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ON FAMILIES OF SYSTEMS: POINTWISE-LOCAL-GLOBAL ISOMORPHISM PROBLEMS

MICHIEL HAZEWINKEL and ANNA-MARIA PERDON

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ERASMUS UNIVERSITY ROTTERDAM P.O. BOX 1738. 3000 DR. ROTTERDAM. THE NETHERLANDS.

ON FAMILIES OF SYSTEMS: POINTWISE-LOCAL-GLOBAL ISOMORPHISM PROBLEMS.*)

by

Michiel Hazewinkel and Anna-Maria Perdon

Dept. Math., Erasmus Univ. Rotterdam, Rotterdam, The Netherlands

Ist. di Mat. Appl. Univ. di Padova, Padova, Italy

ABSTRACT

Let Σ and Σ' be two families of linear dynamical systems, or, almost equivalently, let Σ and Σ' be two systems over a ring. This paper addresses itself to the question, what, if anything, can be said about the relations between Σ and Σ' if it is known that Σ and Σ' are pointwise isomorphic for all or almost all of the parameter values.

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1. INTRODUCTION

(and motivational remarks for studying families rather than single systems).

A linear dynamical system is a system of differential or difference equations

$$(1.1)\qquad \begin{array}{ll} \dot{x} = Fx + Gu & x(t+1) = Fx(t) + Gu(t) \\ y = Hx & y(t) = Hx(t) \end{array}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, i.e. we have state space dimension n, m inputs and p outputs. The theory of linear dynamical systems deals with various properties of and constructions pertaining to such sets of equations, with the coefficients, i.e. the entries of the matrices F, G, H, assumed known. Yet in many circumstances these coefficients are imperfectly known at best and it becomes important to examine what happens to various notions and constructions as the coefficients vary (slightly).

To make things more precise let Q be a topological space. Roughly a family of linear dynamical systems over Q consists of a collection of such equations (1.1), one for each $q \in Q$, such that the matrices F,G,H depend continuously on the parameter q. More generally (and also more properly), a family over Q consists of a vectorbundle E over Q (of dimension n), a vectorbundle endomorphism $F: E \to E$ and two vectorbundle homomorphisms $G: Q \times \mathbb{R}^m \to E$, $H: E \to Q \times \mathbb{R}^p$. The two definitions agree locally (i.e.) over small enough open subsets of Q and for the purposes of this paper the first definition mostly suffices.

In the discrete time case (i.e. the difference equation case) one can consider systems of equations

$$(1.2)\qquad x(t+1) = Fx(t) + Gu(t),\ y(t) = Hx(t)$$

where now the matrices F,G,H can have their coefficients in any ring R (and t = 0,1,2, ..., say). For each prime ideal $\mathfrak{p}$ of R let $R(\mathfrak{p})$ be the quotient field of the integral domain $R/\mathfrak{p}$. This gives us a family of systems

$$(1.3) \qquad x(t+1) = F(\mathfrak{p})x(t) + G(\mathfrak{p})u(t), \; y(t) = H(\mathfrak{p})x(t)$$

which is the local algebraic-geometric analogue of the topological concept of a family introduced above. The main goal of the theory of families of systems is now to develop techniques and prove theorems which do for families all the nice things one can do for a single linear dynamical systems as e.g.

- realization theory for a family of input/output maps (cf. also [3,4,9])
- pole placement and stabilization by feedback (cf. also [4,10,18,20])
- decomposition (e.g. construction of the "canonical" completely reachable subsystem (cf [9,8])
- controllability subspaces and their applications
- disturbancedecoupling

The general philosophy of/and motivation for the study of families of (linear) dynamical systems rather than single ones is discussed more extensively in [9,8]. Results pertaining to different aspects than those of the present paper are in [10,11].

In view of the reinterpretation (sketched above) of a system (1.2) over a ring R as an algebraic-geometric family of systems over Spec(R), the general project encompasses trying to do all the things listed above for systems over rings, and this constitutes an important bit of motivation for studying families of systems.

A related, and important, bit of motivation comes from linear delay differential dynamical systems as e.g.

$$(1.4) \qquad \begin{aligned} \dot{x}_1(t) &= x_1(t) + x_2(t-1) + u(t-1) \\ \dot{x}_2(t) &= x_1(t-1) + u(t) \\ y(t) &= x_1(t) + x_2(t-2) \end{aligned}$$

Introducing the delay operator σ, $\sigma x(t) = x(t-1)$, we can write (1.4) formally as a linear system over the ring $R[\sigma]$, viz.

$$(1.5) \qquad \begin{aligned} \dot{x}(t) &= F(\sigma)x(t) + G(\sigma)u(t) \\ y(t) &= H(\sigma)x(t) \end{aligned}$$

where F(σ), G(σ), H(σ) are the following matrices with coefficients in the ring of polynomials $\mathbb{R}[\sigma]$

$$F(\sigma) = \begin{pmatrix} 1 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad G(\sigma) = \begin{pmatrix} \sigma \\ 1 \end{pmatrix}, \quad H(\sigma) = (1,\sigma^2).$$

As it turns out this rather formal looking procedure is most useful, [13]. For instance in a very nice paper [12], Ed Kamen has worked out some of the relationships between the spectral properties of (1.4) and the commutative algebra which goes into the study of (1.5). And using this, and the reinterpretation of (1.5) as a family of systems, Chris Byrnes [4] has been able to do things about the feedback stabilization theory of (1.4).

Other bits of motivation for studying families come e.g. from identification theory [7] and the study of high-gainfeedback systems, [14].. In both these cases it is important to know in what ways a family of systems can suddenly degenerate, which is the subject matter of [11] and also of the present paper (theorems 2.8 and 2.9).

Ideally one would like to write down explicit local (uni)versal deformations for each systems as Arnol'd did for matrices in [1]. On general principles one expects that this is possible and for pairs of matrices (F,G), i.e. "input systems" or "control systems" this has recently been done by Tannenbaum [17].

To extend these constructions à la Arnol'd of versal deformations to the case of triples of matrices may involve non trivial difficulties. A reason for thinking this is that the stabilizer subgroup (cf. 3.2. below for a definition) of a system which is completely observable <u>or</u> completely reachable is trivial. Yet there is no (fine) moduli space for families of co or cr systems as examples 2.5, 2.6 show. I.e. the stabilizer subgroup, which is at the heart of Arnol'd's constructions may be an insufficient guide in the setting of triples of matrices. For completely reachable or completely observable systems universal deformations result from the fine moduli spaces of [5.6]. And in fact the original starting point for this paper was the far too optimistic idea that these moduli spaces might quite well be extendable to some extent. Thus the main problem considered in this paper became: Given two families of linear dynamical systems Σ, Σ' over a manifold Q. Suppose that pointwise the systems Σ_q, Σ'_q are isomorphic for all or almost all $q \in Q$. What can be said about the relation between Σ and Σ' as families and what can be said about the relations between Σ_q and Σ'_q at the remaining points of Q.

The first question is of course entirely analogous to the one studied by Wasow [19], and later in an algebraic setting by Ohm and Schneider [15], with respect to similarity of families of matrices which depend (holomorphically) on a parameter.

2. ALMOST EVERYWHERE ISOMORPHIC FAMILIES OF SYSTEMS.

We use the abbreviations cr for completely reachable and co for completely observable. Recall that the system (1.1) is cr iff the matrix

$$R(F,G) = (G \quad FG \ldots F^n G) \tag{2.1}$$

is of full rank n, and that (1.1) is co iff the matrix Q(F,H) is of full rank n. Here Q(F,H) is defined as

$$Q(F,H)^T = (H^T \quad F^T H^T \ \ldots \ F^{nT} H^T) \tag{2.2}$$

where the symbol T means "transposes".

Let $L_{m,n,p}$ be the space of all linear dynamical systems (1.1) of state space dimension n and with m inputs and p outputs. I.e. $L_{m,n,p}$ is the space of all triples of matrices (F,G,H) over $\mathbb{R}$ of dimensions nxn, nxm, pxn respectively. We give $L_{m,n,p}$ the corresponding topology, i.e. the topology of $\mathbb{R}^{n(n+m+p)}$. For the purposes of this paper a family of systems over a topological space Q is simply a continuous map $Q \to L_{m,n,p}$. A more general (and better) definition of family of systems is given in [9,8] and there the reader will also find a discussion of the reasons why the present definition is inadequate in some contexts. The theorems of the present paper extend with no trouble to this more general setting. This is automatic for the local theorems (3.3) and (3.4), because locally (i.e. over small enough open neighbourhood the naive definition and the proper one agree. For the global versions of theorems 2.3, 2.4, 2.8, 2.9 it suffices to appeal to the same rigidity phenomenon (= uniqueness of (iso)morphisms if they exists at all) which is the basis of the corresponding local results.

If $\Sigma = (F,G,H)$ is a family of linear dynamical systems over a topological space Q we denote with $\Sigma(q)$ the system $(F(q),G(q),H(q))$. Completely analogously if $\Sigma = (F,G,H)$ is a)discrete time) system over a ring R then $\Sigma(\mathfrak{p}) = (F(\mathfrak{p}), G(\mathfrak{p}), H(\mathfrak{p}))$ is the induced system over $R(\mathfrak{p})$, the quotient field of $R/\mathfrak{p}$.

2.3. Theorem. Let Σ and Σ' be two families over a topological space Q. Let $U_1 = \{q \in Q: \Sigma(q)$ and $\Sigma'(q)$ are both cr$\}$ and $U_2 = \{q \in Q: \Sigma(q)$ and $\Sigma'(q)$ are both co$\}$. Suppose that $U_1 \cup U_2 = Q$ and suppose that $\Sigma(q)$ and $\Sigma'(q)$ are pointwise isomorphic for a dense set Z of points q in Q. Then Σ and Σ' are isomorphic as families over Q, (which, by definition, means that there is a continuous map $Q \to GL_n(\mathbb{R})$, $q \mapsto S(q)$, such that $F'(q) = S(q)F(q)S(q)^{-1}$, $G'(q) = S(q)G(q)$, $H'(q) = H(q)S(q)^{-1}$ for all $q \in Q$).

It follows in particular that $\Sigma(q)$ and $\Sigma'(q)$ are also isomorphic in all the remaining points, i.e. the points of $Q \setminus Z$. The (local) algebraic geometric version of this theorem is

2.4. Theorem. Let Σ and Σ' be two systems over a ring R. Let $U_1 = \{\mathfrak{p} \in \mathrm{Spec}(R) | \Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are both cr$\}$, $U_2 = \{\mathfrak{p} \in \mathrm{Spec}(R) | \Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are both co$\}$. Suppose that $U_1 \cup U_2 = \mathrm{Spec}(R)$ and that there is a dense subset $Z \subset \mathrm{Spec}(R)$ such that $\Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are isomorphic for all $\mathfrak{p} \in Z$. Then Σ and Σ' are isomorphic as systems over R.

This means in particular that if R is an integral domain and $\Sigma = (F,G,H)$, $\Sigma' = (F',G',H')$ are two n-dimensional systems over R which are isomorphic over K, the quotient field of R, and if moreover for all maximal ideals $\mathfrak{m} \subset R$ we have that the rank of both $R(F,G)$, $R(F',G')$ or of both $Q(F,H)$, $Q(F',H')$ stays n mod $\mathfrak{m}$, then Σ and Σ' are also isomorphic as systems over R.

Both theorems 2.3 and 2.4 are almost trivial consequences of the existence of fine moduli spaces for cr families and for co families. These exist both in the topological case (cf [5]) and the algebraic geometric case (cf [6] and especially [8]).

The proofs of theorems 2.3 and 2.4 now go as follows. (We write out the details in the topological case only). Recall that the fine moduli space M^{cr} is the quotient space $L^{cr}_{m,n,p}/GL_n(\mathbb{R})$. Now let $\Sigma: S \to L^{cr}_{m,n,p}$ be a family of cr systems. Assign to Σ the composed map $S \to L^{cr}_{m,n,p} \to M^{cr}$. which assign to $s \in S$ the point of M^{cr} corresponding to $\Sigma(s)$ (= the orbit of $\Sigma(s)$). Then part of the fine moduli property of M^{cr} says that two systems over S are isomorphic (as systems) iff they give rise to the same map $S \to M^{cr}$ (This part of the fine moduli theorem is in fact almost trivial). Thus under the hypothesis of theorem 2.3 the families Σ and Σ' give rise to the same continuous map

$$U_1 \cap Z \to M^{cr}$$

and because $U_1 \cap Z$ is dense in U_1 these two maps agree on all of U_1 which (by the fine moduli property) means that Σ and Σ' are isomorphic over U_1, i.e. there exists a continuous map

$$\phi_1 : U_1 \to GL_n(\mathbb{R})$$

such that for all $q \in U_1$

$$\Sigma'(q) = \Sigma(q)^{\phi_1(q)}$$

where Σ^S is short for (SFS^{-1}, SG, HS^{-1}) if $\Sigma = (F,G,H)$, $S \in GL_n(\mathbb{R})$, the group of invertible nxn matrices.

Similarly there exists a fine moduli space for families of co systems M^{co} which similarly permits us to conclude that Σ and Σ' are isomorphic over U_2, so that there is a continuous map

$$\phi_2 : U_2 \to GL_n(\mathbb{R})$$

such that

$$\Sigma'(q) = \Sigma(q)^{\phi_2(q)} \quad , \quad q \in U_2$$

Now systems which are cr or co enjoy the following rigidity property: if they are isomorphic the isomorphism is unique. Indeed if (F,G,H), $(F',G',H') \in L_{m,n,p}$ are isomorphic via $S \in GL_n(\mathbb{R})$ then S satisfies

$$S\,R(F,G) = R(F',G') \quad \text{and} \quad Q(F,H)S^{-1} = Q(F',H')$$

and if (F,G,H) and (F',G',H') are both cr or if both are co then these relations determine S uniquely.

It follows that in the setting above $\phi_1(q) = \phi_2(q)$ for all $q \in U_1 \cap U_2$. I.e. ϕ_1 and ϕ_2 agree on $U_1 \cap U_2$ proving that Σ and Σ' are isomorphic over all of Q.

The proof of theorem 2.4, the algebraic geometric version is completely analogous: it suffices essentially to replace the words "continuous map" with "morphism of algebraic varieties" everywhere in the above.
The trouble with theorems 2.3 and 2.4 is that, unless one demands something like pointwise isomorphism everywhere, or cr everywhere, or co everywhere, the condition $U_1 \cup U_2 = Q$ cannot be stated in terms of the separate families Σ and Σ'. So one is lead to ask whether not a condition like everywhere co or cr would be sufficient.
It is not, as is more or less predictable from the wellknown fact that as a rule it is perfectly possible for two nonisomorphic systems Σ and Σ' over an integral domain R to become isomorphic over the quotient field, [16]. The simplest such example is undoubtedly the following one dimensional one over $\mathbb{R}[\sigma]$.

$$(2.5) \qquad \begin{array}{l} \Sigma : F = 1,\ G = \sigma,\ H = 1 \\ \Sigma': F' = 1,\ G' = 1,\ H' = \sigma \end{array}$$

Considered as families over $Q = \mathbb{R}$, parametrized by σ, we have that Σ is co everywhere and cr everywhere except in 0, while Σ' is cr everywhere and co everywhere except in 0. Thus $U_1 = U_2 = \mathbb{R} \setminus \{0\}$. Also $\Sigma(q)$ and $\Sigma'(q)$ are isomorphic for all $q \neq 0$. But of course Σ and Σ' are not isomorphic as families nor as systems over the ring $\mathbb{R}[\sigma]$.

Another example, which is slightly more illustrative of what goes on is given by the families

$$(2.6) \qquad \begin{array}{l} \Sigma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1 & 1\\ \sigma & b\end{pmatrix}, (1,0)\right) \\ \Sigma' = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1 & \sigma\\ 1 & b\end{pmatrix}, (1,0)\right) \end{array}$$

which have essentially the same properties as the families (2.5). And here we note that though $\Sigma(0)$ and $\Sigma'(0)$ are of course not isomorphic, they are also not totally unrelated. In fact they agree on the completely reachable subsystem of $\Sigma(0)$. (For a more precise description of what this means, cf below). Note also that these examples largely destroy all hope about extending the fine moduli spaces $M^{cr}_{m,n,p}$ and $M^{co}_{m,n,p}$ a bit.

2.7. Morphisms. Let Σ and Σ' be two families over Q. A morphism $\Sigma \to \Sigma'$ over Q then consist of a continuous map $\psi : Q \to M^{nxn}$ the space of n x n matrices such that for all $q \in Q$, $\psi(q)G(q) = G'(q)$, $F'(q)\psi(q) = \psi(q)F(q)$, $H'(q)\psi(q) = H(q)$.
Completely analogously a morphism $\Sigma \to \Sigma'$ between two systems over a ring R is an n x n matrix T such that $TG = G'$, $F'T = TF$, $H'T = H$.
Using this notion one can now state the two following (dual) "mildness of degeneracy" results.
2.8. Theorem. Let Σ and Σ' be two families over Q. Suppose that $\Sigma(q)$ is cr for all $q \in Q$. Suppose moreover that $\Sigma'(q)$ and $\Sigma(q)$ are isomorphic for all q in a dense subset Z of Q. Then there is a morphism $T: \Sigma \to \Sigma'$ over Q such that $T(q): \Sigma(q) \to \Sigma'(q)$ is an isomorphism for all $q \in Z$ and such that $T(q): \Sigma(q) \to \Sigma'(q)$ maps the state space of $\Sigma(q)$ onto the completely reachable subspace of the state space of $\Sigma'(q)$ for all $q \in Q$.
2.9. Theorem. Let Σ and Σ' be two families over Q. Suppose that $\Sigma(q)$ is co for all $q \in Q$. Suppose moreover that $\Sigma'(q)$ and $\Sigma(q)$ are isomorphic for all q in a dense subset Z of Q. Then there is a morphism $T: \Sigma' \to \Sigma$ over Q such that $T(q): \Sigma(q) \to \Sigma'(q)$ is an isomorphism for all $q \in Z$ and such that for all $q \in Q \setminus Z$ two states x, x' in state space of $\Sigma'(q)$ are indistinguishable (by means of observations) if and only if their difference $x - x'$ is in $\mathrm{Ker}(T\ (q))$.
There are of course the obvious analogous results for systems over rings. In this case 2.8 says, among other things, that the system over a ring R which is cr everywhere is maximal in the lattice of all realizations over R which realize the same input/output behaviour; similarly 2.9 says that the everywhere co realization is the minimal element of this lattice. Cf. [21] for a discussion of the lattice of realizations of a linear response map over a ring.
2.10. Proof of theorem 2.8. Let $q \in Q$. Because Σ is cr in q there exists a nice selection (cf. [5]) and an open subset $U \subset Q$ containing q such that $R(F(q'),G(q'))_\alpha$ is invertible for all $q' \in U$. Now let $z_1, z_2, \ldots$ be a sequence of points of $Z \cap U$ converging to q.

Define the matrix T(q) as the limit

$$T(q) = \lim_{i\to\infty} R(F'(z_i),\ G'(z_i))_\alpha\ R(F(z_i),G(z_i))_\alpha^{-1}$$

It is not difficult to check that T(q) does not depend on the choice of α or on the choice of the sequence $z_1, z_2, \ldots$

Now for all i we have $z_i \in Z$ so that $\Sigma'(z_i)$ and $\Sigma(z_i)$ are isomorphic, say by $S_i \in GL_n(\mathbb{R})$. Then S_i satisfies

$$S_i R(F(z_i),G(z_i)) = R(F'(z_i),G'(z_i))$$

so that

$$S_i = R(F(z_i),G'(z_i))_\alpha \; R(F(z_i),\; G(z_i)_\alpha^{-1}$$

Writing out that S_i is an isomorphism we find

$$S_i F(z_i) = F'(z_i)S_i,\; S_i G(z_i) = G'(z_i),\; H'(z_i)S_i = H(z_i)$$

and taking the limit for $i \to \infty$ we find the relations

$$T(q)F(q) = F'(q)T(q), T(q)G(q) = G'(q),\; H'(q)T(q) = H(q)$$

so that $T(q)$ is a morphism $\Sigma(q) \to \Sigma'(q)$. It is easy to check that $T(q)$ depends continuously on q so that the $T(q)$ combine to define a morphism $T: \Sigma \to \Sigma'$. If $q \in Z$ then $T(q)$ is of course the unique isomorphism $\Sigma(q) \to \Sigma'(q)$. The relations written out above which are satisfied by $T(q)$ imply

$$T(q)R(F(q),G(q)) = R(F'(q),G'(q))$$

and, using that $(F(q),G(q))$ is completely reachable, it follows that the completely reachable subspace of $(F'(q),G'(q))$ is equal to the image of $T(q)$ (because the completely reachable subspace of a system (F,G,H) is the image of the map $R(F,G): \mathbb{R}^{(n+1)m} \to \mathbb{R}^n$)

2.11. The proof of theorem 2.9 is similar (or use duality).

2.12. Example. Let Σ and Σ' be two families over Q, which are pointwise isomorphic over a dense subset Z of Q. Then, without any further assumptions, we know of course that for all $q \in Q$, $\Sigma(q)$ and $\Sigma'(q)$ are related in the sense that their cr and co subquotients are isomorphic. This follows from the continuity of the Laplace transform. Beyond this there seems little one can say (without making some sort of stableness hypothesis as in 2.8 and 2.9 above), as the following example shows.

$$\Sigma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1 & a\\ \sigma & 2\end{pmatrix}, (\sigma,1)\right) \tag{2.13}$$

$$\Sigma' = \left(\begin{pmatrix}\sigma\\1\end{pmatrix}, \begin{pmatrix}1-\sigma a & \sigma^2 a\\ -a & \sigma a+2\end{pmatrix}, (0 \quad \sigma)\right)$$

These families are pointwise isomorphic for all $\sigma \neq 0$. But for $\sigma = 0$ there is not even a morphism $\Sigma(0) \to \Sigma'(0)$, in fact there is not a morphism between the input parts of the completely reachable subsystems of $\Sigma(0)$ and $\Sigma'(0)$.

3. EVERYWHERE POINTWISE ISOMORPHIC FAMILIES OF SYSTEMS.

Now let Σ and Σ' be families of systems over Q (resp. Spec(R)) which are pointwise isomorphic everywhere. Then it does not necessarily follow that Σ and Σ' are isomorphic as families over Q (resp. are isomorphic as systems over R), as the following example shows.

3.1. Example. Consider the two families over $\mathbb{R}$ (or the two systems over $\mathbb{R}[\sigma]$) defined by

$$\Sigma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\ \sigma^2 & 1\end{pmatrix}, (1,2)\right)$$

$$\Sigma' = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\ \sigma & 1\end{pmatrix}, (1,2\sigma)\right)$$

These two families are pointwise isomorphic for all σ(resp. the systems $\Sigma(\mathfrak{p})$, $\Sigma'(\mathfrak{p})$ are isomorphic for all prime ideals $\mathfrak{p} \subset \mathbb{R}[\sigma]$) but they are not isomorphic as families over $\mathbb{R}$ (resp. as systems over $\mathbb{R}[\sigma]$); indeed Σ and Σ' are not isomorphic in any neighbourhood of 0 (resp. not isomorphic over any localization $\mathbb{R}[\sigma]_f$ of $\mathbb{R}[\sigma]$ for which $f(0) \neq 0$).

So we shall need some sort of extra condition to insure that pointwise isomorphism implies isomorphism as families.

3.2. Stabilizer subgroups. Let Σ be a family over Q. Then for each $q \in Q$ we define

$$N(q) = \{ S \in GL_n(\mathbb{R}) : SF(q) = F(q)S,\ SG(q) = G(q), H(q)S = H(q) \}.$$

This is the stabilizer subgroup in $GL_n(\mathbb{R})$ of the system $\Sigma(q)$. The

Lie algebra of N(q) is

$$L(q) = \{T \in M^{n\times n} | TF(q) = F(q)T,\ TG(q) = 0,\ H(q)T = 0\}$$

We use r(q) to denote the dimension of N(q) which is of course equal to the dimension of L(q). Completely analogously one defines in the case of a system $\Sigma = (F,G,H)$ over a ring R the subgroup $N(\mathfrak{p})$ of $GL_n(R(\mathfrak{p}))$ consisting of all invertible matrices S over the field $R(\mathfrak{p})$ (= quotient field of $R/\mathfrak{p}$), such that $SF(\mathfrak{p}) = F(\mathfrak{p})S$, $SG(\mathfrak{p}) = G(\mathfrak{p})$, $H(\mathfrak{p})S = H(\mathfrak{p})$, and $L(\mathfrak{p})$ as the Lie algebra of all n x n matrices T with coefficients in $R(\mathfrak{p})$ such that $TF(\mathfrak{p}) = F(\mathfrak{p})T$, $TG(\mathfrak{p}) = 0$, $H(\mathfrak{p})T = 0$.

3.3. Differentiable families of systems. Topologically the space of all n dimensional systems with m inputs and p outputs is homeomorphic with $\mathbb{R}^{n(n+m+p)}$, cf. section 2 above. We now give $L_{m,n,p}$ also the differentiable structure of $\mathbb{R}^{n(n+m+p)}$. Now let Q be a differentiable manifold. Then a family of systems $\Sigma : Q \to L_{m,n,p}$ is a differentiable family of systems if the map Σ is differentiable. Two differentiable families of systems Σ and Σ' are isomorphic as differentiable families if there is a differentiable map $\phi : Q \to GL_n(\mathbb{R})$ such that $\Sigma(q)^{\phi(q)} = \Sigma'(q)$ for all $q \in Q$. Here, of course, $GL_n(\mathbb{R})$ is given the differentiable structure of an open subset of $\mathbb{R}^{n^2}$. The space of orbits M^{cr} of completely reachable systems has a natural differentiable structure and with this structure it is a fine moduli space for the appropriate notion (based on vectorbundles) of differentiable families of cr systems (in the differentiable category), cf. [5,8].

3.4. Theorem. Let Σ and Σ' be two differentiable families over the differentiable manifold Q. Suppose that Σ and Σ' are pointwise isomorphic everywhere. Suppose moreover that $r(q) = \dim N(q)$ (= dim L(q)) is constant in some neighbourhood U of $q_o \in Q$. Then there is a (possibly smaller) neighbourhood V of q_o such that Σ and Σ' are isomorphic as differentiable families over V.

The proof is not difficult (and more or less standard). Consider the map ϕ: $GL_n(\mathbb{R}) \times Q \to L_{m,n,p} \times Q$ given by $(S,q) \to (\Sigma(q)^S, q)$. It follows from the assumption of constancy of the dimension of N(q) that

$d\phi$ is of constant rank, so that ϕ is a submersion onto its image. In particular ϕ locally admits sections; i.e. if $(\Sigma_o, q_o) \in \mathrm{Im}\phi$ then there is an open neighbourhood U of (Σ_o, q_o) and a differentiable map $s: U \to GL_n(\mathbb{R}) \times Q$ such that $\phi \circ s = \mathrm{id}$. Now consider $\psi : Q \to L_{m,n,p} \times Q$ given by $\psi(q) = (\Sigma'(q), q)$; this is simply the graph of Σ'. By assumption for each q we know that $\psi(q) \in \mathrm{Im}\phi(GL_n(\mathbb{R}) \times \{q\})$ and the fibre of ϕ over $\psi(q)$ is precisely $\Phi(q) \times \{q\}$ where $\Phi(q)$ is set of all possible isomorphisms $\Sigma(q) \to \Sigma'(q)$. (Of course $\Phi(q)$ is a left coset of $N(q)$. Now let s be a local section of ϕ defined in some neighbourhood of $(\Sigma'(q_o), q_o)$. Restricting s to the graph of Σ' (i.e. the image of ϕ) gives us a map $U_o \to GL_n(\mathbb{R}) \times U_o$ of the form $q' \mapsto (S(q'), q')$ (because s is a section). The map $q' \mapsto S(q')$ is then the desired isomorphism $\Sigma \to \Sigma'$ (over U_o). For this proof at least, some sort of differentiability restriction is necessary. There are analogous theorems for holomorphic families and real analytic families. The corresponding theorem for systems over rings is

3.5. <u>Theorem</u>. Let Σ and Σ' be two systems over a ring R. Suppose that $\Sigma(\mathfrak{p})$ and $\Sigma'(\mathfrak{p})$ are isomorphic for all prime ideals $\mathfrak{p}$ contained in some open subset U of Spec(R). Suppose moreover that $r(\mathfrak{p}) = \dim N(\mathfrak{p})$ is constant for some neighbourhood U' of $\mathfrak{p}_o \in U$. Then there exists an open neighbourhood $V = \mathrm{Spec}(R_f)$, $f \in R$, of $\mathfrak{p}_o$ such that Σ and Σ' are isomorphic as systems over R_f (or, equivalently, as families over V). For both these theorems it is in general not true that Σ and Σ' are necessarily isomorphic over all of Q (resp. isomorphic as systems over R) as the following example shows.

3.6. <u>Example</u>. Consider the following two systems, either as families over $\mathbb{R}$ or as systems over the ring $\mathbb{R}$

$$\Sigma = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \sigma \\ 0 & \sigma^2 \end{pmatrix}, (\sigma^2 - 1, -\sigma) \right)$$

$$\Sigma' = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \sigma+2 \\ 0 & \sigma^2 \end{pmatrix}, (\sigma^2 - 1, -\sigma - 2) \right)$$

These two families are pointwise isomorphic everywhere; the dimension of the stabilizer subgroups is 1 everywhere; in addition one has that rank $R(F(\sigma), G(\sigma))$ and rank $Q(F(\sigma), H(\sigma))$ are also equal to 1 everywhere. As families the two systems are isomorphic over $\mathbb{R} \setminus \{-1\}$ and also over $\mathbb{R} \setminus \{1\}$. As systems over rings they are isomorphic over

$\mathbb{R}[\sigma]_{\sigma-1}$ and $\mathbb{R}[\sigma]_{\sigma+1}$, but not, as is easily checked, as systems over $\mathbb{R}[\sigma]$ itself. The systems Σ and Σ' are not even isomorphic as differentiable (or topological families. Indeed such an isomorphism must necessarily be of the form

$$\sigma \mapsto \begin{pmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{pmatrix}$$

because the isomorphism matrices must take $\binom{1}{0} = G(\sigma)$ into $\binom{1}{0} = G'(\sigma)$. Here $c_{12}(\sigma)$, $c_{22}(\sigma)$ are continuous functions of σ such that $c_{22}(\sigma)$ is nowhere zero on $\mathbb{R}$. From

$$\begin{pmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{pmatrix} \begin{pmatrix} 1 & \sigma \\ 0 & \sigma^2 \end{pmatrix} = \begin{pmatrix} 1 & \sigma+2 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{pmatrix}$$

$$(\sigma^2-1,-\sigma-2)\begin{pmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{pmatrix} = (\sigma^2-1,-\sigma)$$

one then sees that the sole remaining condition on the $c_{12}(\sigma)$, $c_{22}(\sigma)$ is that

$$(*) \qquad (\sigma^2-1)c_{12}(\sigma) = c_{22}(\sigma)(\sigma+2) - \sigma$$

This means that $3c_{22}(1) = 1$ and $c_{22}(-1) = -1$. But there is no real continuous function assuming these values in 1 and -1 and which is nonzero everywhere.

And of course the matrix

$$\begin{pmatrix} 1 & c_{12}(\sigma) \\ 0 & c_{22}(\sigma) \end{pmatrix}$$

defines an isomorphism over the ring $\mathbb{R}[\sigma]$ if and only if $c_{22}(\sigma)$ is a nonzero constant which is also incompatible with (*).

The main ingredient of the proof of theorem 3.5 is the following generalization of the central lemma of [15].

3.7. <u>Lemma</u>. Let R be a ring without nilpotents, let A be an $m \times n$ matrix with coefficients in R and let $a \in R^m$. Consider the equation $Ax = a$. Suppose that the equation $A(\mathfrak{p})y = a(\mathfrak{p})$ over the field $R(\mathfrak{p})$ can be solved for all prime ideals $\mathfrak{p}$. Suppose moreover that $r(\mathfrak{p}) = \operatorname{rank} A(\mathfrak{p})$ is constant (as a function of $\mathfrak{p}$). Then $Ax = a$ is solvable over R. Moreover if $\mathfrak{m}$ is a maximal ideal of R and $y(\mathfrak{m})$ is any pregiven solution of $A(\mathfrak{m})y = a(\mathfrak{m})$, then there is a solution x of $Ax = a$ over R such that $x \equiv y(\mathfrak{m}) \bmod \mathfrak{m}$. Finally if $\mathfrak{p}$ is a prime ideal and $y(\mathfrak{p})$ is any given solution of $A(\mathfrak{p})y = a(\mathfrak{p})$ then there is an $f \in R \setminus \mathfrak{p}$ and a solution of $Ax = a$ over R_f such that $x \equiv y(\mathfrak{p}) \bmod \mathfrak{p}R_f$.

Proof. Let $P = \operatorname{Im}(A)$, and let $Q = R^m/\operatorname{Im}(A)$. Let $\mathfrak{p}$ be a prime ideal of R and consider the localized morphism of modules

$$A_{\mathfrak{p}} : R^n_{\mathfrak{p}} \to R^m_{\mathfrak{p}}$$

$A_{\mathfrak{p}}$ takes $R^n_{\mathfrak{p}}$ into $R^m_{\mathfrak{p}}$. Let $A(\mathfrak{p})$ be the induced quotient map

$$A(\mathfrak{p}) : R(\mathfrak{p})^n \to R(\mathfrak{p})^m$$

where $R(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is the quotient field of $R/\mathfrak{p}$. By premultiplying and postmultiplying $A(\mathfrak{p})$ with invertible matrices S,T we can see to it that $A(\mathfrak{p})$ is of the form

$$(*) \qquad \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \end{pmatrix}$$

(where there are $r = r(\ )$ 1's for the first r diagonal entries and zero's everywhere else). Let $\tilde{S},\tilde{T}$ be any invertible matrices over R which reduce to S and T mod $\mathfrak{p}R_{\mathfrak{p}}$. Then $\tilde{S} A_{\mathfrak{p}} \tilde{T}$ looks like $(*)$ with the 1's replaced by $1 + a_{ii}$, $a_{ii} \in \mathfrak{p}R_{\mathfrak{p}}$ and the o's replaced by a_{ij}, $a_{ij} \in \mathfrak{p}R_{\mathfrak{p}}$. Because the $1 + a_{ii}$ are invertible in R further pre-and postmultiplication with invertible matrices gives $A_{\mathfrak{p}}$ the form

$$(**) \qquad \left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline & & & * & \cdots & * \\ & 0 & & \vdots & & \vdots \\ & & & * & \cdots & * \end{array}\right)$$

where all the $*$-elements are in $\mathfrak{p}A_{\mathfrak{p}}$. But the rank hypothesis says that the rank of the matrix ($**$) considered as a matrix over the quotient field of R is also r. Because R has no nilpotents it follows that all the $*$-elements in ($**$) are zero. And from this it is of course immediate that $Q_{\mathfrak{p}} = \mathrm{Coker}(A_{\mathfrak{p}}: R^n_{\mathfrak{p}} \to R^m_{\mathfrak{p}})$ is free of rank $m - r$.

It follows that Q is a projective R-module (because Q is locally free, cf. [2], Ch. II, §5) and hence a direct summand of some free R-module R^p

$$i : Q \to R^p$$

Now consider the image $\bar{a}$ of a in $Q = R^m/\mathrm{Im}(A)$. The solvability of $A(\mathfrak{p})y = a(\mathfrak{p})$ means that $\bar{a}$ maps to zero under $Q \to Q(\mathfrak{p}) = Q \otimes R/\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. So for all coordinates $i_1(\bar{a}), \ldots, i_p(\bar{a})$ of $i(\bar{a})$ we have that $i_s(\bar{a}) \equiv 0 \bmod \mathfrak{p}$ for all $\mathfrak{p}$, i.e. $i_s(\bar{a}) \in \mathfrak{p}$ all $\mathfrak{p}$. Because R has no nilpotents this means that $i_s(\bar{a}) = 0$, $s = 1, \ldots, p$ and hence $\bar{a} = 0$ proving that $Ax = a$ is solvable over R.
Now let $y(\mathfrak{m})$ be any pregiven solution of $A(\mathfrak{m})y = a(\mathfrak{m})$ where $\mathfrak{m}$ is a maximal ideal of R. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & C & \to & R^n & \xrightarrow{A} & P & \to & 0 \\ & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 0 & \to & C(\mathfrak{m}) & \to & R(\mathfrak{m})^n & \to & P(\mathfrak{m}) & \to & 0 \end{array}$$

where C is the kernel of $A: R^n \to R^m$. The module P is also projective as the kernel of $R \to Q$. It follows that the lower sequence is also exact. Some diagram chasing, using that j' is surjective now readily proves the second assertion of the lemma.

Indeed let x_1 be any solution of $Ax = a$. Then $x_1(\mathfrak{m})$ is also a solution of $A(\mathfrak{m})y = a(\mathfrak{m})$. It follows that $A(\mathfrak{m})(x_1(\mathfrak{m}) - y(\mathfrak{m})) = 0$ so that by the exactness of the lower sequence of the diagram above $x_1(\mathfrak{m}) - y(\mathfrak{m}) \in C(\mathfrak{m})$. Now let $x_2 \in C$ be such that $j'(x_2) = x_1(\mathfrak{m}) - y(\mathfrak{m})$. Because $x_2 \in C = \mathrm{Ker}(A)$, $x = x_1 - x_2$ is also a solution of $Ax = a$. Moreover $x(\mathfrak{m}) = j(x) = j(x_1) - j(x_2) = x_1(\mathfrak{m}) - (x_1(\mathfrak{m}) - y(\mathfrak{m})) = y(\mathfrak{m})$, so that this solution does indeed specialize to the given one $\bmod\ \mathfrak{m}$.

If $\mathfrak{p} \subset R$ is prime, one argues exactly the same. The only extra difficulty is that $j': C \to C(\mathfrak{p})$ is not necessarily surjective. However, if $z \in C(\mathfrak{p})$ is any element, then there always is an $f \in R \setminus \mathfrak{p}$ such that z is in the image of $C_f \to C(\mathfrak{p})$.

3.9. Proof of theorem 3.5. Given the lemma, the proof of theorem 3.5 is entirely straightforward. Indeed one considers the linear map $A: R^k \to R$ given by $X \mapsto (XF-F'X, XG, H'X)$ where $k = n^2$ and X is a k-vector written as an n x n matrix. Here $\ell = n^2 + nm + np$. Now let $a \in R^\ell$ be the vector $(0,G',H)$. The constancy of dim $N(\mathfrak{p})$ = dim $L(\mathfrak{p})$ means that rank $A(\mathfrak{p})$ = constant. Now let $\mathfrak{p}_o$ be any prime ideal and $S(\mathfrak{p}_o)$ an invertible matrix over $R(\mathfrak{p}_o)$ taking $\Sigma(\mathfrak{p}_o)$ into $\Sigma'(\mathfrak{p}_o)$. Then $S(\mathfrak{p}_o)$ solves $A(\mathfrak{p}_o)y = a(\mathfrak{p}_o)$. So by the lemma there is a solution S over R_f for some $f \in R \setminus \mathfrak{p}_o$ of $Ax = a$ which moreover agrees with $S(\mathfrak{p}_o)$ mod $\mathfrak{p}_o$. Because $S(\mathfrak{p}_o)$ is invertible S is invertible over $R_{ff'}$ for some suitable $f' \in R \setminus \mathfrak{p}_o$.

3.10. Examples. It does not appear that the condition that the dimension of the stabilizer subgroups N(q) remains constant as q varies has much to do with conditions which seem systemtheoretically more natural like rank R(F(q),G(q)) is constant. Consider for example the family

$$\Sigma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & \sigma\end{pmatrix}, (0,2)\right)$$

For this family over $\mathbb{R}$ one has rank(R(F(q)),G(q)) = 1 = rank(Q(F(σ),H(σ))) for all $\sigma \in \mathbb{R}$, but dim N(σ) = 1 if σ = 1 and dim N(σ) = 0 otherwise. On the other hand the family

$$\Sigma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0 & 1\\\sigma & 1\end{pmatrix}, (1,0)\right)$$

has dim N(σ) = 0 everywhere but rank(R(F(σ),G(σ))) = 2 if $\sigma \neq 0$ and = 1 if σ = 0 (and rank(Q(F(σ), H(σ)) = 2 everywhere.

4. CONCLUSIONS.

The main questions studied in this paper were:

(1) Given two families of system Σ and Σ' which are pointwise isomorphic. Are they then also isomorphic as families?

(2) Given two families of systems Σ and Σ' over Q which are pointwise isomorphic over Q or some dense subset Z of Q. What can be said about the relation between $\Sigma(q)$ and $\Sigma'(q)$ at the points of $Q \setminus Z$.

Question (1) received a positive answer which specializes to a theorem of Wasow's [19] for holomorphic families of matrices under similarity. It seems also likely that the theorem is best possible in the sense that if Σ is a family such that dim N(q) is not constant then there is a family Σ' which is pointwise isomorphic to Σ everywhere but not isomorphic as families in any neighbourhood of a point q where dim N(q) suddenly increases. As to question (2), they are definite relations between $\Sigma(q)$ and $\Sigma'(q)$ if either Σ or Σ' is cr or co in a neighbourhood of q. If not than a number of examples show that the ways in which a family of systems can degenerate do not depend only on the isomorphism classes of the systems involved but also on the systems themselves (apart from the subquotients which are recoverable from the transferfunctions (cf. also [7,11]). Thus one has here the usual scaling and singular perturbation phenomena. It remains to construct local versal deformation of non cr and non co systems.

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