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based on Minkowski addition

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Similarity and Symmetry Measures for Convex Sets Based on Minkowski Addition

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Abstract

This paper discusses similarity and symmetry measures for convex shapes. Their definition is based on Minkowski addition and the Brunn-Minkowski inequality. All measures considered are invariant under translations; furthermore, they may also be invariant under rotations, multiplications, reflections, or the class of all affine transformations. The examples discussed in this paper allow efficient algorithms if one restricts oneselves to convex polygons. Although it deals exclusively with the 2-dimensional case, many of the theoretical results carry over almost directly to higher-dimensional spaces. Some results obtained in this paper are illustrated by experimental data.

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Keywords & Phrases: similarity measure, symmetry measure, convex set, convex polygon, Minkowski addition, perimetric measure/representation, affine transformation, rotation, reflection, multiplication, similitude, volume, mixed volume, Brunn-Minkowski inequality, critical angle, image normalization, canonical form, ellipse of inertia, principal axis, covariance matrix, symmetry of a shape, (strongly) cyclic transformation, skew symmetry, polygon decomposition.

1. Introduction

The problem of shape similarity has been extensively investigated in both machine vision and biological vision. Although for human perception, different features such as shape, color, reflectance, functional information play an important role while comparing objects, in machine vision usually only geometric properties of shapes are used to introduce shape similarity. In the literature, one finds two concepts for expressing the similarity of shapes: distance functions measuring assimilarity and similarity measures expressing how similar two shapes are. In this paper we shall work with similarity measures.

In practice, similarity approaches have to be invariant under certain classes of transformations, e.g. similitudes (i.e., translations, rotations, change of scale). Affine transformations are

also of great practical value as these can approximate shape distortions arising when an object is observed by a camera under arbitrary orientations with respect to the image plane [20]. A well-known method to develop a similarity approach which is invariant under a given class of transformations is to perform a shape normalization first [17, 27, 29]. In Section 6 of this paper we discuss one particular method based on the ellipse of inertia.

In the literature, one finds several different methods for comparing shapes. Among the best known ones are matching techniques [29]. We mention here also contour matching [2, 32], structural matching [4] (which is based on specific structural features), and point set matching [37]. In several approaches, one uses the Hausdorff distance for point sets to describe similarity [19]. An interesting construction of a similitude invariant distance function for polygonal shapes is given in [2]; here one computes the L_2-distance of the so-called turning functions representing the boundary of the polygons. Several authors use the concept of a scale space to develop a multiresolution similarity approach [6, 26]. Finally, Fourier descriptors derived from contour representations have been used by various authors to describe shape similarity and symmetry [21, 22, 28, 39]. See [36] for a comprehensive discussion.

Similarity measures can be used to compute "how symmetric a given shape is", e.g. with respect to reflection in a given line. For many objects, presence or absence of symmetry is a major feature, and therefore the problem of object symmetry identification is of great interest in image analysis and recognition, computer vision and computational geometry. Unfortunately, in many practical cases, exact symmetry does not occur or, if it does, is disturbed by noise. In such circumstances it is useful to define measures of symmetry which give quantitative information about the amount of symmetry of a shape. There exists a vast literature dealing with all kinds of symmetry of shapes and (grey-scale) images: central symmetry [9, 13, 25], reflection symmetry [3, 8, 23], rotation symmetry [7, 35], skew symmetry [5, 10]; see also [24, 38].

In this paper we will discuss a class of similarity and symmetry measures which is based on Minkowski addition, the Brunn-Minkowski inequality, and the theory of mixed volumes. Most of our results apply to arbitrary compact convex sets (in $\mathbb{R}^2$), but when we want to develop efficient algorithms, we shall restrict ourselves to convex polygons.

We conclude with an overview of this paper. We start with some notations and recall some basic concepts in Section 2. In Section 3 we give short treatment of the theory of mixed volumes, the Brunn-Minkowski inequality, and some derived inequalities. A formal definition of similarity measures can be found in Section 4, where we also present some examples based on Minkowski addition. In Section 5 we investigate similarity measures for convex polygons which are invariant under rotations and multiplications, and we present an algorithm to compute such measures efficiently. An affine invariant similarity measure is presented in Section 6. To define it, we introduce an image normalization (canonical form) based on the ellipse of inertia known from classical mechanics. Symmetry measures are introduced in Section 7; there we also give several examples, some of them based on similarity measures. Any convex polygon can be decomposed as the Minkowski sum of two parts: a symmetric part and an asymmetric part. In Sections 8 and 9 we present algorithms for such decompositions, in Section 8 for rotation symmetry, and in Section 9 for reflection symmetry. Finally, in Section 10 we illustrate our theoretical findings with some experimental results.

2. Preliminaries

In this section we present some basic notation and other prerequisites needed in the sequel of the paper. By $\mathcal{K}(\mathbb{R}^2)$, or briefly $\mathcal{K}$, we denote the family of all nonempty compact subsets of $\mathbb{R}^2$. Provided with the Hausdorff distance [30] this is a metric space. The compact, convex

subsets of $\mathbb{R}^2$ are denoted by $\mathcal{C} = \mathcal{C}(\mathbb{R}^2)$, and the convex polygons by $\mathcal{P}(\mathbb{R}^2)$ or just $\mathcal{P}$. In this paper, we are not interested in the location of a shape $A \subseteq \mathbb{R}^2$; in other words, two shapes A and B are said to be *equivalent* if they differ only by translation. We denote this as $A \equiv B$. The *Minkowski sum* of two sets $A, B \subseteq \mathbb{R}^2$ is

$$A \oplus B = \{a + b \mid a \in A, \ b \in B\}.$$

It is well-known [30] that every element A of $\mathcal{C}$ is uniquely determined by its *support function* given by:

$$h(A, u) = \sup\{\langle a, u \rangle \mid a \in A\}, \qquad u \in S^1.$$

Here $\langle a, u \rangle$ is the inner product of vectors a and u, and S^1 denotes the unit circle. It is also known that [30]:

$$h(A \oplus B, u) = h(A, u) + h(B, u), \qquad u \in S^1, \tag{2.1}$$

for $A, B \in \mathcal{C}$. The *support set* $F(A, u)$ of A at $u \in S^1$ consists of all points $a \in A$ for which $\langle a, u \rangle = h(A, u)$.

A polygon $P \subseteq \mathbb{R}^2$ can be represented uniquely by specifying the position of one of its vertices and the lengths and directions of all of its edges. Below, p_i will denote the length of edge i and u_i is the vector orthogonal to this edge; see Figure 1. By $\angle u_i$ we denote the angle between the positive x-axis and u_i. Since we are not interested in the location of P, it is sufficient to give the sequence $(u_1, p_1), (u_2, p_2), \ldots, (u_n, p_n)$, where $n = n_P$ is the number of vertices of P. We will call this sequence the *perimetric representation* of P. In Figure 1 we give an illustration.
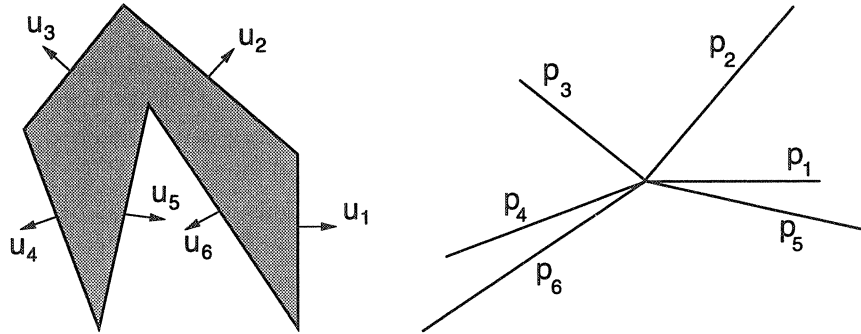

Fig. 1.   Perimetric representation of a polygon.

We denote this sequence by $M(P)$. If the polygon is convex, then the order of (u_i, p_i) does not have to be specified in advance, since in this case the normal vectors have to be ordered counter-clockwise. In this case we can think of $M(P)$ as a set. But we can also use the so-called *perimetric measure* $M(P, \cdot)$ as an alternative representation [25]:

$$M(P, u) = \begin{cases} p_i, & \text{if } u = u_i, \\ 0, & \text{otherwise.} \end{cases}$$

We point out that the perimetric measure is a special case of the concept of *area measure* [30]. It is evident that for every convex polygon P, we have the identity

$$\sum M(P, u)u = 0,$$

where the sum is taken over all u for which $M(P, u) \neq 0$. In fact, this relation expresses that the contour of P is closed.

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It is well-known that the Minkowski addition of two convex polygons can be computed by merging both perimetric representations; see e.g. [11, 14]. Mathematically, this amounts to the following relation:

$$M(P \oplus Q, u) = M(P, u) + M(Q, u), \quad \text{for } P, Q \in \mathcal{P} \text{ and } u \in S^1. \tag{2.2}$$

In the second part of this section we consider *affine transformations* on $\mathbb{R}^2$. The reader may refer to [34] for a comprehensive discussion. The group of all affine transformations on $\mathbb{R}^2$ is denoted by G'. If $g \in G'$ and $A \in \mathcal{K}$, then $g(A) = \{g(a) \mid a \in A\}$. We write $g \equiv g'$ if $g(A) \equiv g'(A)$ for every $A \in \mathcal{K}$. This is equivalent to saying that $g - g'$ is a translation. We denote by G the subgroup of G' containing all linear transformations, i.e., transformations g with $g(0) = 0$.

2.1. Lemma. *For any two sets $A, B \subseteq \mathbb{R}^d$ and for every $g \in G$, we have*

$$g(A \oplus B) = g(A) \oplus g(B). \tag{2.3}$$

We introduce the following notations for subsets of G:

    I: isometries (distance preserving transformations);

    R: rotations about the origin;

    M: multiplications with respect to the origin by a positive factor;

    L: (line) reflections (lines passing through the origin);

    S: similitudes (rotations, reflections, multiplications).

Observe that I, R, M and S are subgroups of G. For every transformation $g \in G$ we can compute its *determinant* 'det g' which is, in fact, the determinant of the matrix corresponding with g. Note that this value is independent of the choice of the coordinate system. If g is an isometry then $|\det g| = 1$; the converse is not true, however. If H is a subgroup of G, then H_+ denotes the subgroup of H containing all transformations with positive determinant. For example, $I_+ = R$ and S_+ comprises all multiplications and rotations. If H is a subgroup of G, then the set $\{mh \mid h \in H, \ m \in M\}$ is also a subgroup, which will be denoted by MH.

Denote by r_θ the rotation around the origin over an angle θ, and by ℓ_α the reflection with respect to the line through the origin which makes an angle α with the positive x-axis. The following relations hold:

$$\ell_\alpha r_\theta = \ell_{\alpha - \theta/2}$$
$$r_\theta \ell_\alpha = \ell_{\alpha + \theta/2}$$
$$\ell_\beta \ell_\alpha = r_{2\beta - 2\alpha}$$

In what follows, the topology on $\mathcal{K}$ is the one induced by the Hausdorff metric, also called *myopic topology* [16]. At several instances in this paper we shall need the following concept.

2.2. Definition. Let $H \subseteq G$ and $\mathcal{J} \subseteq \mathcal{K}$. We say that H is $\mathcal{J}$-*compact* if, for every $A \in \mathcal{J}$ and every sequence $\{h_n\}$ in H, the sequence $\{h_n(A)\}$ has a limit point of the form $h(A)$, where $h \in H$.

It is easy to verify that R is $\mathcal{K}$-compact. However, the subcollection $\{r^m \mid m \in \mathbb{Z}\}$, where $r = r_\theta \in R$ is a rotation with θ/π irrational, is not $\mathcal{K}$-compact. The following result is easy to prove.

2.3. Lemma. *Assume that H is $\mathcal{J}$-compact and let $f : \mathcal{J} \to \mathbb{R}$ be a continuous function. If $A \in \mathcal{J}$ and $f_0 := \sup_{h \in H} f(h(A))$ is finite, then there exists an element $h_0 \in H$ such that $f(h_0(A)) = f_0$.*

3. Mixed volumes and the Brunn-Minkowski inequality

In this section we present a brief account of the theory of volumes and mixed volumes of compact sets (also called 'mixed areas' in the 2-dimensional case). For a comprehensive treatment the reader may consult the book of Schneider [30]. The *volume* (or *area*) of a compact set A will be denoted by $V(A)$. It is well-known that for every affine transformation g the following relation holds:

$$V(g(A)) = |\det g| \cdot V(A). \tag{3.1}$$

The *mixed volume* $V(A, B)$ of two compact, convex sets $A, B \subseteq \mathbb{R}^2$ is implicitly defined by the following formula for the volume of $A \oplus B$:

$$V(A \oplus B) = V(A) + 2V(A, B) + V(B). \tag{3.2}$$

See Figure 2 for an illustration. It has the following properties (A and B being arbitrary compact, convex sets):

$$V(A, B) = V(B, A) \geq 0; \text{ if } V(A) > 0 \text{ and } V(B) > 0 \text{ then } V(A, B) > 0; \tag{3.3}$$

$$V(A, A) = V(A); \tag{3.4}$$

$$V(\lambda A, B) = \lambda V(A, B) \text{ for every } \lambda > 0; \tag{3.5}$$

$$V(g(A), g(B)) = |\det g| \cdot V(A, B), \text{ for every affine transformation } g; \tag{3.6}$$

$$V(A_1 \oplus A_2, B) = V(A_1, B) + V(A_2, B); \tag{3.7}$$

$$V(A, B) \text{ is continuous in } A \text{ and } B \text{ with respect to the Hausdorff metric.} \tag{3.8}$$

Note for example that (3.6) is a straightforward consequence of (2.3) and (3.1)-(3.2).

In this paper the following well-known inequality plays a central role. See Hadwiger [15] or Schneider [30] for a comprehensive discussion.

3.1. Brunn-Minkowski inequality. *For two arbitrary compact sets $A, B \subseteq \mathbb{R}^2$ the following inequality holds:*

$$V(A \oplus B)^{\frac{1}{2}} \geq V(A)^{\frac{1}{2}} + V(B)^{\frac{1}{2}}, \tag{3.9}$$

with equality if and only if A and B are convex and homothetic modulo translation, i.e., $B \equiv \lambda A$ for some $\lambda > 0$.

The Brunn-Minkowski inequality (3.9) in combination with (3.2) yields the following inequality for mixed volumes:

$$V(A, B) \geq V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}, \tag{3.10}$$

and as before equality holds iff A and B are convex and $B \equiv \lambda A$ for some $\lambda > 0$. This latter inequality is called *Minkowski's inequality*.

Using the fact that for two arbitrary real numbers x, y one has $(x + y)^2 \geq 4xy$, with equality iff $x = y$, one derives from (3.9) that:

$$V(A \oplus B) \geq 4V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}, \tag{3.11}$$

with equality iff $A \equiv B$ and both sets are convex.

The mixed volume of two convex polygons P, Q can be easily computed using support functions and perimetric representations. Assume that the perimetric representation of Q is given by the sequence (v_j, q_j), $j = 1, 2, \ldots, n_Q$. Furthermore, if $h(P, \cdot)$ is the support function of P, then

$$V(P, Q) = \frac{1}{2} \sum_{j=1}^{n_Q} h(P, v_j) q_j. \tag{3.12}$$

6
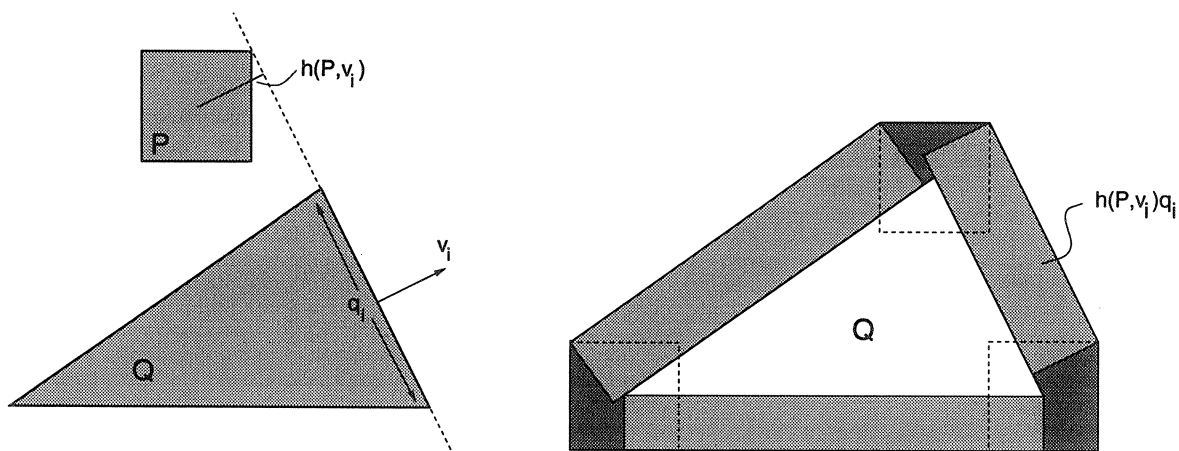
Fig. 2. The right figure is $P \oplus Q$. The sum of the volumes of the light grey regions equals $2V(P,Q)$, the sum of the volumes of the dark grey regions equals $V(P)$.

See Figure 2 for an illustration of this formula.

Note that with this formula the additivity of $V(P,Q)$ as stated in (3.7) follows immediately from the additivity of the support function; see (2.1). Furthermore, (3.12) in combination with (3.3) shows that $V(P,Q)$ is increasing in both arguments. In fact, this observation holds for arbitrary compact, convex sets, i.e.,

$$V(A,B) \leq V(A',B') \quad \text{if } A \subseteq A', \ B \subseteq B'. \tag{3.13}$$

We conclude this section with a formula for the computation of the volume of a 2-dimensional polygon (not necessarily convex) using its perimetric representation. Several formulas for calculating volumes of polyhedra are known in the literature [1]. Let the vertices (ordered counter-clockwise) of a polygon P be given by $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Then

$$V(P) = \frac{1}{2} \sum_{i=1}^{n} \det \begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix} = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i). \tag{3.14}$$

Here $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. One can rewrite (3.14) as

$$V(P) = \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i+1})(y_i + y_{i+1}). \tag{3.15}$$

Refer to [1] for some further information about these formulas.

If P is a polygon with perimetric representation $(u_1, p_1), (u_2, p_2), \ldots, (u_n, p_n)$, then the vertices (x_i, y_i) are given by (putting $x_1 = y_1 = 0$)

$$x_i = -\sum_{j=1}^{i-1} p_j \sin \alpha_j, \quad y_i = \sum_{j=1}^{i-1} p_j \cos \alpha_j, \tag{3.16}$$

where $\alpha_j = \angle u_j$. Here P needs not be convex. Substituting these values into (3.15) we get

$$V(P) = \sum_{j=1}^{n} p_i \sin \alpha_i \sum_{j=1}^{i} p_j \cos \alpha_j - \frac{1}{2} \sum_{i=1}^{n} p_i^2 \sin \alpha_i \cos \alpha_i. \tag{3.17}$$

This is the formula which we will use in the sequel of this paper.

4. Similarity measures

One of the goals of this paper is to find a tool which enables us to compare different shapes, but in such a way that this comparison is invariant under a given group H of transformations and can be computed efficiently. For example, if we take for H all rotations, then our comparison should return the same outcome for A and B as for A and $r(B)$, where r is some rotation.

Towards this goal one could try to find a distance function (or metric) $d(A, B)$ which equals zero if and only if $B \equiv h(A)$ for some $h \in H$. Many authors, however, rather work with so-called *similarity measures* than with distance functions. In this paper we will follow this convention.

4.1. Definition. Let H be a subgroup of G and $\mathcal{J} \subseteq \mathcal{K}$. A function $\sigma : \mathcal{J} \times \mathcal{J} \to [0, 1]$ is called an *H-invariant similarity measure* on $\mathcal{J}$ if
  (1) $\sigma(A, B) = \sigma(B, A)$;
  (2) $\sigma(A, B) = \sigma(A', B')$ if $A \equiv A'$ and $B \equiv B'$;
  (3) $\sigma(A, B) = \sigma(h(A), B)$, $h \in H$;
  (4) $\sigma(A, B) = 1 \iff B \equiv h(A)$ for some $h \in H$;
  (5) σ is continuous in both arguments with respect to the Hausdorff metric.
When H contains only the identity mapping, then σ will be called a *similarity measure*.

Although not stated explicitly in the definition above, it is also required that $\mathcal{J}$ is invariant under H, that is, $h(A) \in \mathcal{J}$ if $A \in \mathcal{J}$ and $h \in H$.

4.2. Examples. (a) Let $\mathcal{J} \subseteq \mathcal{K}$ consist of all regular sets in $\mathcal{K}$, i.e., all sets which are the closure of their interior. Given $A \in \mathcal{J}$, we denote by A_0 the translate which has its centroid positioned at the origin. Then

$$\sigma(A, B) = \exp\big(-\text{area}(A_0 \, \triangle \, B_0)\big),$$

is a similarity measure on $\mathcal{J}$; here $A_0 \, \triangle \, B_0$ is the symmetric set difference of A_0 and B_0.
  (b) Denote by d_H the Hausdorff metric on $\mathcal{K}$. The function

$$\sigma(A, B) = \exp\big(-d_H(A_0, B_0)\big),$$

defines a similarity measure on $\mathcal{K}$; here A_0 has the same meaning as in (a).

4.3. Remark. If σ given by Definition 4.1 satisfies the inequality

$$\sigma(A, C) \geq \sigma(A, B)\sigma(B, C),$$

then the function $d(A, B) = -\log(\sigma(A, B))$ constitutes a metric on $\mathcal{J}$ modulo translations and transformations $h \in H$. That is, d satisfies the triangle inequality.

If σ is a similarity measure on $\mathcal{J}$ and H is a $\mathcal{J}$-compact subgroup of G, then

$$\sigma'(A, B) = \sup_{h \in H} \sigma(h(A), B)$$

defines an H-invariant similarity measure on $\mathcal{J}$. Unfortunately, σ' is difficult to compute in many practical situations. Below, however, we present two cases (with $\mathcal{J} = \mathcal{C}$) for which this can be done efficiently if one restricts attention to convex polygons.

Let H be a given subgroup of G, and define

$$\sigma_1(A, B) = \sup_{h \in H} \frac{4|\det h|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A \oplus h(B))}, \tag{4.1}$$

and

$$\sigma_2(A, B) = \sup_{h \in H} \frac{|\det h|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A, h(B))}. \tag{4.2}$$

4.4. Proposition. *If H is a C-compact subgroup of G, then*

(a) σ_1 is an H-invariant similarity measure on C;

(b) σ_2 is an MH-invariant similarity measure on C.

PROOF. We prove (a). The proof of (b) goes along the same lines. Conditions (1), (2) and (5) in Definition 4.1 are straightforward. First we prove (3). Using (3.1) and (2.3) we get

$$
\begin{aligned}
\sigma_1(h(A), B) &= \sup_{h' \in H} \frac{4|\det h'|^{\frac{1}{2}} V(h(A))^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(h(A) \oplus h'(B))} \\
&= \sup_{h' \in H} \frac{4|\det h'|^{\frac{1}{2}} |\det h|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V\left(h(A \oplus h^{-1} h'(B))\right)} \\
&= \sup_{h' \in H} \frac{4|\det h'|^{\frac{1}{2}} |\det h|^{-\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A \oplus h^{-1} h'(B))} \\
&= \sup_{h' \in H} \frac{4|\det h^{-1} h'|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A \oplus h^{-1} h'(B))} \qquad (\text{ putting } h'' = h^{-1} h') \\
&= \sup_{h'' \in H} \frac{4|\det h''|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A \oplus h''(B))} \\
&= \sigma_1(A, B). \qquad —
\end{aligned}
$$

Finally we prove (4). It is easy to see that $\sigma_1(A, B) = 1$ if $B \equiv h(A)$. To prove the converse, assume that $\sigma_1(A, B) = 1$. Since H is C-compact, we conclude from Lemma 2.3 that there exists an $h \in H$ such that

$$
\frac{4|\det h|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A \oplus h(B))} = 1,
$$

that is,

$$
V(A \oplus h(B)) = 4V(A)^{\frac{1}{2}} V(h(B))^{\frac{1}{2}}.
$$

In (3.11) we have seen that this implies that $A \equiv h(B)$. This concludes the proof. ∎

The following example shows that compactness cannot be missed. Suppose that H consists of all multiplications with rational factors. Now, if α is irrational, then, for an arbitrary $A \in C$,

$$
\begin{aligned}
\sigma_1(A, \alpha A) &= \sup_{\lambda \in \mathbb{Q}, \, \lambda > 0} \frac{4\alpha \lambda V(A)^{\frac{1}{2}} V(A)^{\frac{1}{2}}}{V(A \oplus \alpha \lambda A)} \\
&= \sup_{\lambda \in \mathbb{Q}, \, \lambda > 0} \frac{4\alpha \lambda}{(1 + \alpha \lambda)^2} = 1,
\end{aligned}
$$

but $\alpha A \neq \lambda A$ for $\lambda \in \mathbb{Q}$, the set of all rational numbers.

In the next section we investigate invariance under rotations and multiplications. Here we consider similarity measures which are invariant under the multiplication group.

4.5. Example: invariance under multiplications. First we compute σ_1 if $H = M$, the multiplication group. Since the determinant of the multiplication by λ equals λ^2 we get

$$
\sigma_1(A, B) = \sup_{\lambda > 0} \frac{4\lambda V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A \oplus \lambda B)}
$$

$$
= \sup_{\lambda > 0} \frac{4\lambda V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A) + \lambda^2 V(B) + 2\lambda V(A, B)}
$$

$$
= 4\left[\inf_{\lambda > 0} \left\{ \frac{2V(A, B)}{V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}} + \lambda \cdot \left(\frac{V(B)}{V(A)} \right)^{\frac{1}{2}} + \frac{1}{\lambda} \cdot \left(\frac{V(A)}{V(B)} \right)^{\frac{1}{2}} \right\} \right]^{-1}.
$$

It is easy to see that the infimum is achieved at $\lambda = (V(A)/V(B))^{\frac{1}{2}}$, whence we get

$$\sigma_1(A, B) = 2\left[1 + \frac{V(A, B)}{V(A)^{\frac{1}{2}}V(B)^{\frac{1}{2}}}\right]^{-1}.$$

From Proposition 4.4(b) we see that σ_2 in (4.2) with $H = \{\mathrm{id}\}$ is invariant under multiplications as well. Obviously,

$$\sigma_2(A, B) = \frac{V(A)^{\frac{1}{2}}V(B)^{\frac{1}{2}}}{V(A, B)} = \frac{\sigma_1(A, B)}{2 - \sigma_1(A, B)}.$$

We conclude this section with the following simple but useful result. Recall that ℓ_0 is the line reflection with respect to the x-axis.

4.6. Proposition. *Let σ be a similarity measure on $\mathcal{J}$, and define*

$$\sigma'(A, B) = \max\{\sigma(A, B), \sigma(\ell_0(A), B)\}.$$

(a) *If σ is R-invariant, then σ' is an I-invariant similarity measure.*

(b) *If σ is G_+-invariant, then σ' is a G-invariant similarity measure.*

PROOF. The proofs of (a) and (b) are almost identical. Here we will only prove (b). The properties (1),(2), and (5) of Definition 4.1 are straightforward. We prove (3) and (4).

(3): Let $g \in G$. There are two possibilities: $g \in G_+$ or $g \in G \setminus G_+$. We consider the second case. We can write $g = h\ell_0$ with $h = g\ell_0$, and also $g = \ell_0 h'$ with $h' = \ell_0 g$; then $h, h' \in G_+$. Now

$$\begin{aligned}
\sigma'(g(A), B) &= \max\{\sigma(g(A), B), \sigma(\ell_0(g(A)), B)\} \\
&= \max\{\sigma(h(\ell_0(A)), B), \sigma(h'(A), B)\} \\
&= \max\{\sigma(\ell_0(A), B), \sigma(A, B)\} \\
&= \sigma'(A, B),
\end{aligned}$$

which had to be demonstrated.

(4): Assume $\sigma'(A, B) = 1$, then either $\sigma(A, B) = 1$ or $\sigma(\ell_0(A), B) = 1$. In the first case we have $B \equiv g(A)$, for some $g \in G_+$, and in the second case $B \equiv g\ell_0(A)$ for some $g \in G_+$. Therefore, $B \equiv g(A)$ for some $g \in G$. ∎

5. Rotations and multiplications

In this section we consider similarity measures on $\mathcal{P}$ which are S_+-invariant, i.e., invariant under rotations and multiplications. Towards this goal, we will use the similarity measures defined in (4.1)–(4.2) with $H = S_+$ and $H = R$, respectively. In these expressions, the terms $V(P \oplus r_\theta(Q))$ and $V(P, r_\theta(Q))$ play an important role. Let the perimetric representations of the convex polygons P and Q be given by (u_i, p_i), $i = 1, 2, \ldots, n_P$, and (v_j, q_j), $j = 1, 2, \ldots, n_Q$, respectively. To compute $V(P, r_\theta(Q))$, we use formula (3.12):

$$V(P, Q) = \frac{1}{2} \sum_{j=1}^{n_Q} h(P, v_j) q_j.$$

The support set $F(P, r_\theta(v_j))$ consists of a vertex of P unless θ satisfies $r_\theta(v_j) \in \{u_1, u_2, \ldots, u_{n_P}\}$. Angles θ for which this holds (i.e., $r_\theta(v_j) = u_i$) are called *critical angles*. The set of all critical angles for P and Q is given by

$$\{(\angle u_i - \angle v_j) \bmod 2\pi \mid i = 1, 2, \ldots, n_P \text{ and } j = 1, 2, \ldots, n_Q\},$$
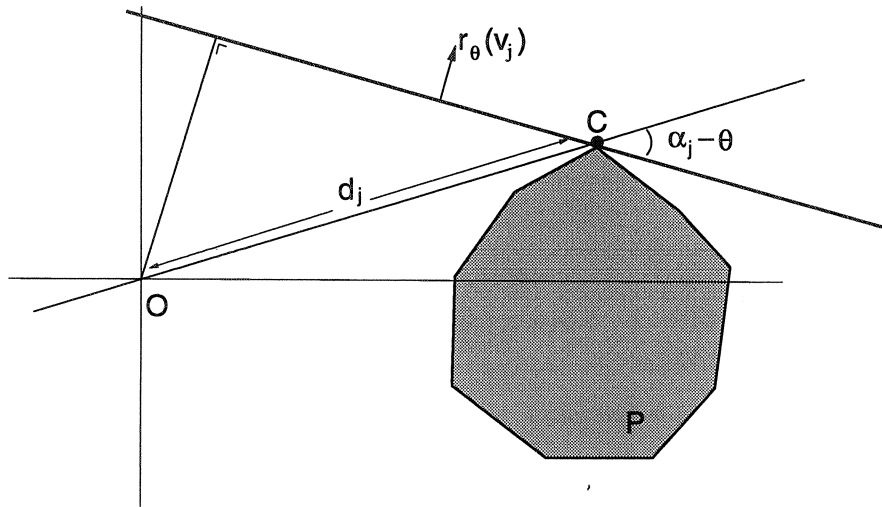
Fig. 3. The support set $F(P, r_\theta(v_j))$ consists of the vertex C.

where $\angle u$ denotes the angle of vector u with the positive x-axis. We denote the critical angles by

$$0 \le \theta_1^* < \theta_2^* < \cdots < \theta_N^* < 2\pi.$$

It is evident that $N \le n_P n_Q$.

Now, fix i and choose $\theta \in (\theta_k^*, \theta_{k+1}^*)$. We have seen that the support set $F(P, r_\theta(v_j))$ consists of a vertex C of P; see Figure 3.

Let $\alpha_j - \theta$ be the angle between the line through C with normal vector $r_\theta(v_j)$ and the line through O and C. It follows that $h(P, r_\theta(v_j)) = d_j \sin(\alpha_j - \theta)$, where $d_j = |OC|$. Thus

$$V(P, r_\theta(Q)) = \frac{1}{2} \sum_{j=1}^{n_Q} d_j q_j \sin(\alpha_j - \theta).$$

Taking the second derivative with respect to θ we find

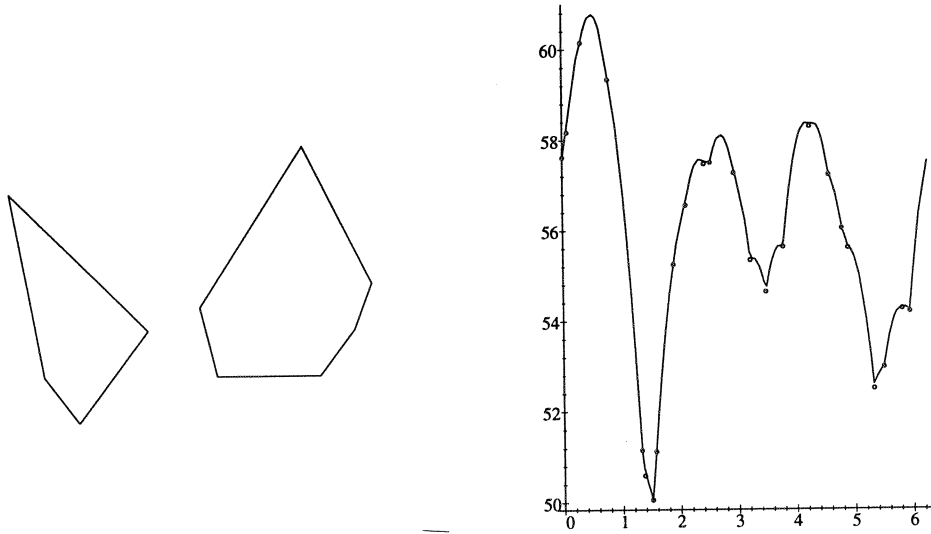$$V''(P, r_\theta(Q)) = -V(P, r_\theta(Q)) < 0.$$

Since

$$V(P \oplus r_\theta(Q)) = V(P) + V(Q) + 2V(P, r_\theta(Q)),$$

we find a similar result for $\theta \mapsto V(P \oplus r_\theta(Q))$. Thus we arrive at the following result.

5.1. Proposition. *The volume $V(P \oplus r_\theta(Q))$ and the mixed volume $V(P, r_\theta(Q))$ are functions of θ which are piecewise concave on $(\theta_k^*, \theta_{k+1}^*)$, for $k = 1, 2, \ldots, N - 1$. Here $0 \le \theta_1^* < \theta_2^* < \cdots < \theta_N^* < 2\pi$ are the critical angles of the two convex polygons P and Q.*

This result is illustrated in Figure 4.

Consider the S_+-invariant similarity measure obtained from (4.1) by choosing $H = S_+$.

Fig. 4. Left: convex polygons P and Q. Right: the function $\theta \mapsto V(P \oplus r_\theta(Q))$ is piecewise concave. The o's indicate the location of the critical angles.

Then

$$\sigma_1(P,Q) = \sup_{\lambda>0,\ \theta\in[0,2\pi)} \frac{4\lambda V(P)^{\frac{1}{2}} V(Q)^{\frac{1}{2}}}{V(P \oplus \lambda r_\theta(Q))}$$

$$= 4\left[\inf_{\lambda>0,\ \theta\in[0,2\pi)} \frac{V(P \oplus \lambda r_\theta(Q))}{\lambda V(P)^{\frac{1}{2}} V(Q)^{\frac{1}{2}}}\right]^{-1}$$

$$= 4\left[\inf_{\lambda>0,\ \theta\in[0,2\pi)} \frac{V(P) + \lambda^2 V(Q) + 2\lambda V(P, r_\theta(Q))}{\lambda V(P)^{\frac{1}{2}} V(Q)^{\frac{1}{2}}}\right]^{-1}$$

$$= 4\left[\inf_{\lambda>0}\left\{\frac{1}{\lambda}\cdot\left(\frac{V(P)}{V(Q)}\right)^{\frac{1}{2}} + \lambda\cdot\left(\frac{V(Q)}{V(P)}\right)^{\frac{1}{2}}\right\} + \inf_{\theta\in[0,2\pi)} \frac{2V(P, r_\theta(Q))}{V(P)^{\frac{1}{2}} V(Q)^{\frac{1}{2}}}\right]^{-1}.$$

Thus, in order to compute $\sigma_1(P,Q)$ we have to minimize two expressions, one in λ and one in θ. The first expression achieves its minimum at $\lambda = \left(V(P)/V(Q)\right)^{\frac{1}{2}}$, the second at one of the critical angles associated with P, Q.

The similarity measure σ_2 given by (4.2) results in

$$\sigma_2(P,Q) = \sup_{\theta\in[0,2\pi)} \frac{V(P)^{\frac{1}{2}} V(Q)^{\frac{1}{2}}}{V(P, r_\theta(Q))}. \tag{5.1}$$

From Proposition 4.4 we know that σ_2 is S_+-invariant, too. As above, the maximum is attained at one of the critical angles associated with P, Q. Like in Example 4.5 we get

$$\sigma_2(P,Q) = \frac{\sigma_1(P,Q)}{2 - \sigma_1(P,Q)}.$$

In Section 3 we have given some formulas for the computation of (mixed) volumes of convex polygons. The expression in (3.17) uses the perimetric representation, and we use it to get the following result.

5.2. Proposition. *Given the perimetric representation of the convex polygons P and Q, the time complexity of computing the similarity measures σ_1 and σ_2 is $\mathcal{O}(n_P n_Q(n_P + n_Q))$; here n_P, n_Q are the number of vertices of P and Q, respectively.*

PROOF. We restrict ourselves to σ_1; the result for σ_2 then follows immediately. Let $\theta = \theta_k^*$ be a critical angle for the pair P, Q. The perimetric measure of $r_\theta(Q)$ is given by $\{(r_\theta(v_j), q_j) \mid j = 1, 2, \ldots, n_Q\}$. The angles $\angle v_j + \theta$ can be ordered in $\mathcal{O}(n_Q)$ steps. Merging of the perimetric measures of P and $r_\theta(Q)$ in $\mathcal{O}(n_P + n_Q)$ steps yields the perimetric measure of $S = P \oplus r_\theta(Q)$, which we denote by $\{(w_i, s_i) \mid i = 1, 2, \ldots, n_S\}$, where $n_S \le n_P + n_Q$. By (3.17), the volume of S is

$$V(S) = \sum_{j=1}^{n_S} s_j t_j \sin \angle w_j - \frac{1}{2} \sum_{j=1}^{n_S} s_j^2 \sin \angle w_j \cos \angle w_j,$$

where

$$t_j = \sum_{i=1}^{j} s_i \cos \angle w_i.$$

All values t_j are computed in $\mathcal{O}(n_S)$ steps, and therefore $V(S)$ is calculated in $\mathcal{O}(n_S)$ steps. Since there exist at most $n_P n_Q$ critical angles, the result follows. ∎

If we choose $H = S$, we get

$$\sigma_1(P, Q) = 4 \left[\inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \cdot \left(\frac{V(P)}{V(Q)} \right)^{\frac{1}{2}} + \lambda \cdot \left(\frac{V(Q)}{V(P)} \right)^{\frac{1}{2}} \right\} + \inf_{h \in I} \frac{V(P, h(Q))}{V(P)^{\frac{1}{2}} V(Q)^{\frac{1}{2}}} \right]^{-1}.$$

Using that $\ell_\alpha = r_{2\alpha} \ell_0$, we have

$$\inf_{h \in I} V(P, h(Q)) = \min \left\{ \inf_{\theta \in [0, 2\pi)} V(P, r_\theta(Q)), \inf_{\theta \in [0, 2\pi)} V(P, r_\theta(\tilde{Q})) \right\},$$

where $\tilde{Q} = \ell_0(Q)$. To find the minimum, we need to consider the critical angles of P, Q as well as those of $P, \tilde{Q}$.

6. Affine invariant similarity measure

If $H = G$, then the similarity measures σ_1, σ_2 defined in (4.1) and (4.2), respectively, are affine invariant (that is, invariant under arbitrary affine transformations). Unfortunately, we do not have efficient algorithms to compute them. However, using the approach of Hong and Tan in [17], we are able to define similarity measures which can be computed efficiently, and which are invariant under a large group of affine transformations, namely G_+, the collection of all linear transformations which have a determinant which is positive. In combination with Proposition 4.6, this leads to similarity measures which are G-invariant.

The basic idea is to transform a set A to its so-called canonical form A^* in such a way that two sets A and B are equivalent modulo a transformation in G_+ if and only if A^* and B^* are equivalent modulo rotation. The definition of the canonical form, as discussed by Hong and Tan [17], is based on the concept of the *ellipse of inertia* known from classical mechanics [12]. Note, however, that Hong and Tan [17] use a slightly different approach; they introduce the *moment curve* which is closely related to the ellipse of inertia.
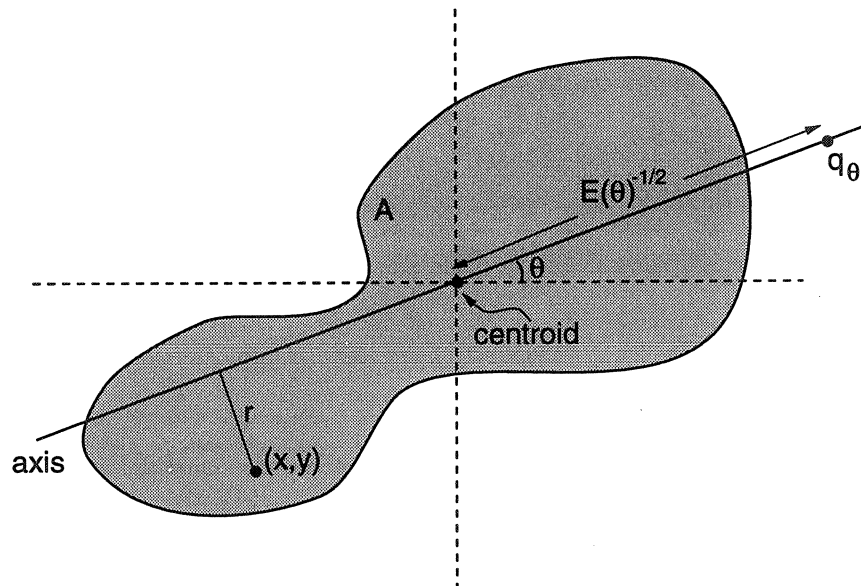
Fig. 5. Computation of moment of inertia with respect to axis through centroid.

Throughout this section we restrict ourselves to the family of compact sets with positive area, in the sequel denoted by $\mathcal{K}_+$. Consider an axis through the centroid of A, and denote, for a point $(x, y) \in A$, by $r(x, y)$ its distance to this axis (see Figure 5).

The *moment of inertia* with respect to the axis is given by:

$$E(\theta) = \int \int_A r(x, y)^2 \, dx \, dy.$$

Here θ denotes the angle between the axis and the x-axis in some fixed coordinate system. An easy calculation shows that

$$E(\theta) = m_{xx} \sin^2 \theta + m_{yy} \cos^2 \theta - 2 m_{xy} \sin \theta \cos \theta$$

$$= \frac{1}{2}(m_{xx} + m_{yy}) + \frac{1}{2}(m_{yy} - m_{xx}) \cos 2\theta - m_{xy} \sin 2\theta$$

(see also [18, p.48–53]). Here

$$m_{xx} = \int \int_A x^2 \, dx \, dy, \qquad m_{yy} = \int \int_A y^2 \, dx \, dy, \qquad m_{xy} = \int \int_A xy \, dx \, dy.$$

The point $q_\theta = E(\theta)^{-\frac{1}{2}}(\cos\theta, \sin\theta)$ on the axis traces an ellipse when θ varies between 0 and 2π, the so-called *ellipse of inertia*. The long and short axes of this ellipse are called the *principal axes*; they can be found by computing the extrema of $E(\theta)$. These extrema are determined by $E'(\theta) = 0$, that is $(m_{xx} - m_{yy})\sin 2\theta = 2m_{xy} \cos 2\theta$. If $m_{xx} \neq m_{yy}$, this yields

$$\tan 2\theta = \frac{2m_{xy}}{m_{xx} - m_{yy}}.$$

Using that $E'(\theta) = (m_{xx} - m_{yy}) \cos 2\theta \cdot (\tan 2\theta - 2m_{xy}/(m_{xx} - m_{yy}))$, it is easy to verify that we have a minimum if

$$(m_{xx} - m_{yy}) \cos 2\theta > 0,$$

and a maximum otherwise. We reach the following conclusion: if $m_{xx} = m_{yy}$ and $m_{xy} = 0$ then $E(\theta) = m_{xx}$ for all θ. In all other cases $E(\theta)$ has a unique minimum θ_0 inside the range $[0, \pi)$; obviously, its maximum lies at $\theta_0 + \pi/2$. The angle θ_0 is the unique solution in $[0, \pi)$ of the equations

$$\sin 2\theta_0 = \frac{2m_{xy}}{\sqrt{4m_{xy}^2 + (m_{xx} - m_{yy})^2}}, \qquad \cos 2\theta_0 = \frac{m_{xx} - m_{yy}}{\sqrt{4m_{xy}^2 + (m_{xx} - m_{yy})^2}}. \qquad (6.1)$$

The ellipse of inertia, which is depicted in Figure 6, has its long axis at θ_0 and its short axis at $\theta_0 + \pi/2$.
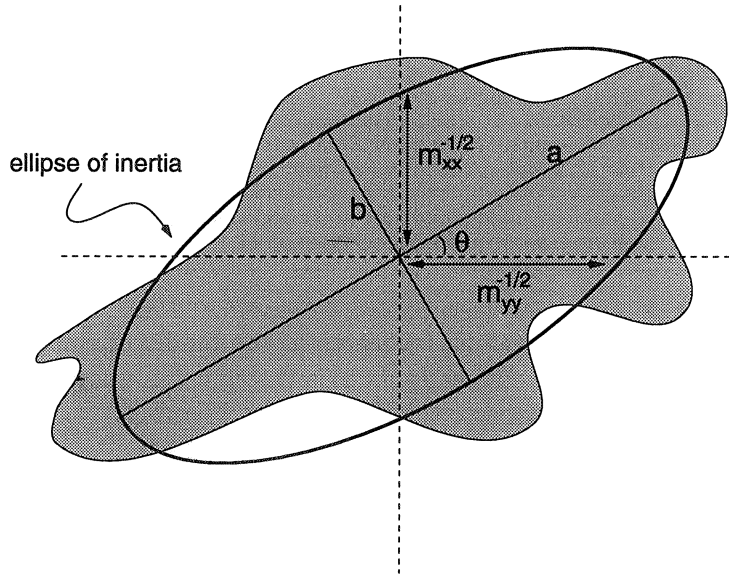

Fig. 6. The ellipse of inertia of a shape.

Let $2a$ and $2b$ be the lengths of these axes, respectively. One easily finds that $a = \left(E(\theta_0)\right)^{-1/2}$ and $b = \left(E(\theta_0 + \pi/2)\right)^{-1/2}$, which yields that

$$a = \left[\frac{1}{2}(m_{xx} + m_{yy}) - \frac{1}{2}\sqrt{(m_{xx} - m_{yy})^2 + 4m_{xy}^2}\right]^{-\frac{1}{2}},$$

$$b = \left[\frac{1}{2}(m_{xx} + m_{yy}) + \frac{1}{2}\sqrt{(m_{xx} - m_{yy})^2 + 4m_{xy}^2}\right]^{-\frac{1}{2}}. \qquad (6.2)$$

A simple calculation shows that

$$\frac{1}{a^2} + \frac{1}{b^2} = m_{xx} + m_{yy}. \qquad (6.3)$$

The following definition is due to Hong and Tan [17].

6.1. Definition. A shape is said to be in *canonical form* if its centroid is positioned at the origin and its ellipse of inertia is a unit circle.

6.2. Proposition. *Every compact set can be transformed into its canonical form by means of a transformation in G_+, namely by a stretching along the long axis of the ellipse of inertia by a factor $b/(ab)^{1/4}$ and along the short axis by a factor $a/(ab)^{1/4}$.*

The proof of this result is based on the observation that, under the transformation $(x, y) \mapsto (\lambda x, \mu y)$, the second moments scale as follows:

$$m'_{xx} = \lambda^3 \mu\, m_{xx}, \quad m'_{yy} = \lambda \mu^3 m_{yy}, \quad m'_{xy} = \lambda^2 \mu^2 m_{xy}.$$

6.3. Proposition. *For every $A \in \mathcal{K}_+$ we have*

$$[g(A)]^* \equiv \begin{cases} A^*, & \text{if } g \in M \\ g(A^*), & \text{if } g \in I. \end{cases}$$

PROOF. The first relation is easy to prove. The second relation is also easy if g is a rotation. Suppose now that ℓ_α is the reflection with respect to the long principal axis. It is evident that $[\ell_\alpha(A)]^* = \ell_\alpha(A^*)$. Every isometry g which is not a rotation, can be decomposed as $g \equiv r_\theta \ell_\alpha$ for some rotation r_θ. Then

$$[g(A)]^* = [r_\theta \ell_\alpha(A)]^* = r_\theta(\ell_\alpha(A)^*) = r_\theta \ell_\alpha(A^*) \equiv g(A^*).$$

This proves the result. ∎

In [27] Pei and Lin use the *covariance matrix* to compute the canonical form of a given shape (note that Pei and Lin speak of *image normalization*).

The covariance matrix C of a compact set A is given by

$$C = \begin{pmatrix} m_{xx} & m_{xy} \\ m_{xy} & m_{yy} \end{pmatrix}.$$

A simple calculation shows that $1/a^2$ and $1/b^2$ are the two eigenvalues of C.

Applying an affine transformation g to A results in a covariance matrix C' of $A' \equiv g(A)$ which is given by

$$C' = \det g \cdot WCW^T, \tag{6.4}$$

where W is the matrix representing (the linear part of) g (in particular, $\det g = \det W$) and W^T is it's transpose. If A is in canonical form, then its covariance matrix is the identity matrix.

6.4. Proposition. *Let A, B be two compact sets with positive area. Then $B \equiv g(A)$ for some $g \in G_+$ iff $B^* = r(A^*)$ for some $r \in R$.*

PROOF. 'if': suppose $B^* = r(A^*)$ for some $r \in R$. Since the canonical transformation, which maps a shape onto its canonical form, is an element of G_+ we have $g_2(B) \equiv r g_1(A)$, that is $B \equiv g_2^{-1} r g_1(A)$, and $g_2^{-1} r g_1$ is an element of G_+.

'only if': suppose $B \equiv g(A)$ for some $g \in G_+$. It follows that $B^* = g'(A^*)$ for some $g' \in G_+$. Since the covariance matrices of A^* and B^* are the identity matrices, we get from (6.4) that

$$I = \det g' \cdot WW^T,$$

where W is the matrix representing g'. It follows immediately that W must represent a rotation. This concludes the proof. ∎

With this result it is easy to construct G_+-invariant similarity measures from R-invariant ones.

6.5. Proposition. *Let $\sigma : \mathcal{K}_+ \times \mathcal{K}_+ \to [0, 1]$ be an R-invariant similarity measure, and define $\sigma^* : \mathcal{K}_+ \times \mathcal{K}_+ \to [0, 1]$ by*

$$\sigma^*(A, B) = \sigma(A^*, B^*),$$

then σ^* is a G_+-invariant similarity measure.

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As the map $A \mapsto A^\bullet$ preserves convexity, we get the same result for shapes in $\mathcal{C}_+ = \mathcal{C} \cap \mathcal{K}_+$ as well as for shapes in $\mathcal{P}$.

To apply these results for convex polygons, there are at least two possibilities. We can compute the canonical shape of the polygon itself or of the set given by its vertices considered as point masses. In the latter case, which is the one considered below, the previous findings remain valid, albeit that integrals have to be replaced by summations. Furthermore, the stretching factors in Proposition 6.2 become b (along the long axis) and a (along the short axis), respectively. Suppose we are given the perimetric representation $M(P) = \{(u_i, p_i) \mid i = 1, 2, \ldots, n_P\}$ of a convex polygon P. The computation of $M(P^\bullet) = \{(u_i^\bullet, p_i^\bullet) \mid i = 1, 2, \ldots, n_P\}$ consists of the following steps (putting $n = n_P$):

1. Fixing the origin at the first vertex of P, we can find the coordinates (x_i, y_i) of the other vertices; see (3.16).
2. The centroid (x_c, y_c) of P is given by

$$x_c = \frac{1}{n} \sum_{i=1}^{n} x_i, \qquad y_c = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

3. The second moments m_{xx}, m_{yy}, m_{xy} are given by

$$m_{xx} = \sum_{i=1}^{n} (x_i - x_c)^2, \qquad m_{yy} = \sum_{i=1}^{n} (y_i - y_c)^2, \qquad m_{xy} = \sum_{i=1}^{n} (x_i - x_c)(y_i - y_c).$$

4. Compute θ_0, a, b from (6.1)-(6.2).
5. Define $\phi_i = \angle u_i - \theta_0$ in such a way that $-\pi/2 \leq \phi_i < 3\pi/2$. Compute $u_i^\bullet$ and $p_i^\bullet$ from

$$\angle u_i^\bullet = \begin{cases} \angle u_i, & \text{if } \phi_i = \pm\frac{\pi}{2} \\ \theta_0 + \arctan(\frac{b}{a}\tan\phi_i), & \text{if } -\pi/2 < \phi_i < \pi/2 \\ \pi + \theta_0 + \arctan(\frac{b}{a}\tan\phi_i), & \text{if } \pi/2 < \phi_i < 3\pi/2 \end{cases}$$

and

$$p_i^\bullet = p_i \big(a^2 \cos^2 \phi_i + b^2 \sin^2 \phi_i\big)^{1/2}.$$

6.6. Example. Consider the rotation invariant similarity measure given by (4.2), i.e.,

$$\sigma(A, B) = \sup_{\theta \in [0, 2\pi)} \frac{V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A, r_\theta(B))}.$$

Define $\sigma^\bullet$ on $\mathcal{P}$ as in Proposition 6.5, then $\sigma^\bullet$ is a G_+-invariant similarity measure. Using Proposition 4.6(b) we obtain a G-invariant similarity measure.

7. Symmetry measures

Exact symmetry only exists in the mathematician's mind. It is never achieved in the real world, neither in nature nor in man-made objects [33]. Thus, in order to access symmetry of objects (convex 2-dimensional polygons in our case), we need a tool to measure the amount of symmetry. Towards that goal Grünbaum [13] introduced the concept of a *symmetry measure*; refer to [7, 8] for some other references. Below we give a formal definition of this concept. But first we recall

some basic terminology. We will restrict attention to the 2-dimensional case, but most of what we say carries over immediately to higher dimensions.

The *symmetry group* of a set $A \subseteq \mathbb{R}^2$ consists of all $g \in G$ such that $g(A) \equiv A$. The use of the word 'group' is justified by the observation that these transformations constitute a subgroup of G. An element in this subgroup is called a *symmetry* of A. An element $g \in G$ for which $g^m = \mathrm{id}$ for some finite $m \geq 1$ is called a *cyclic transformation of order m*. Sometimes we write $m = m_g$ to denote the dependence on g. It is evident that $|\det g| = 1$ if g is cyclic.

In this paper we are mostly interested in symmetries of a given shape which are cyclic. However, as shown in Example 7.1(b), there may also exist symmetries which are not cyclic.

7.1. Examples. (a) Not every cyclic transformation is an isometry. For example, $(x, y) \mapsto (2y, x/2)$ is cyclic of order 2, but it is not an isometry.

(b) Not every symmetry is cyclic. Let B be the unit disk in $\mathbb{R}^2$ and let $A := g(B)$ for some $g \in G$. It is obvious that $g r_\theta g^{-1}$ is a symmetry of A for every $\theta \in [0, 2\pi]$, since r_θ is a symmetry of B. In most cases, however, this symmetry is not cyclic. Let, for example, g be the transformation $(x, y) \mapsto (2x, y)$. Then A is the ellipse $x^2/4 + y^2 = 1$. For every θ with θ/π irrational, the transformation $g r_\theta g^{-1}$ is a non-cyclic, non-isometric symmetry of the ellipse.

If H is a subgroup of G, then the set of cyclic transformations in H is denoted by $C(H)$. It is easy to see that

$$h^{-1} e h \in C(H) \text{ if } e \in C(H), \ h \in H.$$

In general, $C(H)$ is not a subgroup.

Let $e \in G$ be a cyclic transformation of order m. We define the mapping $e^* : \mathcal{K} \to \mathcal{K}$ by

$$e^*(A) = \big(A \oplus e(A) \oplus \cdots \oplus e^{m-1}(A)\big)/m.$$

It is easy to see that $e^*(A)$ is e-symmetric, and we call this set the *e-symmetrization* of A. Observe that e^* is <u>not</u> an affine transformation. As a matter of fact, e^* is defined for shapes rather than for points.

Every line reflection ℓ_α is a cyclic transformation of order 2. The corresponding symmetrization of a set A, that is $(A \oplus \ell_\alpha(A))/2$, is called *Blaschke symmetrization* of A [30].

7.2. Proposition. *If e is a cyclic transformation and let $A \in \mathcal{C}$, then $V(e^*(A)) \geq V(A)$. Furthermore, the following statements are equivalent:*
 (1) $e(A) \equiv A$, *i.e.*, e *is a symmetry of A;*
 (2) $e^*(A) \equiv A$;
 (3) $V(e^*(A)) = V(A)$.

PROOF. From the Brunn-Minkowski inequality (3.9) it follows that

$$V(e^*(A))^{\frac{1}{2}} \geq \frac{1}{m}\big[V(A)^{\frac{1}{2}} + V(e(A))^{\frac{1}{2}} + \cdots + V(e^{m-1}(A))^{\frac{1}{2}}\big] = V(A)^{\frac{1}{2}},$$

where we have used that $V(e^k(A)) = V(A)$. This proves the inequality. Furthermore, equality holds iff $e(A) \equiv \lambda A$, as can be shown from the Brunn-Minkowski inequality. Thus $A \equiv e^m(A) \equiv \lambda^m A$. This yields that $\lambda = 1$ and that e is a symmetry of A. Thus we have shown the equivalence (1) $\Leftrightarrow$ (3). Therefore we have to prove that (1) $\Leftrightarrow$ (2). The implication '$\Rightarrow$' is obvious. To prove '$\Leftarrow$', assume that $e^*(A) \equiv A$. Using (2.3), we find that $e e^* = e^*$, and thus $e(A) \equiv e(e^*(A)) \equiv e^*(A) \equiv A$. This concludes the proof. ∎

It is easy to express $V(e^*(A))$ in terms of mixed volumes. Assume that e is a cyclic transformation of order m. Then, writing $A^i = e^i(A)$, we find

$$
\begin{aligned}
V(e^*(A)) &= V\Big(\frac{1}{m}(A^0 \oplus A^1 \oplus \cdots \oplus A^{m-1})\Big) \\
&= \frac{1}{m^2} V(A^0 \oplus A^1 \oplus \cdots \oplus A^{m-1}) \\
&= \frac{1}{m^2}\Big[V(A^0) + \cdots + V(A^{m-1}) + 2\sum_{i=1}^{m-1}\sum_{j=0}^{i-1} V(A^i, A^j)\Big].
\end{aligned}
$$

Using that $V(A^i) = V(A)$ and $V(A^i, A^j) = V(A^{i-j}, A)$, we get that

$$
\begin{aligned}
V(e^*(A)) &= \frac{1}{m}\Big[V(A) + \frac{2}{m}\sum_{i=1}^{m-1}\sum_{j=0}^{i-1} V(A^{i-j}, A)\Big] \\
&= \frac{1}{m}\Big[V(A) + \frac{2}{m}\sum_{k=1}^{m-1}(m-k)V(A^k, A)\Big].
\end{aligned}
$$

Let $1 \le k \le m$; we write $k \triangleleft m$ if the greatest common divisor of k and m equals 1. If e is a cyclic transformation of order m and $k \triangleleft m$, then e^k is a cyclic transformation of order m, and $(e^k)^* = e^*$. It is easy to see that every cyclic rotation of order m is of the form $(r_{2\pi/m})^k$, where $k \triangleleft m$.

Symmetry measures were introduced by Grünbaum [13] for point symmetries. Here we will generalize this definition to arbitrary families of cyclic transformations.

7.3. Definition. Let E be a given collection of cyclic transformations and $\mathcal{J} \subseteq \mathcal{K}$. A function $\mu : \mathcal{J} \times E \to [0,1]$ is called an *E-symmetry measure* on $\mathcal{J}$ if, for every $e \in E$, the function $\mu(\cdot, e)$ is continuous on $\mathcal{J}$ with respect to the Hausdorff topology, and if

(1) $\mu(A, e) = \mu(A', e)$ if $A \equiv A'$;
(2) $\mu(A, e) = \mu(e^k(A), e)$, $k \ge 1$;
(3) $\mu(A, e) = 1$ iff A is e-symmetric.

Suppose that, in addition, the following property holds:

(4) if e has order m and $k \triangleleft m$, then $\mu(A, e) = \mu(A, e^k)$;

then μ is called a *consistent E-symmetry measure*.

Let $H \subseteq G$ be such that $heh^{-1} \in E$ if $e \in E$ and $h \in H$; we say that μ is *H-invariant* if

(5) $\mu(A, e) = \mu(h(A), heh^{-1})$, $h \in H$.

Note that in this definition we have restricted ourselves to cyclic transformations.

7.4. Example. It is easy to show that

$$
\mu(A, r_\theta) = \frac{V(A)^{\frac{1}{2}} V(r_\theta(A))^{\frac{1}{2}}}{V(A, r_\theta(A))} = \frac{V(A)}{V(A, r_\theta(A))}
$$

defines a symmetry measure for all cyclic rotations r_θ (i.e., θ/π rational). This symmetry measure is invariant under similitudes. It is not consistent, however.

The consistency condition (4) has the following intuitive interpretation. Suppose that a shape A is (nearly) symmetric with respect to rotation over $2\pi/m$, then it is also (nearly) symmetric with respect to rotation over an angle $2k\pi/m$, where $1 \le k \le m$. Moreover, if $k \triangleleft m$, then the converse also holds.

There are at least two different ways to make an E-symmetry measure consistent. Our next result, the proof of which is straightforward, shows how this can be done.

7.5. Proposition. *If μ is an E-symmetry measure, then*

$$\mu_{\min}(A, e) := \min_{k \vartriangleleft m_e} \mu(A, e^k)$$

$$\mu_\pi(A, e) := \prod_{k \vartriangleleft m_e} \mu(A, e^k)$$

both define a consistent E-symmetry measure. If μ is H-invariant, then $\mu_{\min}$ and μ_π are H-invariant as well.

It is easy to see that $\mu_{\min} = \mu$ iff μ is consistent. The next result shows how one can obtain symmetry measures from similarity measures.

7.6. Proposition. *Let H be a subgroup of G and $E \subseteq C(H)$ such that*

$$eh = he, \text{ for } h \in H \text{ and } e \in E. \tag{7.1}$$

If σ is an H-invariant similarity measure, then μ given by

$$\mu(A, e) = \sigma(A, e^*(A)) \tag{7.2}$$

is a consistent H-invariant E-symmetry measure.

PROOF. Observe that, under the given assumptions, $heh^{-1} \in E$ for $e \in E$ and $h \in H$. The nontrivial part of the proof consists of showing that $\mu(A, e) = 1$ implies that A is e-symmetric. Therefore, assume that $\sigma(A, e^*(A)) = 1$, i.e., $e^*(A) \equiv h(A)$ for some $h \in H$. Then

$$h(A) \equiv e^*(A) \equiv ee^*(A) \equiv eh(A) \equiv he(A).$$

Applying h^{-1} at both sides yields $A \equiv e(A)$. ∎

Remarks. (a) If we do not assume the conditions in (7.1), the equality $e^*(A) \equiv h(A)$ yields that $h(A) \equiv e^*(A) \equiv ee^* \equiv eh(A)$, hence $A \equiv h^{-1}eh(A)$. This implies that $h(A)$ is e-symmetric.

(b) It is tempting to replace (7.2) by: $\mu(A, e) = \sigma(A, e(A))$. However, such a definition does not allow us to consider invariance under groups H which contain e. For, $\sigma(A, e(A)) = 1$ if σ is H-invariant and $e \in H$.

The following example is based on the similarity measure σ_2 given by (4.2).

7.7. Example. Let E consist of the rotations $e_m = r_{2\pi/m}$, where m is a positive integer. Furthermore, let $H = S_+$. It is clear that condition (7.1) in Proposition 7.6 holds, hence

$$\mu(A, e_m) = \sup_{\theta \in [0, 2\pi)} \frac{V(A)^{\frac{1}{2}} V(e_m^*(A))^{\frac{1}{2}}}{V(A, r_\theta e_m^*(A))}$$

defines an S_+-invariant E-symmetry measure.

There are other construction methods for symmetry measures besides those based on similarity measures. Below we present several examples of symmetry measures based on Minkowski addition.

In Proposition 7.2 we have seen that $V(e^*(A)) \geq V(A)$ if e is a cyclic transformation. Let E be a collection of cyclic transformations; we define

$$\mu_1(A, e) = \frac{V(A)}{V(e^*(A))}, \quad A \in C, \ e \in E. \tag{7.3}$$

In Proposition 7.8 below we show that μ_1 defines a consistent E-symmetry measure. There is alternative way to define a symmetry measure using mixed volumes. It is based upon the observation (see (3.10)) that

$$V(A, e(A)) \geq V(A)^{\frac{1}{2}} V(e(A))^{\frac{1}{2}} = V(A),$$

if e is a cyclic transformation. We define

$$\mu_2(A, e) = \frac{V(A)}{V(A, e(A))}. \tag{7.4}$$

Note that in Example 7.4 we have discussed the case where E comprises all cyclic rotations.

At first sight, it seems possible to define yet another symmetry measure by replacing $V(A, e(A))$ by $V(A, e^*(A))$ in (7.4). However, a simple calculation using properties (3.6)–(3.7) shows that $V(A, e^*(A)) = V(e^*(A), e^*(A))$, and thus, using (3.4), one gets that

$$V(e^*(A)) = V(A, e^*(A)).$$

Therefore, such a definition would coincide with μ_1 in (7.3).

7.8. Proposition. *Let E be a given collection of cyclic transformations, then μ_1 and μ_2 given by (7.3) and (7.4), respectively, are E-symmetry measures. The measure μ_1 is consistent.*

Suppose, furthermore, that $H \subseteq G$ is such that $heh^{-1} \in E$ if $e \in E$ and $h \in H$; then μ_1 and μ_2 are H-invariant.

PROOF. The proof of (1)–(2) in Definition 7.3 is easy in both cases. The proof of (3) for μ_1 follows easily from Proposition 7.2. For μ_2 one has to use that equality in (3.10) holds iff $B \equiv \lambda A$ for some $\lambda > 0$. The consistency of μ_1 follows from the observation that $(e^k)^* = e^*$ if $k \triangleleft m$. We prove H-invariance for μ_1; the proof for μ_2 is very similar. We must show that $\mu_1(A, e) = \mu_1(h(A), heh^{-1})$. It is easy to verify that $(heh^{-1})^* = he^*h^{-1}$. This yields that

$$\mu_1(h(A), heh^{-1}) = \frac{V(h(A))}{V(he^*h^{-1}h(A))} = \frac{|\det h| \cdot V(A)}{|\det h| \cdot V(e^*(A))} = \mu_1(A, e).$$

This concludes the proof. ∎

If e is a finite-order rotation or a reflection, and if P is a convex polygon whose perimetric representation is given, then it is easy to compute the perimetric representation of $e^*(P)$ by merging the perimetric representations of $e^i(P)$; see Section 2. This also leads to an efficient computation of the symmetry measure μ_1.

7.9. Example: rotations. Let E consist of all cyclic rotations. Then μ_1 given by (7.3) is a consistent S-invariant E-symmetry measure, S being the group of similitudes. Because of the consistency of μ_1, it suffices to consider $E = \{r_{2\pi/m} \mid m = 1, 2, \ldots\}$. Given a polygon P and a rotation r over the angle $2\pi/m$, for some $m \geq 1$; the r-symmetrization $r^*(P)$ is a polygon which is symmetric under rotations of order m. If $M(P, u)$ is the perimetric measure of P, then we can use (2.2) to find the perimetric measure of $r^*(P)$:

$$M(r^*(P), u) = \frac{1}{m} \sum_{i=0}^{m-1} M(r^i(P), u)$$

$$= \frac{1}{m} \sum_{i=0}^{m-1} M(P, r^i(u)).$$

It is obvious that

$$\angle(r^i(u)) = \angle u + \frac{2\pi i}{m} \bmod 2\pi.$$

Using formula (3.17), we can compute μ_1 directly. Table 2 in Section 10 contains the outcomes for a given collection of convex polygons.

7.10. Example: line reflections. In this example we restrict ourselves to convex polygons. If E consists of all line reflections, then μ_1 given by (7.3) defines an S-invariant E-symmetry measure. For a line reflection ℓ_α we find

$$\mu_1(P, \ell_\alpha) = \frac{V(P)}{V(\frac{1}{2}(P \oplus \ell_\alpha(P)))} = \frac{2V(P)}{V(P) + V(P, \ell_\alpha(P))}. \tag{7.5}$$

Like in the previous example, we can compute the perimetric measure $M(\ell_\alpha^*(P), u)$ if the perimetric measure of P is given:

$$M(\ell_\alpha^*(P), u) = \frac{1}{2}[M(P, u) + M(P, \ell_\alpha(u))],$$

and

$$\angle\ell_\alpha(u) = (2\alpha - \angle u) \bmod 2\pi.$$

In Table 3 in Section 10 we compute the symmetry measure μ_1 for several convex polygons for the angles $\alpha = 0°, 30°, 45°, 60°, 90°$.

The symmetry measure μ_2 given by (7.4) amounts to

$$\mu_2(P, \ell_\alpha) = \frac{V(P)}{V(P, \ell_\alpha(P))}.$$

Thus we get that

$$\mu_2(P, \ell_\alpha) = \frac{\mu_1(P, \ell_\alpha)}{2 - \mu_1(P, \ell_\alpha)}.$$

In most of the literature, one does not compute the symmetry measure for specific line reflections ℓ_α, but rather the maximum over all lines. In our setting this leads to the following definition. A function $\iota : \mathcal{K} \to [0, 1]$ is called an *index of reflection symmetry* if ι is continuous, and $\iota(A) = 1$ if and only if A is reflection symmetric with respect to some line. If μ is a measure of reflection symmetry, such as μ_1 in (7.5), then

$$\iota(A) := \sup_{\alpha \in [0, \pi)} \mu(A, \ell_\alpha)$$

is an index of reflection symmetry.

The computation of this index can be done efficiently because of the following observations. Since $V(P, \ell_\alpha(P)) = V(P, r_{2\alpha}(\tilde{P}))$, with $\tilde{P} = \ell_0(P)$, we conclude from Proposition 5.1 that $\alpha \mapsto V(P, \ell_\alpha(P))$ is piecewise concave on $(\alpha_k^*, \alpha_{k+1}^*)$, where the angles $2\alpha_k^*$ are the critical angles of $P, \tilde{P}$ lying between 0 and 2π. Thus every α_k^* is of the form $\frac{1}{2}(\angle u_i + \angle u_j)$ with $i, j \in \{1, 2, \ldots, n_P\}$, and where P has perimetric representation $\{(u_i, p_i) \mid i = 1, 2, \ldots, n_P\}$. This yields that the minimum of $\alpha \mapsto V(P, \ell_\alpha(P))$ is achieved at one of the angles α_k^*. Using the same argument as in Proposition 5.2, one finds that the index can be computed in $\mathcal{O}(n_P^3)$ time. In Table 3 we also give the index as well as the angle of the reflection axis for which the index (maximum) is attained.

7.11. Example: skew-symmetry. A shape A is said to be *skew-symmetric* if there exists an affine transformation $g \in G_+$ such that $g(A)$ is reflection symmetric with respect to some line. In this example we show that one can use the notion of canonical shapes (Section 6) to find 'how skew-symmetric' a given shape is.

Suppose that A is skew-symmetric; then $g(A)$ is reflection symmetric for some $g \in G_+$. The symmetry line of $g(A)$ coincides with one of the axes of inertia, and therefore it is also a symmetry axis of $(g(A))^*$. As this latter shape is a rotation of A^* (see Proposition 6.4), we conclude that A^* is reflection symmetric, too. Conversely, if A^* is reflection symmetric, then A is skew-symmetric (for, A^* is the result of two stretchings along the principal axes of the ellipse of inertia of A). Thus we find that A is skew-symmetric if and only if A^* is reflection symmetric. This yields immediately that we obtain an index of skew symmetry from any index of reflection symmetry applied to the canonical shapes; see Example 7.10.

8.   Polygon decomposition: the strongly cyclic case

Recall that $M(P, \cdot)$ is the perimetric measure of P. If $P \in \mathcal{P}$ and $g \in I$, then

$$M(g(P), u) = M(P, g^{-1}(u)). \tag{8.1}$$

Thus it follows that P is e-symmetric if and only if $M(P, e^k(u))$ does not depend on k, for every $u \in S^1$.

8.1. Definition. Let e be an isometry. A convex polygon P is said to be *totally e-asymmetric* if there does not exist a nontrivial e-symmetric polygon Q and a polygon R such that $P = Q \oplus R$.

8.2. Proposition. *Let e be a cyclic transformation of order m. If*

$$\min_{k=0,1,\ldots,m-1} M(P, e^k(u)) = 0, \quad \textit{for every } u \in S^1, \tag{8.2}$$

then the polygon P is totally e-asymmetric.

PROOF. Assume that e is cyclic and that (8.2) holds. Suppose that $P = Q \oplus R$ with Q e-symmetric. Since $M(P, u) = M(Q, u) + M(R, u)$, we get that

$$M(P, e^k(u)) \geq M(Q, e^k(u)) = M(Q, u).$$

But this contradicts (8.2); we conclude that P is totally e-asymmetric. ∎

In this section, transformations e which have the property

$$x + e(x) + \cdots + e^{m-1}(x) = 0, \quad \text{for every } x \in \mathbb{R}^2, \tag{8.3}$$

are of special interest. It is easy to see that (8.3) implies that e is a cyclic transformation of order m. The converse is not true, however. A transformation e satisfying (8.3) is called a *strongly cyclic transformation of order m*. Finite-order rotations are strongly cyclic, whereas line reflections are cyclic, but not strong. If e is strongly cyclic and $g \in G$, then $g^{-1}eg$ is strongly cyclic, too.

Assume now that e is a strongly cyclic transformation of order m, and that P is a convex polygon which is not totally e-asymmetric. Define

$$M(u) := \min_{k=0,1,\ldots,m-1} M(P, e^k(u)), \quad \text{for every } u \in S^1. \tag{8.4}$$

Let u be such that $M(u) \neq 0$; as P is not totally e-asymmetric, such a u does exist. Now $M(e^k(u)) = M(u)$, and

$$\sum_{k=0}^{m-1} M(e^k(u)) e^k(u) = M(u) \sum_{k=0}^{m-1} e^k(u) = 0,$$

since e is strongly cyclic. This yields that

$$\sum_{u \in S^1} M(u) u = 0,$$

thus M is the perimetric measure of an e-symmetric polygon P_s^e. It is obvious that $M(P, u) - M(u) \geq 0$, with equality everywhere iff $P = P_s^e$. Suppose $M \neq M(P, \cdot)$; we get that $M(P, \cdot) - M(\cdot)$ is the perimetric measure of a convex polygon, which we denote by Q. Now, for every $u \in S^1$,

$$\min_{k=0,1,\ldots,m-1} M(Q, e^k(u)) = \min_{k=0,1,\ldots,m-1} \left[M(P, e^k(u)) - M(e^k(u)) \right]$$

$$= \min_{k=0,1,\ldots,m-1} M(P, e^k(u)) - M(u) = 0.$$

This yields that Q is totally e-asymmetric. We write $P_a^e := Q$. Observe that $P_a^e = P$ if P is totally e-asymmetric. The following result has been established.

8.3. Proposition. *If e is a strongly cyclic transformation of order m and if P is an arbitrary convex polygon, then P can be decomposed as*

$$P \equiv P_s^e \oplus P_a^e, \tag{8.5}$$

where P_s^e is e-symmetric and P_a^e is totally e-asymmetric. The perimetric measures of P_s^e and P_a^e are respectively given by

$$M(P_s^e, u) = \min_{k=0,1,\ldots,m-1} M(P, e^k(u))$$

$$M(P_a^e, u) = M(P, u) - M(P_s^e, u). \tag{8.6}$$

The polygon P is totally e-asymmetric (i.e., $P_a^e \equiv P$) if and only if (8.2) holds. Note that in the latter case $P_s^e \equiv \{0\}$.

See Figure 10 for an illustration. The decomposition in (8.5) is a generalization of a result by Matheron and Serra in [25] where they consider the central symmetric case.

The decomposition in (8.5) suggests that

$$\mu_3(P, e) = \frac{V(P_s^e)}{V(P)}, \tag{8.7}$$

is a symmetry measure. However, it is easy to construct examples which show that this function is not continuous with respect to P. Yet, we can prove the following result.

8.4. Proposition. *Let e be a strongly cyclic transformation of order m. The function μ_3 in (8.7), where P_s^e is given by (8.6), has properties (1)–(5) of Definition 7.3 (definition of a consistent E-symmetry-measure on $\mathcal{P}$).*

Suppose, furthermore, that $H \subseteq I$ is such that $heh^{-1} \in E$ if $e \in E$ and $h \in H$. Then μ_3 is H-invariant.

PROOF. It is easy to verify that μ_3 satisfies (1)–(3) of Definition 7.3. To establish consistency, assume that $k \vartriangleleft m$. Then

$$M(P_s^{e^k}, u) = \min_{l=0,1,\ldots,m-1} M(P, (e^k)^l(u))$$

$$= \min_{l'=0,1,\ldots,m-1} M(P, e^{l'}(u))$$

$$= M(P_s^e, u).$$

It remains to prove H-invariance. Observe that

$$\mu_3(h(P), heh^{-1}) = \frac{V\left([h(P)]_s^{heh^{-1}}\right)}{V(h(P))}.$$

We show that $[h(P)]_s^{heh^{-1}} = h(P_s^e)$. Namely

$$M([h(P)]_s^{heh^{-1}}, u) = \min_{k=0,1,\ldots,m-1} M(h(P), (heh^{-1})^k(u))$$

$$= \min_{k=0,1,\ldots,m-1} M(h(P), he^k h^{-1}(u))$$

$$= \min_{k=0,1,\ldots,m-1} M(P, e^k h^{-1}(u))$$

$$= M(P_s^e, h^{-1}(u))$$

$$= M(h(P_s^e), u).$$

This yields that

$$\mu_3(h(P), heh^{-1}) = \frac{V(h(P_s^e))}{V(h(P))} = \frac{V(P_s^e)}{V(P)} = \mu_3(P, e).$$

This concludes the proof. ∎

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9. Polygon decomposition for line reflections

When we consider line reflections, the decomposition problem is more difficult. Namely, in this case, the function M given by (8.4) is not the perimetric measure of a convex polygon, in general. Here we shall describe an algorithm which, for a given line reflection $e = \ell_\alpha$, yields a unique decomposition

$$P \equiv P_s^e \oplus P_a^e, \tag{9.1}$$

such that P_s^e is ℓ_α-symmetric and has largest possible area. The basic idea is captured by Figure 7. The line $L_\alpha^\perp$ which is orthogonal to L_α separates the plane in a left part H_α^- and a right part H_α^+; see Figure 7(a). Furthermore, we put $H_\alpha^0 = L_\alpha^\perp$. We are interested in all directions $u \in S^1$ in the support of $M(P, \cdot)$ for which $\ell_\alpha(u) = u'$ lies in the support as well. In Figure 7(b) we have drawn all these vectors. The vectors u_{+i} and $u'_{+i} = \ell_\alpha(u_{+i})$ $(i = 1, 2, \ldots, k)$ lie in H_α^+, and the vectors u_{-i} and $u'_{-i} = \ell_\alpha(u_{-i})$ $(i = 1, 2, \ldots, l)$ lie in H_α^-. If there exist vectors in H_α^0 with the given properties, they will be denoted by u_0 and u'_0. The vector u_{+1} is the vector in H_α^+ which makes the largest angle with the line L_α. Furthermore, it is possible that $\angle u_{+k} = \angle u'_{+k} = \alpha$ and that $\angle u_{-l} = \angle u'_{-l} = \alpha + \pi$.
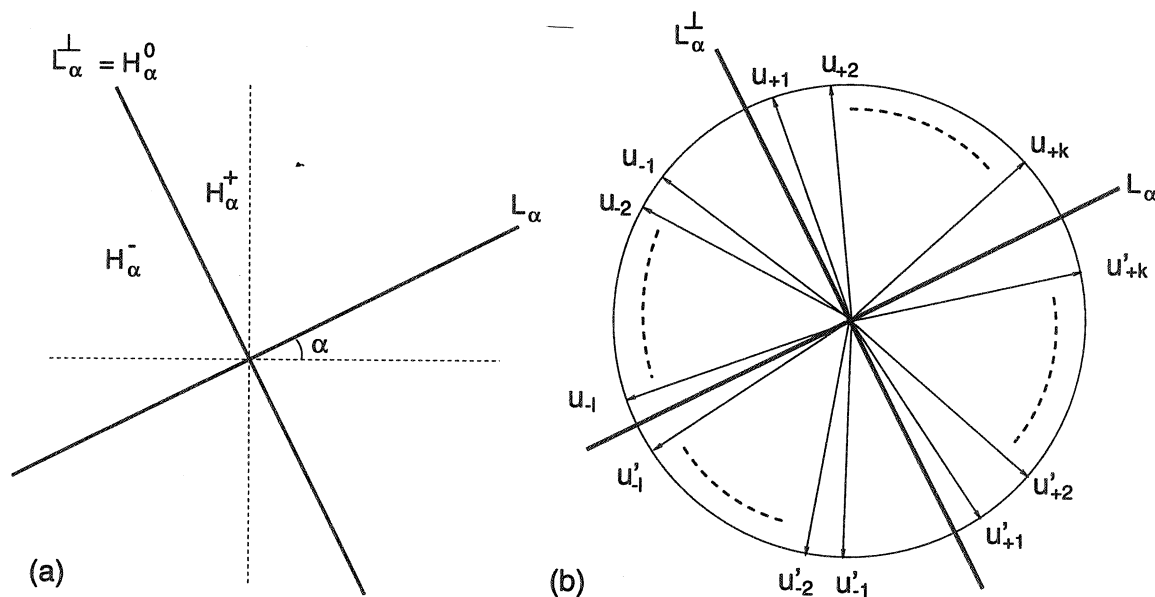

Fig. 7. L_α is the line of reflection. The line $L_\alpha^\perp$ separates vectors u in the perimetric representation for which also $u' = \ell_\alpha(u)$ is present, into two subclasses: u_{+i}, u'_{+i} at the right and u_{-j}, u'_{-j} at the left. More details can be found in the text.

9.1. Proposition. *Given a line reflection $e = \ell_\alpha$ and a convex polygon P, there exists a solution of (9.1) if and only if $k \geq 1$, $l \geq 1$, and the set comprising $u_{+i}, u'_{+i}, u_0, u'_0, u_{-j}, u'_{-j}$, with $i = 1, \ldots, k$ and $j = 1, \ldots, l$, contains at least three different vectors.*

A proof is implicitly contained in Algorithm 9.2 below.

The basic idea now is to choose pairs u_{+i}, u'_{+i} and lengths $p_{+i} = p'_{+i}$ as well as pairs u_{-i}, u'_{-i} and lengths $p_{-i} = p'_{-i}$ such that

$$\sum p_{+i}(u_{+i} + u'_{+i}) = -\sum p_{-i}(u_{-i} + u'_{-i}). \tag{9.2}$$

Notice that $u_{+i} + u'_{+i}$ is directed along L_α in the positive direction, whereas $u_{-i} + u'_{-i}$ is directed along L_α in the negative direction. Relation (9.2) expresses that the collection consisting of the

pairs $(u_{+i}, p_{+i}), (u'_{+i}, p_{+i}), (u_{-j}, p_{-j}), (u'_{-j}, p_{-j})$, along with $(u_0, p_0), (u'_0, p_0)$ (if present), defines a perimetric measure. Since

$$\|u_{+i} + u'_{+i}\| < \|u_{+(i+1)} + u'_{+(i+1)}\|$$

for every i (the same holds for u_{-i}), our algorithm starts with the smallest indices in order to obtain a maximal area for P_s^e. We define p_{+i}, $1 \leq i \leq k$, as follows:

$$p_{+i} = \begin{cases} \min\{M(P, u_{+i}), M(P, u'_{+i})\}, & \text{if } u_{+i} \neq u'_{+i} \\ \frac{1}{2}M(P, u_{+i}), & \text{if } u_{+i} = u'_{+i} \end{cases}$$

(Observe that $u_{+i} = u'_{+i}$ implies $i = k$). The values p_{-i}, $1 \leq i \leq l$, are defined similarly, and $p_0 = \min\{M(P, u_0), M(P, u'_0)\}$, if u_0 does occur. Furthermore, we define

$$S_+ = \sum_{i=1}^{k} p_{+i} \cdot \|u_{+i} + u'_{+i}\| \quad \text{and} \quad S_- = \sum_{i=1}^{l} p_{-i} \cdot \|u_{-i} + u'_{-i}\|.$$

We describe our algorithm for the case that $S_+ \geq S_-$; the case $S_+ < S_-$ is analogous. As we observed in Section 2, we can use a set instead of a sequence for the perimetric representation of a convex polygon.

9.2. Algorithm. (Case that $S_+ \geq S_-$)

$M := \{(u_{-i}, p_{-i}), (u'_{-i}, p_{-i}) \mid i = 1, 2, \ldots, l\}$; add $(u_0, p_0), (u'_0, p_0)$ to M if present;
$S := 0$;
$i := 1$;
$\triangle S := p_{+1} \cdot \|u_{+1} + u'_{+1}\|$;
<u>while</u> $(S + \triangle S \leq S_-)$ {
        $M := M \cup \{(u_{+i}, p_{+i}), (u'_{+i}, p_{+i})\}$;
        $S := S + \triangle S$;
        $i := i + 1$;
        $\triangle S := p_{+i} \cdot \|u_{+i} + u'_{+i}\|$;
}
$p := (S_- - S)/\cos(\angle u_i - \alpha)$; (compute the remainder)
$M := M \cup \{(u_{+i}, p), (u'_{+i}, p)\}$

The perimetric measure M associated with the resulting perimetric set M represents an ℓ_α-symmetric convex polygon P_s^e, and the difference $M(P, \cdot) - M(\cdot)$ the asymmetric part P_a^e.

It remains to be shown that Algorithm 9.2 yields the unique decomposition with P_s^e having maximal area. This is demonstrated by the following two observations. First we explain that, starting with a perimetric set M (first line of Algorithm 9.2) the algorithm yields the polygon with maximal area whose perimetric set contains M. The set M yields a left part of an ℓ_α-symmetric polygon. Our algorithm extends this polygon rightwards in a symmetric fashion, but it does so by choosing a path from point A to the line L_α which has smallest descent, thus maximizing the area. This means that our algorithm is optimal if we can show that the initial choice for M is optimal; see Figure 8.

However, any other choice for M in combination with our algorithm leads to a perimetric measure M' which is smaller than the perimetric measure M obtained from Algorithm 9.2: $M' \leq M$. This implies, however, that the area of the corresponding polygon is smaller, too.

Thus we have shown that Algorithm 9.2 yields the decomposition in (9.1) where P_s^e has maximal area. See Figure 10 for an illustration.
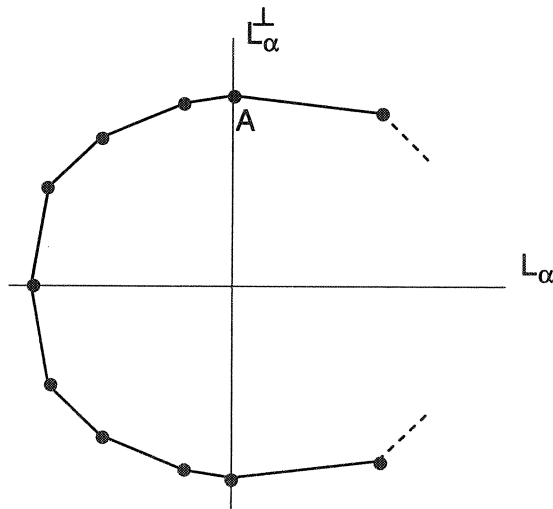
Fig. 8. Construction of reflection symmetric
part with maximal area.

However, for most angles α the condition in Proposition 9.1 will not be satisfied. To find an upper estimate for the number of angles which have to be checked if P contains $n = n_P$ vertices, we have to consider the angles $\alpha_{i,j} = \frac{1}{2}(\angle u_i + \angle u_j) \bmod \pi$, with $1 \le j \le i \le n$. An angle α is a candidate solution if there exists at least two pairs i_1, j_1 and i_2, j_2 such that $\alpha_{i_1,j_1} = \alpha_{i_2,j_2} = \alpha$. Furthermore, it is not allowed that both $i_1 = j_1$ and $i_2 = j_2$. An upper bound for the number of candidates is $\frac{1}{2}\sum_{i=1}^{n} i = \frac{1}{4}n(n+1)$, where $n = n_P$.

Just like in Proposition 8.4, we can use the decomposition in (9.1) to define a non-continuous L-symmetry measure

$$\mu_3(P, e) = \frac{V(P_s^e)}{V(P)}.$$

(Recall that L contains all line reflections). It is clear that $h\ell_\alpha h^{-1} \in L$ for every $\ell_\alpha \in L$ and $h \in I$. In the same way as we did in the proof of Proposition 8.4 we can show that μ_3 is I-invariant. In fact, we use that Algorithm 9.2 is "invariant" under isometries.

Before we conclude this section, we point out that in [31], Schneider discusses decomposition results for convex polyhedra based on Minkowski addition.

10. Experimental results and conclusions

In this section the results obtained previously will be applied to some concrete examples. We consider four, more or less regular, shapes, namely: a triangle, a square, a tetragon with one reflection axis, and a regular octagon. These shapes, along with their canonical forms, are depicted in Figure 9. In this figure, we depict four other convex polygons (and their canonical forms), namely: P, a reflection of P denoted by P_{refl}, a distortion of P denoted by Q (the lower three points have been shifted in the x-direction), and an affine transformation of Q denoted by Q_{aff}.

In Table 1 we compute the similarity measure σ_2 given by (5.1) which is S_+-invariant. In the first row we compute $\sigma_2(Q, R)$, where R is one of the other polygons depicted in Figure 9. In the third row we compute the values $\sigma_2(Q_{\text{aff}}, R)$. The second row contains the values $\sigma_2^*(Q, R)$, where σ_2^* is the G_+-invariant similarity measure obtained from Proposition 6.5. Observe that we do not compute $\sigma_2^*(Q_{\text{aff}}, R)$, since these values are identical to $\sigma_2^*(Q, R)$. In Table 1 we also give the angle at which the maximum in expression (5.1) is achieved.

triangle   triangle$^{\circledast}$ square   square$^{\circledast}$

tetragon   tetragon$^{\circledast}$ octagon   octagon$^{\circledast}$

P $P^{\circledast}$ P_{refl} $P^{\circledast}_{\mathrm{refl}}$

Q $Q^{\circledast}$ Q_{aff} $Q^{\circledast}_{\mathrm{aff}}$
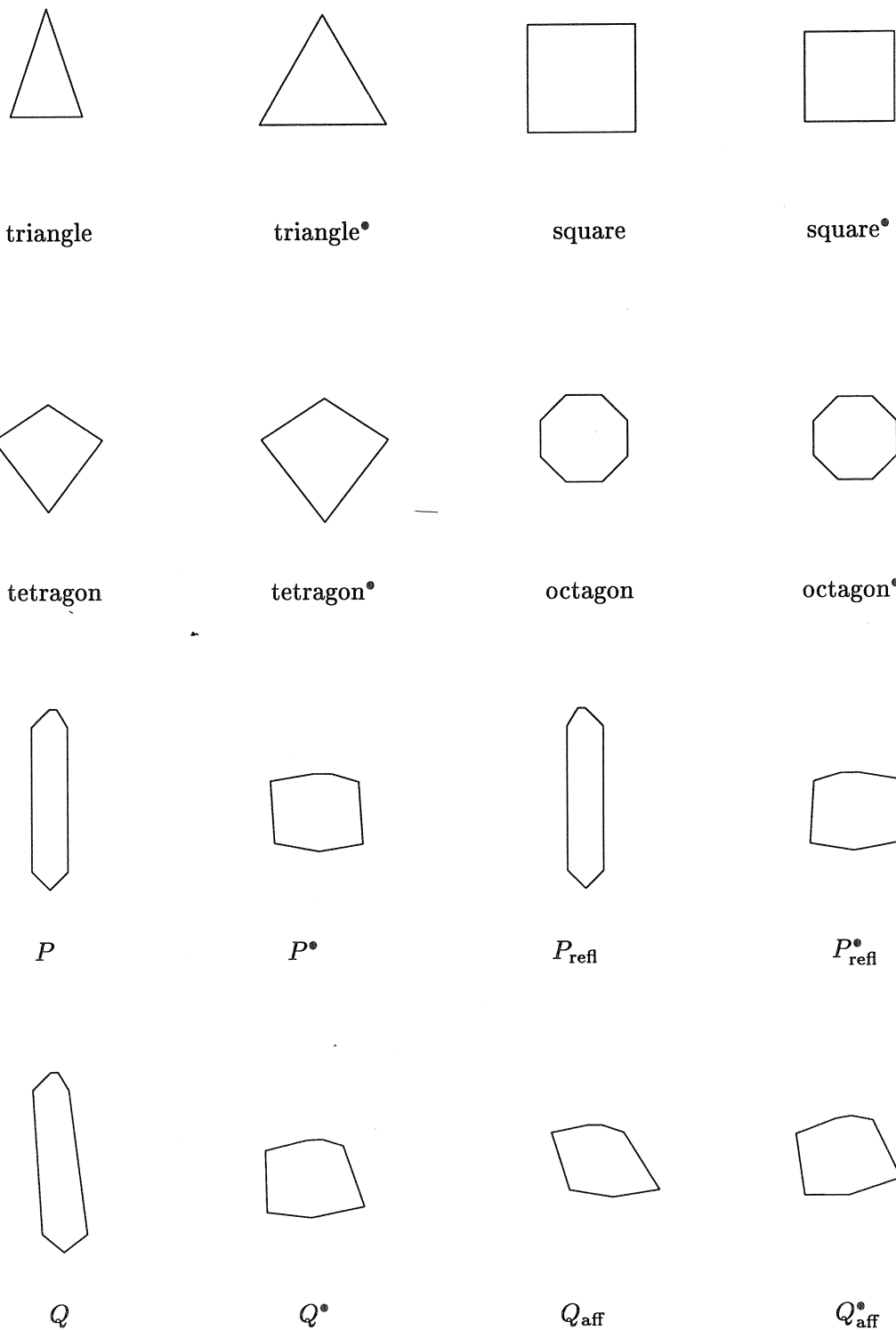
Fig. 9. Polygons used in this section. Note that $Q^{\circledast}_{\mathrm{aff}}$ is a rotation of $Q^{\circledast}$.

similarity measures	polygons					
	P	P_{refl}	triangle	square	tetragon	octagon
$\sigma_2(Q,\cdot)$	0.941	0.933	0.674	0.724	0.692	0.725
angle	3.6°	185.2°	348.7°	7.1°	326.7°	3.6°
$\sigma_2^{\bullet}(Q,\cdot)$	0.949	0.933	0.768	0.907	0.920	0.898
angle	3.8°	184.9°	233.4°	11.5°	322.1°	281.5°
$\sigma_2(Q_{\text{aff}},\cdot)$	0.759	0.749	0.753	0.845	0.847	0.862
angle	260.6°	80.6°	242.1°	11.3°	68.9°	31.9°

Table 1. Similarity measures for polygons in Figure 9.

Table 2 and Table 3 are concerned with symmetry measures for rotations and reflections, respectively. In Table 2 we illustrate the measure μ_1 discussed in Example 7.9 for $m = 2, 3, \ldots, 8$, corresponding with rotations over $360°/m$. Observe that $\mu_1(Q, r_\pi) = \mu_1(Q_{\text{aff}}, r_\pi)$. In fact, it is easy to see that both μ_1 and μ_2 defined in (7.3) and (7.4), respectively, satisfy

$$\mu_i(A, r_\pi) = \mu_i(g(A), r_\pi),$$

for every shape A and every affine transformation g.

rotation order	polygons				P	Q	Q_{aff}
	triangle	square	tetragon	octagon			
2	0.667	1	0.857	1	0.995	0.943	0.943
3	0.696	0.804	0.825	0.954	0.503	0.548	0.805
4	0.571	1	0.847	1	0.530	0.562	0.809
5	0.577	0.792	0.780	0.950	0.490	0.528	0.786
6	0.557	0.804	0.769	0.954	0.503	0.543	0.790
7	0.563	0.789	0.788	0.949	0.485	0.524	0.778
8	0.550	0.828	0.771	1	0.504	0.532	0.777

Table 2. Rotation symmetry measure; see Example 7.9.

Table 3 shows the reflection symmetry measure of Example 7.10 for five different reflection axes. Furthermore, the two bottom rows capture the maximum over all axes (i.e., the index of reflection symmetry; see Example 7.10) and the angle at which this maximum is attained.

reflection axis	polygons				P	Q	Q_{aff}
	triangle	square	tetragon	octagon			
0°	0.667	1	0.857	1	0.995	0.864	0.864
30°	0.754	0.845	0.863	0.965	0.568	0.646	0.828
45°	0.667	1	0.911	1	0.531	0.572	0.885
60°	0.619	0.845	0.828	0.965	0.582	0.597	0.950
90°	1	1	1	1	0.995	0.867	0.867
maximum	1	1	1	1	0.995	0.987	0.965
angle	90°	0°	90°	0°	90°	95.3°	149.3°

Table 3. Reflection symmetry measure; see Example 7.10
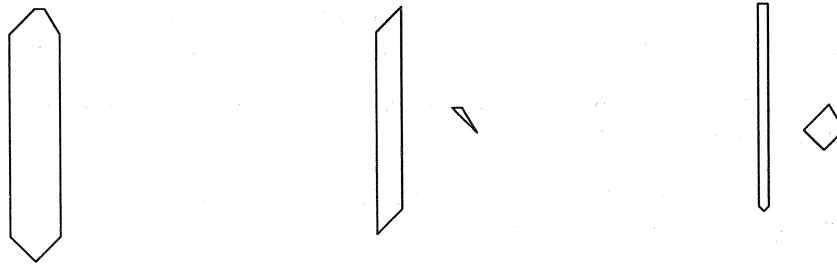
Fig. 10. Decomposition of polygon P (left) in its symmetric and asymmetric part. The middle figure shows the decomposition with respect to rotation symmetry (rotation over 180°, i.e., central symmetry), the right figure shows the decomposition with respect to reflection symmetry (reflection in y-axis).

In Sections 8–9 we have discussed the decomposition of polygons in a symmetric and an asymmetric part. The algorithms given there are illustrated in Figure 10 for the polygon P of Figure 9.

At this point it is yet unclear to us how useful our results are in comparison with those obtained by other researchers. Of course, it depends largely upon the application at hand. As our approach is based on the area of shapes, it will be difficult to compare it with boundary-oriented approaches, such as boundary matching.

Furthermore, we have restricted ourselves here to the 2-dimensional case. However, our theoretical results are more general. We intend to investigate the 3-dimensional case in the near future.

Acknowledgements

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