - X_{(2m)} \to \log 2$ *in probability e.g.* $(n \to \infty)$.

Further we list a well-known result (see L. DE HAAN 1984).

LEMMA 2.2: *Suppose (1.1) holds and define* $U := (\frac{1}{1-F})^{\leftarrow}$ *(the inverse function). Then for* $x, y > 0, y \neq 1$

$$\lim_{t \to \infty} \frac{U(tx) - U(t)}{U(ty) - U(t)} = \frac{x^\gamma - 1}{y^\gamma - 1} \text{ locally uniformly } \left(:= \frac{\log x}{\log y} \text{ for } \gamma = 0\right).$$

THEOREM 2.1: *(weak consistency) If (1.1) holds,* $m(n) \to \infty$ *and* $m(n)/n \to 0$ $(n \to \infty)$, *then* $\hat{\gamma}_n \to \gamma$ *in probability* $(n \to \infty)$.

PROOF: Let $A_1, A_2, \ldots$ be i.i.d. exponential random variables and let $\{A_{(m)}\}$ be the descending order statistics of $A_1, A_2, \ldots, A_n$. Then $\{X_{(m)}\}_{m=1}^n \overset{d}{=} \{U(e^{A_{(m)}})\}_{m=1}^n$. Note that $m(n)/n \to 0$ implies $e^{A_{(m)}} \to \infty$ a.s. $(n \to \infty)$. Now

$$\frac{U(e^{A_{(m)}}) - U(e^{A_{(2m)}})}{U(e^{A_{(2m)}}) - U(e^{A_{(4m)}})} = \frac{U(e^{A_{(2m)}} \cdot e^{A_{(m)} - A_{(2m)}}) - U(e^{A_{(2m)}})}{U(e^{A_{(2m)}}) - U(e^{A_{(2m)}} \cdot e^{A_{(4m)} - A_{(2m)}})} \to \frac{2^\gamma - 1}{1 - 2^{-\gamma}} = 2^\gamma$$

in probability by corollary 2.1 and lemma 2.2. The result follows. □

THEOREM 2.2: *(strong consistency). If (1.1) holds,* $m(n)/n \to 0$ *and* $m(n)/\log\log n \to \infty (n \to \infty)$, *then*

$$\hat{\gamma}_n \to \gamma \text{ a.s.} \qquad (n \to \infty).$$

PROOF: The conditions on the sequence $m(n)$ imply $A_{(m)} + \log \frac{m(n)}{n} \to 0$ a.s.

(J. WELLNER 1978, corrollary 4). Hence $A_{(m)}-A_{(2m)}\to\log 2$ $(n\to\infty)$ a.s. The rest of the proof is as before. We thank R. Helmers for making us aware of the Wellner reference. □

THEOREM 2.3: *(Asymptotic normality) Suppose U has a positive derivative and suppose there exists a function a such that for $x>0$ and $\gamma\in\mathbb{R}$ (with either choice of sign)*

$$\lim_{t\to\infty}\frac{(tx)^{1-\gamma}U'(tx)-t^{1-\gamma}U'(t)}{a(t)} = \pm\log x$$

(Π- variation, notation $\pm t^{1-\gamma}U'(t)\in\Pi(a)$), then

$$\sqrt{m}(\hat{\gamma}_n-\gamma)$$

has asymptotically a normal distribution with mean zero and variance $\gamma^2(2^{2\gamma+1}+1)/\{2(2^\gamma-1)\log 2\}^2$ for sequences $m=m(n)$ satisfying $m(n)=o(n/g^{\leftarrow}(n))$ where $g(t):=\frac{1}{2}t^{3-2\gamma}\{U'(t)/a(t)\}^2$.

Before we prove this theorem we will formulate the conditions on U in terms of the distribution function F and its density. The proof of theorem 2.4 will be given in section 3 (theorem 3.1, 3.3 and 3.8; lemma 3.3).

THEOREM 2.4: *Suppose U has a positive derivative U'. Equivalent are (with either choice of sign)*

a. $\pm t^{1-\gamma}U'(t)\in\Pi$.

b. *for $\gamma>0$: $\pm t^{1+1/\gamma}F'(t)\in\Pi$,*
for $\gamma<0$: $U(\infty):=\lim_{t\to\infty}U(t)<\infty$ and $\mp t^{-1-1/\gamma}F'(U(\infty)-t^{-1})\in\Pi$,
for $\gamma=0$: let $f_0=(1-F)/F'$ and $x^*:=\sup\{x|F(x)<1\}$. There exists a positive function α with $\alpha(t)\to 0$ $(t\uparrow x^*)$ such that for $x>0$

$$\lim_{t\uparrow x^*}\frac{\dfrac{1-F(t+xf_0(t))}{1-F(t)}-e^{-x}}{-\alpha(t)} = \pm\frac{x^2}{2}e^{-x}.$$

In case $\gamma=0$ the following condition is sufficient for (b): suppose F is three times differentiable, $\pm f_0'>0$, $\lim_{t\to\infty}f_0''(t)f_0(t)/f_0'(t)=0$ and $\lim_{t\to\infty}f_0'(t)=0$ then (b) holds with $\gamma=0$.

PROOF of theorem 2.3: Assume for the moment that $+\ t^{1-\gamma}U'(t)\in\Pi$. This implies $F\in D(G_\gamma)$. Write $V(t):=U(e^t)$. We have

$$\frac{V'(t)-e^{-\gamma x}V'(t+x)}{\alpha(t)}\to -x \qquad \text{locally uniformly},$$

for some positive function α satisfying $\alpha(t+x)\sim e^{\gamma x}\alpha(t)$ locally uniformly and $\alpha(t)/V'(t)\to 0$ $(t\to\infty)$. Now

$$V(t+x)-V(t)-e^{\gamma x}V(t)+e^{\gamma x}V(t-x) =$$

$$= \int_0^x\{V'(t+s)-e^{\gamma x}V'(t+s-x)\}ds =$$

$$= \alpha(t)\int_0^x\frac{V'(t+s)-e^{\gamma x}V'(t+s-x)}{\alpha(t+s)}\cdot\frac{\alpha(t+s)}{\alpha(t)}ds,$$

hence locally uniformly

$$\lim_{t\to\infty}\frac{V(t+x)-V(t)-e^{\gamma x}V(t)+e^{\gamma x}V(t-x)}{\alpha(t)} = x\cdot\frac{e^{\gamma x}-1}{\gamma}. \qquad (2.1)$$

We write as in the proof of theorem 2.1

$$\frac{X_{(m)}-X_{(2m)}}{X_{(2m)}-X_{(4m)}}-2^{\gamma}=\frac{V(\{A_{(m)}-A_{(2m)}\}+A_{(2m)})-V(A_{(2m)})}{V(A_{(2m)})-V(\{A_{(4m)}-A_{(2m)}\}+A_{(2m)})}-2^{\gamma}=$$
$$\frac{V(\{A_{(m)}-A_{(2m)}\}+A_{(2m)})-V(A_{(2m)})-2^{\gamma}V(A_{(2m)})-2^{\gamma}V(\{A_{(4m)}-A_{(2m)}\}+A_{(2m)})}{V(A_{(2m)})-V(A_{(4m)})}. \tag{2.2}$$

In view of the result of lemma 2.1 we introduce

$Q_n:=\sqrt{2m}(A_{(m)}-A_{(2m)}-\log 2)$ and

$R_n:=\sqrt{4m}(A_{(2m)}-A_{(4m)}-\log 2)$.

Note that Q_n and R_n are independent and asymptotically standard normal.

We start by evaluating the denominator of (2.2) asymptotically. Note that $t^{1-\gamma}U'(t)\in\Pi$ implies $V'(t+x)\sim e^{\gamma x}V'(t)$ locally uniformly ($t\to\infty$). Hence

$$V(A_{(2m)})-V(A_{(4m)})=V'(A_{(2m)})\int_{-\log 2-R_n/\sqrt{4m}}^{0}\frac{V'(A_{(2m)}+s)}{V'(A_{(2m)})}ds\sim V'(A_{(2m)})\gamma^{-1}(1-2^{-\gamma})$$

in probability ($n\to\infty$), with the usual convention $\log 2=:\dfrac{1-2^{-\gamma}}{\gamma}$ when $\gamma=0$.

For the numerator of (2.2) we proceed as follows:

$$\sqrt{m}\frac{V(A_{(2m)}+\log 2+\frac{Q_n}{\sqrt{2m}})-V(A_{(2m)})-2^{\gamma}V(A_{(2m)})+2^{\gamma}V(A_{(2m)}-\log 2-\frac{R_n}{\sqrt{4m}})}{V'(A_{(2m)})}=$$
$$=\sqrt{m}\int_0^{Q_n/\sqrt{2m}}\frac{V'(A_{(2m)}+\log 2+s)}{V'(A_{(2m)})}ds+\sqrt{m}\,2^{\gamma}\int_{-R_n/\sqrt{4m}}^{0}\frac{V'(A_{(2m)}-\log 2+s)}{V'(A_{(2m)})}ds+$$
$$+\sqrt{m}\frac{V(A_{(2m)}+\log 2)-V(A_{(2m)})-2^{\gamma}V(A_{(2m)})+2^{\gamma}V(A_{(2m)}-\log 2)}{V'(A_{(2m)})}.$$

Now $V'(t+x)\sim e^{\gamma x}V'(t)$ locally uniformly ($t\to\infty$), hence the sum of the first two terms converges in distribution to $2^{\gamma-½}Q-2^{-1}R$ where Q and R are independent and standard normal. Our aim is to make the last term negligible by choosing the sequence $m(n)$ appropriately.

Using (2.1) we get that the last term converges to $-\gamma^{-1}(\log 2)(2^{\gamma}-1)$ for any sequence $m(n)$ with if

$$\sqrt{m}\sim\frac{V'(A_{(2m)})}{\sqrt{2}\,\alpha(A_{(2m)})}\quad,\ n\to\infty. \tag{2.3}$$

We now investigate what sequences $m(n)$ satisfy (2.3).

Note that (see e.g. N.V. SMIRNOV 1967)

$$A_{(2m)}+\log\frac{2m(n)}{n}\to 0 \text{ in probability, } n\to\infty,$$

so that (2.3) reads

$$\sqrt{2m}\sim\frac{V'(-\log\frac{2m(n)}{n})}{\sqrt{2}\;\alpha(-\log\frac{2m(n)}{n})}=\frac{(\frac{n}{2m})^{1-\gamma}U'(\frac{n}{2m})}{\sqrt{2}\;a(\frac{n}{2m})}$$

where a is the auxiliary function for $t^{1-\gamma}U'(t)\in\Pi$, or

$$n\sim\tfrac{1}{2}(\tfrac{n}{2m})^{3-2\gamma}\{U'(\tfrac{n}{2m})/a(\tfrac{n}{2m})\}^{2}=:g(\tfrac{n}{2m})$$

with $g \in RV_1$ (for definition see theorem 2.5 below). The function g has an asymptotic inverse $g^{\leftarrow} \in RV_1$. So (2.3) is equivalent to

$$m(n) \sim \frac{n}{2g^{\leftarrow}(n)} (n \to \infty) \tag{2.4}$$

and the latter sequence is RV_0. Thus the sequences $m(n)$ for which the condition holds, tend to infinity rather slowly.

Let $m_0(n)$ be the sequence of integers defined by

$$m_0(n) := [n/2g^{\leftarrow}(n)] \,. \tag{2.5}$$

We claim that the statement of the theorem holds for any sequence of integers $m(n) \to \infty$ satisfying

$$m(n) = o(m_0(n)) \ , \ n \to \infty. \tag{2.6}$$

To see this recall that

$$\sqrt{2m_0} \frac{V(\log\frac{n}{2m_0}+\log 2) - V(\log\frac{n}{2m_0}) - 2^{\gamma} V(\log\frac{n}{2m_0}) + 2^{\gamma} V(\log\frac{n}{2m_0}) - \log 2)}{V'(\log\frac{n}{2m_0})} \to \frac{2^{\gamma}-1}{\gamma} \log 2, \ n \to \infty \,.$$

Since (2.6) makes $\sqrt{m}$ of smaller order than $\sqrt{m_0}$ and $\log(\frac{n}{m})$ of no smaller order than $\log\frac{n}{m_0}$, we must have

$$\sqrt{m} \frac{V(\log\frac{n}{2m}+\log 2) - V(\log\frac{n}{2m}) - 2^{\gamma} V(\log\frac{n}{2m}) + 2^{\gamma} V(\log\frac{n}{2m} - \log 2)}{V'(\log\frac{n}{2m})} \to 0 \ , \ n \to \infty$$

and the statement of the theorem holds for the sequence $m(n)$. The proof in case $-t^{1-\gamma} U'(t) \in \Pi$ is now obvious. □

The normal distribution satisfies the conditions of theorem 2.4 and we then have asymptotic normality of $\hat{\gamma}_n$ for sequences $m(n) \to \infty$ satisfying $m(n) = o(\log^2 n)$. See the end of section 3. For distributions like the Cauchy distribution we have the following theorem.

THEOREM 2.5 *Suppose that one of the following conditions holds:*

a. *For some $\gamma>0$, $\rho>0$, $c>0$ the function $t^{1+1/\gamma} F'(t) - c$ is of constant sign and*

$$\lim_{t \to \infty} \frac{(xt)^{1+1/\gamma} F'(tx) - c}{t^{1+1/\gamma} F'(t) - c} = x^{-\rho}$$

(regular variation with exponent $-\rho$, notation $\pm\{t^{1+1/\rho} F'(t) - c\} \in RV_{-\rho}$).

b. *For some $\gamma<0$, $\rho>0$ and $c>0$ the function $\pm\{t^{-1-1/\gamma} F'(U(\infty) - t^{-1}) - c\} \in RV_{-\rho}$.*

Then

$$\sqrt{m}\{(\log 2)^{-1} \log (\frac{X_{(m)} - X_{(2m)}}{X_{(2m)} - X_{(4m)}}) - \gamma\}$$

has asymptotically a normal distribution with mean zero and variance $\gamma^2(2^{2\gamma+1}+1)/\{2(2^{\gamma}-1)\log 2\}^2$ for sequences $m = m(n)$ satisfying

$$m(n) = o(n/g^{\leftarrow}(n)) \ , \ n \to \infty \ ,$$

where $g^{\leftarrow}$ is the inverse function of

$$g(t) := \frac{1}{2} t^{3-2\gamma} \{U'(t)/(t^{1-\gamma} U'(t) - c^{\gamma} |\gamma|^{1+\gamma})\}^2 \,.$$

PROOF. Note that $\pm\{t^{1+1/\gamma}F'(t)-c\}\in RV_{-\rho}$ if and only if $\mp\{t^{1-\gamma}U'(t)-c^{\gamma}\gamma^{\gamma+1}\}\in RV_{-\rho\gamma}$, hence $(t\to\infty)$

$$\frac{1}{\gamma\rho}\cdot\frac{V'(t)-e^{-\gamma x}V'(t+x)}{V'(t)-e^{\gamma t}c^{\gamma}\gamma^{1+\gamma}}\to-\frac{e^{-\rho\gamma x}-1}{\rho\gamma} \text{ locally uniformly.}$$

The rest of the proof is similar to that of theorem 2.3 and is omitted. □

REMARK 2.1. Note that $g(t)\in RV_{1+2\rho\gamma}$ so that $t/g^{\leftarrow}(t)\in RV_{2\rho\gamma/(1+2\rho\gamma)}$. So here the asymptotic normality holds for sequences $m(n)$ increasing more rapidly than in the situation of theorem 2.3.

The Cauchy distribution satisfies the conditions of theorem 2.5 and we then have asymptotic normality of $\hat{\gamma}_n$ for sequences $m(n)\to\infty$ satisfying $m(n)=o(n^{4/5})$. We deal with examples more extensively at the end of section 3.

3. Analytical results

The conditions in theorem 2.3 are phrased in terms of U, the inverse functions of $1/(1-F)$. The aim of this section is to formulate these conditions in terms of the distribution function F itself and its density. The main result here is actually in terms of F alone but this result is not immediately applicable for theorem 2.3. It is given for completeness and since it will probably be useful in other contexts as well.

The relation to be studied is $\pm t^{1-\gamma}U'(t)\in\Pi$. We only consider this relation with the + sign except for theorem 3.7. In the other case the relevant formula's in the theorems below should be multiplied by -1. First we consider the case $\gamma>0$.

THEOREM 3.1: *Suppose U has a positive derivative U' and* $\gamma>0$.
Equivalent are

a. $t^{1-\gamma}U'(t)\in\Pi.$ (3.1.1)

b. $-U(t)+\gamma^{-1}tU'(t)\in RV_{\gamma}.$ (3.1.2)

c. $(t^{-\gamma}U(t))'\in RV_{-1}.$ (3.1.3)

d. $t^{1+1/\gamma}F'(t)\in\Pi.$ (3.1.4)

PROOF: (see DE HAAN 1977)

(a)⇔(b): $\int_0^1 s^{1-\gamma}\log s\,ds \leftarrow \int_0^1 \frac{(ts)^{1-\gamma}U'(ts)-t^{1-\gamma}U'(t)}{a(t)}s^{\gamma-1}ds = \frac{U(t)-\gamma^{-1}tU'(t)}{t^{\gamma}a(t)}$

(b)⇔(c): obvious

(b)⇔(d): Replacing t by $1/(1-F(s))\in RV_{1/\gamma}$ in (b) yields

$$\gamma^{-1}\frac{1-F(s)}{F'(s)}-s\in RV_1 \text{ i.e. } \gamma^{-1}(1-F(s))-sF'(s)\in RV_{-1/\gamma}.$$

This is a relation like (b) for U. The equivalence of this relation and (d) also and the converse implication are proved as in the first part of the proof. □

Relation (c) of theorem 3.1 implies $\pm\ t^{-\gamma}U(t)\in\Pi$. The latter relation can also be translated for F even when there is no derivative. That is the content of the next theorem.

THEOREM 3.2: *Equivalent are (for* $\gamma>0$*)*

a. $t^{-\gamma}U(t)\in\Pi.$ (3.2.1)

b. $\quad -t^{1/\gamma}(1-F(s))\in\Pi.$ (3.2.2)

Proof: Suppose for some positive function α and all $x>0$

$$\frac{\frac{U(tx)}{(tx)^\gamma}-\frac{U(t)}{t^\gamma}}{\alpha(t)} \to \log x \ (t\to\infty). \tag{3.3}$$

Then since $t^\gamma\alpha(t)/U(t)\to 0$ $(t\to\infty)$,

$$\log U(tx)-\log U(t)-\gamma\log x = \log(\frac{U(tx)}{x^\gamma U(t)}) \sim \frac{U(tx)}{x^\gamma U(t)}-1 \ (t\to\infty).$$

Hence with $R(t):=\log U(e^t)$ and $a(t):=e^{\gamma t}\alpha(e^t)/U(e^t)$

$$\frac{Q(t+x)-Q(t)-\gamma x}{a(t)} \to x \text{ locally uniformly,}$$

for all x and $a(t)\to 0$ $(t\to\infty)$. Set $S(t):=\log\left\{\frac{1}{1-F(e^t)}\right\}$, then for $\epsilon>0$ and all x

$$\frac{R(S(t)+x)-t-\gamma x}{a(S(t))} \geqslant$$

$$\geqslant \frac{R(S(t)+x)-R(S(t))-\gamma x}{a(S(t))} - \frac{R(S(t))+\epsilon a(S(t))-R(s(t))-\gamma\epsilon a S(t)}{a(S(t))} - \gamma\epsilon.$$

Hence $\liminf\limits_{t\to\infty} \frac{R(S(t)+x)-t-\gamma x}{a(S(t))} \geqslant x-\gamma\epsilon$. This together with a similar upper inequality gives

$$\lim_{t\to\infty} \frac{R(S(t)+x)-t-\gamma x}{a(S(t))} = x \text{ locally uniformly,}$$

hence in particular

$$R(S(t))-t=o(a(S(t)) \ (t\to\infty),$$

i.e. with $V(t):=\frac{1}{1-F(t)}$ (and using $\log y\sim y-1$ for $y\to 1$)

$$U(V(t))-t = o((V(t))^\gamma\alpha(V(t))) \ \ (t\to\infty). \tag{3.4}$$

Further (3.3) is equivalent to

$$U(t)-(\gamma+1)t^{-1}\int_0^t U(y)dy \sim t^\gamma\alpha(t)(\gamma+1)\int_0^1 y^\gamma\log y\,dy = t^\gamma\alpha(t)\frac{1}{\gamma+1} \ \ (t\to\infty).$$

Replacing t by $V(t)$ in this relation and using (3.4) we obtain

$t-\frac{\gamma+1}{V(t)}\int_0^t x\,dV(x) \in RV_1$ since $\alpha\in RV_0$ and $V\in RV_{\gamma-1}$, i.e.

$$-\frac{1}{t}\,(1+\frac{1}{\gamma})\int_0^t V(x)dx + V(t)\in RV_{1/\gamma}$$

which is obviously equivalent to $t^{-1/\gamma}V(t)\in\Pi$ and hence to (b). □

Next we consider the case $\gamma<0$.

Theorem 3.3: *Let $\gamma<0$ and suppose U has a positive derivative U'. Equivalent are, with $U(\infty):=\lim\limits_{t\to\infty}U(t)$,*

a. $t^{1-\gamma}U'(t)\in\Pi$. (3.5.1)

b. $U(\infty)<\infty$ and $-\{U(\infty)-U(t)+\gamma^{-1}tU'(t)\}\in RV_\gamma$. (3.5.2)

c. $U(\infty)<\infty$ and $-(t^{-\gamma}\{U(\infty)-U(t)\})'\in RV_{-1}$. (3.5.3)

d. $U(\infty)<\infty$ and $-t^{-1-1/\gamma}F'(U(\infty)-t^{-1})\in\Pi$. (3.5.4)

PROOF:

(a)⇔(b): $\int_1^\infty s^{1-\gamma}\log s\,ds \leftarrow \int_1^\infty \frac{(ts)^{1-\gamma}U'(ts)-t^{1-\gamma}U'(t)}{a(t)}s^{\gamma-1}ds = \frac{U(\infty)-U(t)+\gamma^{-1}tU'(t)}{t^\gamma a(t)}$.

(b)⇔(c): obvious.

(b)⇔(d): Write $U(\infty)-U(t)=s$ with $U(\infty)-U(t)\in RV_\gamma$ then $t=U^{\leftarrow}(U(\infty)-s)$ and $U^{\leftarrow}(U(\infty)-s^{-1})=1/\{1-F(U(\infty)-s^{-1})\}\in RV_{-1/\gamma}$.

Replacing t by $U^{\leftarrow}(U(\infty)-s^{-1})$ in (3.5.2) yields

$$-s^{-1}-\gamma^{-1}\frac{U^{\leftarrow}(U(\infty)-s^{-1})}{(U^{\leftarrow})'(U(\infty)-s)^{-1}} = -s^{-1}-\gamma^{-1}\frac{1-F(U(\infty)-s^{-1})}{F'(U(\infty)-s^{-1})}\in RV_{-1} \text{ using } (U^{\leftarrow})'=\frac{F'}{(1-F)^2}.$$

Since $F'(U(\infty)-s^{-1}) = \frac{\{1-F(U(\infty)-s^{-1})\}^2}{U'(1/\{1-F(U(\infty)-s^{-1})\})}\in RV_{1+1/\gamma}$, we obtain finally

$$-\{s^{-1}F'(U(\infty)-s^{-1})+\gamma^{-1}\int_s^\infty F'(U(\infty)-u^{-1})u^{-2}du\} =$$

$$-s^{-1}F'(U(\infty)-s^{-1})-\gamma^{-1}\{1-F(U(\infty)-s^{-1})\}\in RV_{1/\gamma}.$$

The implication (d) ⇒ (b) is proved in an analogous way. □

The case $\gamma=0$ is considerably more complicated. We start with a theorem on U.

THEOREM 3.4: *Suppose U has a positive derivative U'.*
Equivalent are:

a. $tU'(t)\in\Pi(a)$. (3.6.1)

b. $tU'(t)-U(t)+\frac{1}{t}\int_0^t U(s)ds\sim a(t)$ $(t\to\infty)$ where a is slowly varying. (3.6.2)

c. $\frac{U(tx)-U(t)-tU'(t)\log x}{a(t)}\to\frac{1}{2}\log^2 x$ $(t\to\infty)$ for $x>0$, where a is a positive function. (3.6.3)

PROOF:

(a)⇔(b):
$$\frac{U(t)-\frac{1}{t}\int_0^t U(s)ds-tU'(t)}{a(t)} = \int_0^1\frac{txU'(tx)-tU'(t)}{a(t)}dx\to$$

$$\to\int_0^1\log x\,dx = -1\quad(t\to\infty).$$

(a)⇒(c): For $x>0$ and $t\to\infty$

$$\frac{U(tx)-U(t)-t\log x\,U'(t)}{a(t)} = \int_1^x\frac{tyU'(ty)-tU'(t)}{a(t)}\frac{dy}{y}\to\int_1^x\frac{\log y}{y}dy\quad(t\to\infty).$$

(c)⇒(a): (As in OMEY and WILLEKENS 1986) For $x,y>0$

$$\frac{U(txy)-U(ty)-U(tx)+U(t)}{a(t)} = \frac{U(txy)-U(t)-tU'(t)\log(xy)}{a(t)} +$$
$$-\frac{U(ty)-U(t)-tU'(t)\log y}{a(t)} - \frac{U(tx)-U(t)-tU'(t)\log x}{a(t)} \to \log x \cdot \log y$$

It follows that for all $x>1$ the function $U(tx)-U(t)$ is in $\Pi(a(t)\log x)$ for $t\to\infty$. Hence $a\in RV_0$. Now for $t\to\infty$

$$\left[\frac{\log xy}{s}\right]^2 \leftarrow \frac{U(txy)-U(t)-t\log(xy)U'(t)}{a(t)} =$$
$$\frac{U(txy)-U(ty)-ty\log x U'(ty)}{a(ty)} \cdot \frac{a(ty)}{a(t)} +$$
$$+\frac{U(ty)-U(t)-t\log y U'(t)}{a(t)} + \log x \frac{tyU'(ty)-tU'(t)}{a(t)}.$$

Since everything else converges, also the last term must converge, hence $tU'(t)\in\Pi(a)$. □

After these preliminary statements on U we show what the translation to the inverse function is going to be in the nice case when one can work with derivatives. This serves as an introduction to the general results given afterwards.

Let Q be a three times differentiable function, then

$$Q(t+x)-Q(t) = xQ'(t) + \frac{x^2}{2}Q''(t) + \frac{x^3}{6}Q'''(t) + \cdots$$

If $Q'(t)>0$ and $Q''(t)/Q'(t)\to 0$ then all terms except the first one are asymptotically negligible: $Q''(t)/Q'(t)\to 0$ implies $Q'(t+x)/Q'(t)\to 1$ $(t\to\infty)$ for all x and hence $\{Q(t+x)-Q(t)\}/Q'(t)\to x$ $(t\to\infty)$ locally uniformly for all x (just integrate). This is basically Π-variation. Suppose next that $Q''(t)>0$ and $Q'''(t)/Q''(t)\to 0$ $(t\to\infty)$, then all terms except the first two ones are asymptotically negligible: $Q'''(t)/Q''(t)\to 0$ implies $Q''(t+x)/Q''(t)\to 1$ $(t\to\infty)$ for all x and hence

$$\{Q(t+x) - Q(t)-xQ'(t)\}/Q''(t) \to \frac{x^2}{2} \quad (t\to\infty) \qquad (*)$$

(use the just mentioned result for Q' instead of Q and integrate).

Now let P be the inverse function of Q. Let $x^*:=\sup\{x|P(x)<1\}$. We expand P as follows (we still suppose $Q'>0$, hence $P'>0$)

$$P(t+x/P'(t)) - P(t) = x + \frac{x^2}{2}\frac{P''(t)}{\{P'(t)\}^2} + \frac{x^2}{6}\frac{P'''(t)}{\{P'(t)\}^3} + \cdots$$

If $P''(t)/\{P'(t)\}^2 \to 0$, then $P'(t+x/P'(t))/P'(t) \to 1$ $(t\uparrow x^*)$ locally uniformly, hence $P(t+x/P'(t))-P(t)\to x$ $(t\uparrow x^*)$ locally uniformly. This is basically Γ-variation. Suppose next $P''(t)>0$, $P''(t)/\{P'(t)\}^2\to 0$ and $P'''(t)/\{P''(t)\cdot P'(t)\}\to 0$ $(t\uparrow x^*)$, then

$$\log P''(t+x/P'(t))-\log P''(t) = \frac{x}{P'(t)}\frac{P'''(t+x\theta/P'(t))}{P''(t+x\theta/P'(t))} \sim$$
$$\sim \frac{xP'''(t+x\theta/P'(t))}{P'(t+x\theta/P'(t))P''(t+x\theta/P'(t))} \to 0 \ (t\uparrow x^*) \text{ locally uniformly}$$

where $\theta=\theta(t,x)\in[0,1]$ and we can prove (see lemma 3.3 below)

$$\frac{P(t+x/P'(t))-P(t)-x}{-P''(t)/\{P'(t)\}^2} \to \frac{x^2}{2} \ (t\uparrow x^*) \text{ locally uniformly.} \qquad (**)$$

Note that the joint statements $Q''(t)/Q'(t)\to 0$ and $Q'''(t)/Q''(t)\to 0$ $(t\to\infty)$ are equivalent to the statements $P''(t)/\{P'(t)\}^2 \to 0$ and $P'''(t)/\{P''(t)\cdot P'(t)\} \to 0$ $(t\uparrow x^*)$.

To relate this to our problem, let $P := -\log(1-F)$ hence $Q = U\circ\exp$. Relation (*) is the same as relation (3.6.3) from theorem 3.4. Note that $P(t+x/P'(t)) - P(t) \to x$ $(t\uparrow x^*)$ means $\{1-F(t+xf_o(t))\}/\{1-F(t)\} \to e^{-x}$ $(t\uparrow x^*)$ with $f_o(t) = \{1-F(t)\}/F'(t)$. Hence (**) can be translated as follows (note $f_0'(t) \to 0$, $(t\uparrow x^*)$)

$$\frac{\dfrac{1-F(t+xf_o(t))}{1-F(t)} - e^{-x}}{f_o'(t)} \sim \frac{e^{-x}\log\left\{\dfrac{1-F(t+xf_o(t))}{(1-F(t))}e^{x}\right\}}{f_o'(t)}$$

$$= \frac{-P(t+x/P'(t)) + P(t) + x}{f_o'(t)}\cdot e^{-x} \to x^2e^{-x}\ (t\uparrow x^*), \text{ locally uniformly.}$$

We shall see that this is basically the relation we get in the general case.

We now work in an order different from what we did for $\gamma \neq 0$ and start with deriving the result with no differentiability assumption. The differentiable case will then be quite obvious.

THEOREM 3.5: *Suppose Q is non-decreasing and $P = Q^{\leftarrow}$. Equivalent are*

a. $$\frac{Q(t+x) - Q(t) - xa_1(t)}{a_2(t)} \to \frac{x^2}{2}\ (t\to\infty) \text{ for all } x \text{ and some positive functions } a_1 \text{ and } a_2 \tag{3.7.1}$$

b. $$\frac{P(t+xf(t)) - P(t) - x}{-\alpha(t)} \to \frac{x^2}{2}\ (t\to\infty) \text{ for all } x \text{ locally uniformly}, \tag{3.7.2}$$

where f and α are positive functions and $\alpha(t)\to 0$ $(t\uparrow x^*)$.

PROOF:

(a)⇒(b): For $\epsilon > 0$ and all x

$$Q(P(t)+x) - t \geq \{Q(P(t)+x) - Q(P(t))\} - \{Q(P(t)+\epsilon\frac{a_2(P(t))}{a_1(P(t))}) - Q(P(t))\}$$

A similar upper inequality is obtained, hence by the local uniformity in (3.7.1) and because $a_2(P(t))/a_1(P(t)) \to 0$ $(t\to\infty)$,

$$\lim_{t\to\infty} \frac{Q(P(t)+x) - t - xa_1(P(t))}{a_2(P(t))} = \frac{x^2}{2}$$

locally uniformly and in particular

$$Q(P(t)) - t = o(a_2(P(t)))\ (t\to\infty). \tag{3.8}$$

Also, with $\alpha(t) = a_2(P(t))/a_1(P(t))$,

$$Q(P(t)+x-\frac{x^2}{2}\alpha(t)) - t - \{x - \frac{x^2}{s}\alpha(t)\}a_1(P(t)) \sim \frac{x^2}{2}a_2(P(t)),$$

hence locally uniformly for $\epsilon > 0$ and t sufficiently large

$$\{Q(P(t)+x-\frac{x^2}{2}\alpha(t)) - t - xa_1(P(t))\}/a_2(P(t)) \leq \epsilon.$$

Then also $P(t)+x-\frac{x^2}{2}\alpha(t) \leq P\,(t+xa_1(P(t))+\epsilon a_2(P(t)))$ or (by substituting $y = x+\epsilon\alpha(t)$)

$$P(t+ya_1(P(t)))-P(t)-y \geqslant -\epsilon\alpha(t) - \frac{(y+\epsilon\alpha(t))^2}{2}\,\alpha(t)\,.$$

A similar lower inequality is readily derived. Relation (b) follows.
(b)⇒(a): The proof follows the same line. For $\epsilon>0$

$$P(Q(t)+xf(Q(t)))-t \geqslant$$

$$\geqslant \{P(Q(t)+xf(Q(t))) - P(Q(t))\} - \{P(Q(t)+\epsilon\alpha(Q(t))f(Q(t))) - P(Q(t))\}$$

In the same manner as above this yields, with $a_1(t) := f(Q(t))$,

$$\frac{P(Q(t)+xa_1(t))-t-x}{-\alpha(Q(t))} \to \frac{x^2}{2}$$

and in particular

$$P(Q(t))-t = o(\alpha(Q(t))) \quad (t\to\infty)\,. \tag{3.9}$$

Also locally uniformly

$$\frac{P(Q(t)+xa_1(t)-\frac{x^2}{2}\alpha(Q(t))a_1(t))-t-x}{-\alpha(Q(t))} \to 0$$

hence for $\epsilon>0$ and sufficiently large t

$$Q(t) + xa_1(t) - \frac{x^2}{2}\alpha(Q(t))a_1(t) \leqslant Q(t+x+\epsilon\alpha(Q(t)))$$

and as above

$$\limsup_{t\to\infty} \frac{Q(t+x)-Q(t)-xa_1(t)}{-\alpha(Q(t))a_1(t)} \leqslant \frac{x^2}{2}.$$

The other inequality is obtained similarly. □

COROLLARY 3.1: *If condition (b) of theorem 3.5 holds, then $\alpha(t+xf(t)) \sim \alpha(t)$ locally uniformly $(t\uparrow x^*)$.*

PROOF: Since $\alpha(t)\to 0$, $P(t+xf(t))-P(t)\to x$ locally uniformly $(t\uparrow x^*)$. We must prove (cf. the first part of the proof of theorem 3.2) that $a_i(P(t)+xf(t)) \sim a_i(P(t))$ locally uniformly for $i=1,2$. Now $a_i(t+x) \sim a_i(t)$ locally uniformly hence $a_i(P(t+xf(t))-P(t)+P(t)) \sim a_i(x+P(t)) \sim a_i(P(t))$ (cf. Omey and Willekens 1986). □

COROLLARY 3.2: *If condition (b) of theorem 3.5 holds, then $\{f(t+xf(t))-f(t)\}/\{-\alpha(t)f(t)\} \to x$ locally uniformly $(t\uparrow x^*)$.*

PROOF: Replace t in (3.7.2) by $t+yf(t)\to\infty$ for some real y, then $(t\uparrow x^*)$

$$\frac{x^2}{2} \leftarrow \frac{P(t+\{y+x\frac{f(t+yf(t))}{f(t)}\}\cdot f(t))-P(t)-\{y+x\frac{f(t+yf(t))}{f(t)}\}}{-\alpha(t)}\cdot\frac{\alpha(t)}{\alpha(t+yf(t))} +$$

$$-\frac{P(t+yf(t))-P(t)-y}{-\alpha(t)}\cdot\frac{\alpha(t)}{\alpha(t+yf(t))} +$$

$$-\frac{x\alpha(t)}{\alpha(t+yf(t))}\{\frac{f(t+yf(t))}{f(t)}-1\}/\alpha(t)\,.$$

Since every other term converges, also the last term must converge, thus giving the statement of the corollary. □

LEMMA 3.1: *Let $P:=-\log(1-F)$. The function P satisfies (3.7.2) of theorem 3.5. if and only if*

$$\frac{\frac{1-F(t+xf(t))}{1-F(t)}-e^{-x}}{-\alpha(t)} \to \frac{x^2}{2}e^{-x}\ (t\uparrow x^*) \text{ locally uniformly}. \tag{3.10}$$

Moreover (3.10) holds with f replaced by $g(t)\sim f(t)$ $(t\uparrow x^*)$ and $\frac{x^2}{2}e^{-x}$ replaced by $(\frac{x^2}{2}+(c-1)x)e^{-x}$ if and only if $\{g(t)-f(t)\}/\{-\alpha(t)f(t)\}\to c$ $(t\uparrow x^*)$.

PROOF: Suppose (3.7.2) holds. Since $\alpha(t)\to 0$, $P(t+xf(t))-P(t)\to x$ $(t\uparrow x^*)$ locally uniformly i.e. $\{1-F(t+xf(t))\}/\{1-F(t)\}\to e^{-x}$ $(t\uparrow x^*)$ locally uniformly. Hence

$$\frac{1-F(t+xf(t))}{1-F(t)}e^{x}-1\sim\log\{\frac{1-F(t+xf(t))}{1-F(t)}e^{x}\} = -P(t+xf(t))+P(t)+x\,.$$

The converse is proved similarly. Now suppose (3.10).

$$\frac{P(t+xg(t))-P(t)-x}{-\alpha(t)} = \frac{P(t+\{\frac{xg(t)}{f(t)}\}f(t))-P(t)-x\frac{g(t)}{f(t)}}{-\alpha(t)} + x\cdot\frac{\frac{g(t)}{f(t)}-1}{-\alpha(t)}\,.$$

Since the first term on the right converges, the convergence of the other terms imply each other. □

REMARK 3.2: f can be called the scale function and $-\alpha$ the reference function for $1-F$.

THEOREM 3.6: *If $P:=-\log(1-F)$ satisfies (3.7.2) then*

$$\frac{\frac{1-F(t+xf_1(t))}{1-F(t)}-e^{-x}}{-\alpha(t)} \to (\frac{x^2}{2}-x)e^{-x}\quad (t\uparrow x^*)$$

locally uniformly with $f_1(t):=\frac{\int_t^{x^*}1-F(s)ds}{1-F(t)}$.

PROOF: Write $U:=(\frac{1}{1-F})^{\leftarrow}$ as before and $Q:=U\circ\exp$. Then Q satisfies (3.7.1). As in Omey and Willekens (1986) one sees that

$$\frac{U(tx)-U(t)-DU(t)\log x}{a_2(t)} \to \frac{1}{2}\log^2 x-\log x$$

locally uniformly $(t\to\infty)$, with $DU(t):=t\int_t^{\infty}U(s)\frac{ds}{s^2}-U(t)=\int_1^{\infty}\{U(ty)-U(t)\}\frac{dy}{y^2}$. It follows that locally uniformly $(t\uparrow x^*)$

$$\frac{\frac{1-F(t+xDU(\frac{1}{1-F(t)}))}{1-F(t)}-e^{-x}}{-\alpha(t)} \to (\frac{x^2}{2}-x)e^{-x}\,.$$

Now $DU(\frac{1}{1-F(t)}) = \frac{1}{1-F(t)}\int_{\frac{1}{1-F(t)}}^{\infty}U(s)\frac{ds}{s^2} - U(\frac{1}{1-F(t)}) =$

$$= \frac{1}{1-F(t)}\int_t^{\infty}ydF(y) - U(\frac{1}{1-F(t)}) = \frac{1}{1-F(t)}\int_t^{\infty}ydF(y) - t+o(\alpha(t)f(t))\,,$$

using (3.8) in the last equality. The result now follows from lemma 3.1. □

COROLLARY 3.3: *Under the conditions of theorem 3.6*

$$\frac{f_1(t+xf_1(t))-f_1(t)}{-\alpha(t)f_1(t)} \to x \text{ locally uniformly } (t\uparrow x^*).$$

PROOF: Corollary 3.2. □

LEMMA 3.2: *If (3.10) holds for F, then the same relation holds with F replaced by* $1-F_1(x):=\max(0,\int_x^{x^*}\{1-F(u)\}du)$ *and f replaced by* $f_1(t):=\frac{\int_t^{x^*}1-F(s)ds}{1-F(t)}$.

PROOF:

$$\{\frac{1-F_1(t+xf_1(t))}{1-F_1(t)}-e^{-x}\}/-\alpha(t) = \frac{f_1(t+xf_1(t))}{f_1(t)}\{\frac{1-F(t+xf_1(t))}{1-F(t)}-e^{-x}\}/-\alpha(t) +$$
$$+ e^{-x}\{f_1(t+xf_1(t))-f_1(t)\}/\{-\alpha(t)f_1(t)\}.$$

Use corollary 3.3. □

Next we proceed to give sufficient conditions in terms of derivatives.

LEMMA 3.3: *Suppose F is three times differentiable and* $F'>0$. *Set* $f_0:=(1-F)/F'$. *If*

$$f_0''(t)f_0(t)/f_0'(t)\to 0 \text{ and } f_0'(t)\to 0 \quad (t\uparrow x^*), \tag{3.11.1}$$

then

$$f_0'(t+xf_0(t))/f_0'(t)\to 1 \ (t\uparrow x^*) \text{ locally uniformly}. \tag{3.11.2}$$

If (3.11.2), then

$$\frac{f_0(t+xf_0(t))-f_0(t)}{f_0'(t)f_0(t)} \to x \text{ locally uniformly } (t\uparrow x^*). \tag{3.11.3}$$

If (3.11.3), then

$$\frac{\frac{1-F(t+xf_0(t))}{1-F(t)}-e^{-x}}{f_0'(t)} \to \frac{x^2}{2}e^{-x} \text{ locally uniformly } (t\uparrow x^*). \tag{3.11.4}$$

PROOF: $f_0'(t)\to 0$ implies $f_0(t)/t\to 0$ if $x^*=\infty$ and $f_0(t)/(x^*-t)\to 0$ if $x^*<\infty$, hence

$$\frac{f_0(t+xf_0(t))}{f_0(t)}-1 = \int_0^x f'(t+uf_0(t))\,du \to 0 \text{ locally uniformly}.$$

Using this we find

$$\log f_0'(t+xf_0(t))-\log f'(t) =$$
$$= xf_0(t)\frac{f_0''(t+x\theta f_0(t))}{f_0'(t+x\theta f_0(t))} \sim x\frac{f_0(t+x\theta f_0(t))f_0''(t+x\theta f_0(t))}{f_0'(t+x\theta f_0(t))}$$

for some $\theta=\theta(t,x)\in[0,1]$. Hence (3.11.1) implies (3.11.2). Further (3.11.3) follows from (3.11.2) by integrating both sides of (3.11.2) over $x\in[0,y]$ and (3.11.4) follows from (3.11.3) by integrating both sides of the relation $f_0'(t)/f_0'(t+xf_0(t)) \to 1$ over $x \in [0,y]$ and using lemma 3.1. □

Necessary and sufficient conditions are contained in the next theorem.

THEOREM 3.7: *Set* $F_0 := F$ *and* $1-F_i(t) := \max\{0, \int_t^{x^*}(1-F_{i-1}(u))du\}$ *for* $i=1,2,\cdots$. *Also set* $f_i(t) := \{1-F_i(t)\}/\{1-F_{i-1}(t)\}$ *for* $i=1,2,\cdots$. *Equivalent are*

a. *For some functions* $f>0$, $\alpha(t)\to 0$ $(t\uparrow x^*)$ and α *of constant sign* $\neq 0$ (3.12.1)

$$\frac{\dfrac{1-F(t+xf(t))}{1-F(t)} - e^{-x}}{-\alpha(t)} \to \frac{x^2}{2}e^{-x} \text{ locally uniformly } (t\uparrow x^*).$$

b. f_2' *is of constant sign* $\neq 0$, $f_3'(t) \sim f_2'(t)$ and $f_3(t) \sim f_2(t)$ $(t\uparrow x^*)$. (3.12.2)

c. $$\frac{f_1(t+xf_1(t))-f_1(t)}{-\alpha(t)f_1(t)} \to x, \qquad (3.12.3)$$

locally uniformly $(t\uparrow x^*)$ *for some function* $\alpha(t)\to 0$ $(t\uparrow x^*)$ *of constant sign* $\neq 0$.

d. $1-F(t) = g_1(t)\exp\int_0^x \frac{ds}{g_2(s)}$ *with* g_1, g_2 *positive, both satisfying* (3.12.3), (3.12.4)

$$g_2(t)\to 0 \text{ and } g_1(t) = g_2(t)\{1-2\alpha(t) + o(g_2(t))\} \quad (t\uparrow x^*).$$

REMARK 3.3. The derivatives are to be taken in the Radon-Nicodym sense, if necessary.

PROOF:
(a)⇒(b): Note that $\alpha(t)\to 0$, $(t\uparrow x^*)$ implies $f_3(t)\sim f_2(t)$ (L. DE HAAN 1970). Now

$$\frac{\dfrac{1-F_1(t+xf_1(t))}{1-F_1(t)} - e^{-x}}{-\alpha(t)} \to \frac{x^2}{2}e^{-x} \ (t\uparrow x^*) \text{ locally uniformly}$$

by theorem 3.6 and lemma 3.2. But then according to theorem 3.6 also

$$\frac{\dfrac{1-F_1(t+xf_2(t))}{1-F_1(t)} - e^{-x}}{-\alpha(t)} \to \left(\frac{x^2}{2}-x\right)e^{-x}.$$

Hence $\dfrac{f_1(t)-f_2(t)}{-\alpha(t)f_2(t)} \to 1$ $(t\uparrow x^*)$ by lemma 3.1.

Repeating this reasoning with $1-F$ replaced by $1-F_1$, and $1-F_1$ replaced by $1-F_2$, we also get

$$\frac{f_2(t)-f_3(t)}{-\alpha(t)f_2(t)} \to 1 \ (t\uparrow x^*).$$

Now note that $f_i'(t) = -1+f_i(t)/f_{i-1}(t)$ for $i\geqslant 2$.

(b)⇒(c):

$$\frac{f_3''f_3}{f_3'} = 2\frac{f_3}{f_2} + \frac{f_3}{f_2}\cdot\frac{f_2}{f_1}\,\frac{f_1-f_2}{f_3-f_2} - \frac{f_3}{f_2}\cdot\frac{f_2}{f_1} \to 0 \ (t\uparrow x^*).$$

Hence $f_3'(t+xf_3(t) \sim f_3'(t)$ locally uniformly $(t\uparrow x^*)$ by lemma 3.3. Using $f_3' = -1+f_3/f_2$ gives

$$\frac{f_3(t+xf_3(t))-f_2(t+xf_3(t))}{f_3'(t)f_3(t)} \to 1 \text{ locally uniformly } (t\uparrow x^*)$$

and hence (using lemma 3.3 again)

$$\frac{f_2(t+xf_3(t))-f_2(t)}{f_3'(t)f_3(t)} = \frac{f_2(t+xf_3(t))-f_3(t+xf_3(t))}{f_3'(t)f_3(t)} +$$
$$+\frac{f_3(t+xf_3(t))-f_3(t)}{f_3'(t)f_3(t)} + \frac{f_3(t)-f_2(t)}{f_3'(t)f_3(t)} \to x \text{ locally uniformly } (t\uparrow x^*).$$

In exactly the same way one then obtains

$$\frac{f_1(t+xf_1(t))-f_1(t)}{f_2'(t)f_1(t)} \sim \frac{f_1(t+xf_3(t))-f_1(t)}{f_3'(t)f_3(t)} \to x \text{ locally uniformly } (t\uparrow x^*).$$

(c)⇒(d):

Take $cg_1 = g_2 = f_1$.

(d)⇒(a): Define $P(t):=\int_0^t \frac{ds}{g_2(s)}$. Straightforward calculation gives

$$\frac{P(t+xg_2(t))-P(t)-x}{-\alpha(t)} \to \frac{1}{2}x^2 \text{ locally uniformly } (t\uparrow x^*)$$

i.e. with $1-F_*(t):= \exp \int_0^t \frac{ds}{g_2(s)}$

$$\frac{\dfrac{1-F_*(t+xg_2(t))}{1-F_*(t)}-e^{-x}}{-\alpha(t)} \to \frac{1}{2}x^2e^{-x} \text{ locally uniformly } (t\uparrow x^*).$$

Next use a decomposition like the one in the proof of lemma 3.2 to obtain
$\frac{1-F(t+xg_2(t))}{1-F(t)}-e^{-x}\sim-\alpha(t)\frac{x^2}{2}e^{-x}$. Finally apply lemma 3.1. □

COROLLARY 3.4: *If (3.12.1) holds, then (3.12.1) also holds with* $-\alpha$ *replaced by* $\frac{f_2}{f_1}-1=f_2'$ (see the first part of the proof of theorem 3.7) *and* f *replaced by* $f_1-\alpha$ (see theorem 3.6 and lemma 3.1).

Finally we turn back to the question how to translate the condition $tU'(t)\in\Pi$ into a condition for the distribution function F and its derivative F'.

THEOREM 3.8: *Set* $F_0 := F$, $1-F_i(t) := \max\{0, \int_t^{x^*}(1-F_{i-1}(u))du\}$, $f_0 = \{1-F\}/F'$ *and* $f_i := \{1-F_i(t)\}/\{1-F_{i-1}(t)\}$ *for* $i = 1,2,\ldots$ *Equivalent are*

a. $tU'(t)\in\Pi$. (3.13.1)

b. $$\frac{\dfrac{1-F(t+xf_0(t))}{1-F(t)}-e^{-x}}{-\beta(t)} \to \frac{x^2}{2}e^{-x} \textit{ locally uniformly } (t\uparrow x^*)$$ (3.13.2)

for some positive function $\beta(t) \to 0$ $(t\uparrow x^*)$.

c. $f_2'(t)\sim f_1'(t)\to 0$ $(t\uparrow x^*)$. (3.13.3)

d. $$\frac{f_0(t+xf_0(t))-f_0(t)}{-\beta(t)f_0(t)} \to x \textit{ locally uniformly } (t\uparrow x^*)$$ (3.13.4) *for some function* $\beta(t)\to 0$ $(t\uparrow x^*)$.

The proof of this theorem will not be given since it follows exactly the same lines as the proof in the general case, but uses theorem 3.4 relation (3.6.3) to get f_0 in relation (3.13.2). The proof here is actually easier since inversion is very simple.

EXAMPLES

1. The distribution functions $F(x):=1-\exp(-x^\alpha)$ satisfy the criterion of lemma 3.3 for all $\alpha>0$, $\alpha\neq 1$.
2. *Normal distribution* Using the previous example for $\alpha=2$ and lemma 3.2 we find that the normal distribution satisfies (3.12.1) with $f(t)=t^{-1}$ and $\alpha(t)=t^{-2}$. By lemma 3.1 the same relation also holds with $f(t)=f_0(t)=e^{t^2/2}\int_t^\infty e^{-s^2/2}ds$ since $\int_t^\infty e^{-s^2/2}ds= e^{-t^2/2}\{t^{-1}-t^{-3}+o(t^{-3})\}$, $t\to\infty$. (see also M. ABRAMOWITZ and I.A. STEGUN (1965) 26.2.12 pag. 932). Hence (theorem 3.8) U satisfies (3.6.3) where $tU'(t)\sim\{U(t)\}^{-1}$ and $a(t)\sim\pm\{U(t)\}^{-3}$ $(t\to\infty)$. It follows that here the function g from theorem 2.3 satisfies $g(t)\sim 2t\log^2 t$ so that the theorem holds for sequences $m=m(n)\to\infty$ satisfying $m(n)=o(log^2 n)$.
3. *Gamma distribution.* The conditions of theorem 3.8 are easily checked using the expansion $(r\neq 1)$ $\int_t^\infty s^{r-1}e^{-s}ds=e^{-t}\{t^{r-1}+(r-1)t^{r-2}+o(t^{r-3})\}$.
4. *Cauchy distribution.* The condition of theorem 2.5 are satisfied with $\rho=2$ and $c=\pi^{-1}$. Then $g(t)\sim ct^5$ so that the theorem holds for sequences $m=m(n)\to\infty$ satisfying $m(n)=o(n^{4/5})$.
5. For the *exponential* and *uniform* distributions we have $t^{1-\gamma}U'(t)\equiv 1$ so that the lefthand side of (2.1) is identically zero. It follows that the conclusion of theorem 2.3 holds for *all* sequences $m=m(n)\to\infty$, $m(n)/n\to 0$ $(n\to\infty)$. The same is true for the generalized Pareto distribution $F_\gamma(x) = 1-(1+\gamma x)^{-1/\gamma}, \gamma\in\mathbb{R}$ and $1+\gamma x\geqslant 0$.
6. *Extreme value distribution*: $G_\gamma(x) = \exp-(1+\gamma x)^{-1/\gamma}$, $\gamma\in\mathbb{R}$. For $\gamma>0$, condition (a) of theorem 2.5 is satisfied with $c=\gamma^{-1-1/\gamma}$ and $\rho=\min(1,1/\gamma)$ and for $\gamma<0$, condition (b) is satisfied with $c=(-\gamma)^{-1-1/\gamma}$ and $\rho=-1/\gamma$. The theorem holds for sequences $m=m(n)\to\infty$ satisfying respectively $m(n)=o(n^{1-1/\gamma\{1+\min(1,\gamma)\}})$ and $m(n)=o(n^{2/3})$. Note that for $\gamma=0$ the tail of $G_0(x)=\exp(-\exp-x)\approx 1-\exp(-x)$ is of the exponential type and so the conclusion of theorem 2.3 holds for all sequences $m=m(n)\to\infty$, $m(n)/n\to 0$, $n\to\infty$. The same is true for the *Logistic distribution.*

REFERENCES

1. M. ABRAMOWITZ and I.A. STEGUN (1965). *Handbook of mathematical functions.* Dover.
2. J.P. COHEN (1986). *Large sample theory for fitting an approximating Gumbel model to maxima.* Sankhyā, series A, 48-3, 372-392.
3. J.P. COHEN (1986). *Fitting extreme value distributions to maxima.* Preliminary report, University of Kentucky.
4. J.L. GELUK and L. DE HAAN (1986). *Regular variation, extensions and Tauberian theorems.* Book manuscript submitted for publication.
5. B.V. GNEDENKO and A.N. KOLMOGOROV (1954). *Limit distributions for sums of independent random variables.* Addison-Wesley, Cambridge (Mass.)
5. L. DE HAAN (1970). *On regular variation and its application to the weak convergence of sample extremes.* Mathematical Center Tracts, Amsterdam.
6. L. DE HAAN (1977). *On functions derived from regularly varying functions.* J. Austral. Math. Soc., series A, XIII -4, 431-438.
7. L. DE HAAN (1984). *Slow variation and characterization of domains of attraction.* Statistical Extremes and Applications, ed. J. Tiago de Oliveira. Reidel, Dordrecht Holland.

8. J. KIEFER (1972). *Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$.* Proc. 6th Berkeley Sympos. Math. Statist. Probab., Univ. of California Press, 1, 227-244.
9. E. OMEY and E. WILLEKENS (1986). *Π-variation with remainder.* Technical Report, University of Leuven.
10. J. PICKANDS III (1975). *Statistical inference using extreme order statistics.* Ann. Statist. 3, 119-131.
11. N.V. SMIRNOV (1967). *Some remarks on limit laws for order statistics.* Theory Probab. Appl. 12, 336-337.
12. R.L. SMITH (1985). *Estimating tails of probability distributions.* Technical Report. Imperial College, London.
13. J.A. WELLNER (1978). *Limit theorems for the ratio of the emperical distribution function to the true distribution function.* Z. Wahrsch. verw. Gebiete. 45, 73-88.