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ON LIE ALGEBRAS AND FINITE DIMENSIONAL FILTERING

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Abstract

A Lie algebra $L(\Sigma)$ can be associated with each nonlinear filtering problem, and the realizability or, better, the representability of $L(\Sigma)$ or quotients of $L(\Sigma)$ by means of vector fields on a finite dimensional manifold is related to the existence of finite dimensional recursive filters. In this paper, the structure and representability properties of $L(\Sigma)$ are analyzed for several interesting and/or well known classes of problems. It is shown that, for certain nonlinear filtering problems, $L(\Sigma)$ is given by the Weyl algebra

$W_n = \mathbb{R} < x_1,\dots,x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} >$. It is proved that neither W_n nor any quotient of W_n can be realized with C^∞ or analytic vector fields on a finite dimensional manifold, thus suggesting that for these problems, no statistic of the conditional density can be computed with a finite dimensional recursive filter. For another class of problems (including bilinear systems with linear observations), it is shown that $L(\Sigma)$ is a certain type of filtered Lie algebra. The algebras of this class are of a type which suggest that "sufficiently many" statistics are exactly computable. Other examples are presented, and the structure of their Lie algebras is discussed.

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I. Introduction

This paper is motivated by the problem of recursively filtering the state x_t of a nonlinear stochastic system, given the past observations $z^t = \{z_s, 0 \le s \le t\}$. The systems we consider satisfy the Ito stochastic differential equations

$$\begin{aligned} dx_t &= f(x_t)dt + G(x_t)dw_t \\ dz_t &= h(x_t)dt + R_t^{\frac{1}{2}} dv_t \end{aligned} \tag{Σ}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $z \in \mathbb{R}^p$, w and v are independent unit variance Wiener processes, and $R > 0$. The optimal (minimum-variance) estimate of x_t is of course the conditional mean $\hat{x}_t \triangleq E[x_t | z^t]$ (also denoted $\hat{x}_{t|t}$ or $E^t[x_t]$); $\hat{x}_t$ satisfies the (Ito) stochastic differential equation [1]-[3]

$$d\hat{x}_t = \hat{f}(x_t) - (\widehat{x_t h^T} - \hat{x}_t \hat{h}^T) R^{-1}(t) \hat{h} dt + (\widehat{x_t h^T} - \hat{x}_t \hat{h}^T) R^{-1}(t) dz_t \tag{1.1}$$

where ^ denotes conditional expectation given z^t and h denotes $h(x_t)$. The conditional probability density $p(t,x)$ of x_t given z^t itself (we will assume that $p(t,x)$ exists) satisfies the stochastic partial differential equation [3],[4]

$$dp(t,x) = L p(t,x) dt + (h(x) - \hat{h}(x))^T \; R^{-1}(t)(dz_t - \hat{h}(x)dt) p(t,x) \tag{1.2}$$

where

$$L(\cdot) = - \sum_{i=1}^{n} \frac{\partial(\cdot f_i)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2(\cdot (GG^T)_{ij})}{\partial x_i \partial x_j} \tag{1.3}$$

is the forward diffusion operator.

Notice that the differential equation (1.1) is in general both infinite dimensional and nonrecursive (because of the occurrence of the expectations $\hat{f}$, $\widehat{xh^T}$, and $\hat{h}$). Equation (1.2) is recursive but of course still infinite dimensional. Aside from the linear-Gaussian case in which the Kalman filter is optimal, there are very few known cases in which the conditional mean, or indeed <u>any</u> nonzero statistic of the conditional distribution, can be computed with a <u>finite dimensional recursive</u> filter (a number of these are summarized in [5]). More precisely, a <u>finite dimensional recursive filter</u> is a stochastic differential equation driven by the observations of the form

$$d\eta_t = a(\eta_t)dt + \sum_{i=1}^{p} b_i(\eta_t)dz_{it}, \tag{1.4}$$

where η evolves on a <u>finite dimensional</u> manifold and a and b_i are sufficiently smooth to insure existence and uniqueness (these conditions will be strengthened later). The conditional statistic $E[c(x_t)|z^t]$ is said to be <u>finite dimensionally computable</u> (FDC) if it can be computed "pointwise" as a function of the state of a finite dimensional recursive filter:

$$\hat{c}(x_t) \triangleq E[c(x_t)|z^t] = \gamma(\eta_t). \tag{1.5}$$

As a practical matter, it is also useful to require that the combined estimator (1.4)-(1.5) yield a statistic $\hat{c}(x_t)$ which is a continuous function of z; we will comment on this later in this section.

Recently, Brockett [6],[7] and Mitter [8],[9] have shown that Lie algebras play an important role in nonlinear recursive estimation theory; the approach of Brockett [6] is the following. Instead of studying the equation (1.2) for the conditional density, we consider the Zakai equation for an unnormalized conditional density $\rho(t,x)$ [10]:

$$d\rho(t,x) = L\rho(t,x)dt + \sum_{i=1}^{p} h_i(x)\rho(t,x)dz_{it} \tag{1.6}$$

where z_i and h_i are the i^{th} components of z and h, and $\rho(t,x)$ is related to (t,x) by the normalization

$$p(t,x) = \rho(t,x)\cdot(\int\rho(t,x)dx)^{-1}. \tag{1.7}$$

The Zakai equation (1.6) looks much simpler than (1.2); indeed, (1.6) is an (infinite dimensional) bilinear differential equation [11] in ρ, with z considered as the input. This is the first indication (given work on the roles of Lie algebras in solving finite dimensional bilinear equations [32], [33]) that the Lie algebraic and differential geometric techniques developed for finite dimensional systems of this type may be brought to bear here. Modulo some conjectured infinite dimensional extensions of some known results in the finite dimensional case (to be discussed below) this can be made more precise as follows: <u>suppose</u> that, for some given initial density, some statistic of the conditional distribution of x_t given z^t can be calculated with a finite dimensional recursive estimator of the form (1.4)-(1.5), where a, b_i, and γ are C^∞ or analytic. Of course, this statistic can also be obtained from $\rho(t,x)$ by

$$\hat{c}(x_t) = \int c(x)\rho(t,x)dx(\int\rho(t,x)dx)^{-1}. \tag{1.8}$$

For the rest of the development, it is more convenient to write (1.4) and (1.6) in Fisk-Stratonovich form (so that they obey the ordinary rules of calculus and so that Lie algebraic calculations involving differential operators can be performed as usual):

$$d\eta_t = \tilde{a}(\eta_t)dt + \sum_{i=1}^{p} b_i(\eta_t)dz_{it}$$

$$d\rho(t,x) = [L - \frac{1}{2}\sum_{i=1}^{p} h_i^2(x)]\rho(t,x)dt + \sum_{i=1}^{p} h_i(x)\rho(t,x)dz_{it} \qquad (1.10)$$

where the i^{th} component $\tilde{a}_i(\eta) = a_i(\eta) - \frac{1}{2}\sum_{j,k} b_{jk}(\eta)\frac{\partial b_{ik}}{\partial \eta_j}(\eta)$ (here b_{jk} is the k^{th} component of b_j).

The two systems (1.9),(1.5) and (1.10),(1.8) are thus two representations of the same mapping from "input" functions z to "outputs" $\hat{c}(x_t)$: (1.10),(1.8) via a bilinear infinite dimensional state equation, and (1.9),(1.5) via a nonlinear finite dimensional state equation. Motivated by the results of [12],[13] for finite dimensional state equations, the major thesis of [6] is that, under appropriate hypotheses, the Lie algebra F generated by $\tilde{a},b_1,\ldots,b_p$ (under the commutator $[a,b] = \frac{\partial a}{\partial \eta} b - \frac{\partial b}{\partial \eta} a$) should be a homomorphic image (quotient) of the Lie algebra $L(\Sigma)$ generated by $e_0 = L - \frac{1}{2}\sum_{i=1}^{p} h_i^2(x)$ and $e_i = h_i(x)$, i=1,...,p (under the commutator $[e_0,e_i] = e_0e_i - e_ie_0$), with $e_0 \to \tilde{a}$ and $e_i \to b_i$, i=1,...,p. On the other hand, if there is a homomorphism ϕ of $L(\Sigma)$ onto a Lie algebra generated by p+1 complete vector fields $\tilde{a},b_1,\ldots,b_p$, on a finite dimensional manifold, then this is an indication (possibly via appropriate globalized and/or integrated infinite dimensional generalizations of some results of [34],[35]) that some conditional statistic may be computable by an estimator of the form (1.9),(1.5). It is not known in what generality such results are valid, especially for cases in which $L(\Sigma)$ is infinite dimensional, and much work remains to be done (the fact that existence of a finite dimensional filter implies the existence of a Lie algebra homomorphism <u>has</u> been made rigorous for a class of estimation problems, including the cubic sensor discussed in Section II, in [36]). Moreover, it <u>is</u> clear (among others, from a number of examples discussed below) that there is a strong relationship in general between the structure of $L(\Sigma)$ and the existence of finite dimensional filters. In this paper, we discuss the properties of $L(\Sigma)$ for some interesting classes of examples. These Lie algebraic calculations give some new insights into certain nonlinear estimation problems and guidance in the search for finite dimensional estimators.

If $L(\Sigma)$ is finite dimensional (this seems to occur only in very special cases [9],[37]), a finite dimensional estimator can in some cases be constructed by integrating the Lie algebra representation [9]. Indeed, if $L(\Sigma)$ or any of its quotients is finite dimensional, then by Ado's Theorem [27, p. 202] this Lie algebra has a faithful finite dimensional representation; thus it can be realized with linear vector fields on a finite dimensional manifold, which may result in a bilinear filter computing some nonzero statistic (see, e.g., [16] and [26] for examples). However, actually computing the mapping from $\rho(t,x)$ to $\hat{c}(x_t)$ (i.e., deciding which statistic the filter computes) is a difficult problem from this point of view; at the moment at least, one must usually use other, more direct, methods, to actually construct this mapping or to derive the filter for a particular conditional statistic (see, e.g., [14]-[17]). Also, just a Lie algebra homomorphism from $L(\Sigma)$ to a Lie algebra of vector fields is not enough. In addition to the homomorphism of Lie algebras, one needs compatibility conditions in terms of isotropy subalgebras [34],[35], or equivalently, in terms of the natural representations of the Lie algebras operating on the spaces of functions on the manifolds involved. Even if $L(\Sigma)$ or its quotients are infinite dimensional, it is still possible that these Lie algebras can be realized by nonlinear vector fields on a finite dimensional manifold. Conditions under which this can be done is an unsolved problem in general;

we prove in Section II that this is <u>not</u> possible for certain classes of Lie algebras. As an almost totally trivial example that two vector fields on a finite dimensional manifold <u>can</u> generate an infinite dimensional Lie algebra, consider the vector fields $a = x^2 \frac{\partial}{\partial x}$ and $b = x^3 \frac{\partial}{\partial x}$ on a one-dimensional manifold; it is easy to see that a and b generate the infinite dimensional Lie algebra of vector fields of the form $x^2 p(x) \frac{\partial}{\partial x}$, where p is a polynomial.

If a statistic $\hat{c}(x_t)$ is finite dimensionally computable, the Lie algebraic approach also gives some insight into the continuity of the estimator. Since there is a Lie algebra homomorphism as discussed above, the vector fields $b_1,\ldots,b_p$ are homomorphic images of the operators $e_1,\ldots,e_p$ which all commute with each other (these are just multiplication operators). Thus $b_1,\ldots,b_p$ also commute, and the results of [18] imply that the filter (1.9) represents a continuous map (in the C^0 and L_p topologies) from the space of "inputs" z to the solutions η. Hence, the estimator (1.9),(1.5) gives a continuous map from z to $\hat{c}(x_t)$; this is a very useful property, indicating the "robustness" of the filter (see also [19],[20]).

Brockett and Clark [38] used this approach to study the estimation of a finite state Markov process observed in additive Brownian motion; the Lie algebraic approach led to the discovery of new low dimensional filters for the conditional distribution, even in some cases when the number of states was arbitrarily large. And even in the extremely well known case of linear systems (Kalman filter), the Lie algebraic approach gives an additional result in that it tells us how to propagate a non-Gaussian initial density [2]. In this case the Lie algebra is finite dimensional; in fact, one finds higher dimensional relatives of the so-called oscillator algebra of some fame in physics (incidentally, this is no accident [9]). In [21], a similar analysis is carried out for an example of the class of estimation problems considered in [14]-[16]; for this class of nonlinear stochastic systems, the conditional mean (and all conditional moments) of x_t given z^t are finite dimensionally computable. For this example, the Lie algebra $L(\Sigma)$ is infinite dimensional but has many finite dimensional quotients corresponding to the Lie algebras of the finite dimensional filters; these are analyzed in detail in [21]. These last two examples, as well as the example of Beneš [17], are special cases of the class considered in Section III.

In Section II, we consider estimation problems for which $L(\Sigma)$ is the Weyl algebra W_n. A number of examples are given and useful properties of the Weyl algebra are derived; some of these results have been obtained independently by Mitter [9]. The major results of Section II are proofs that neither W_n nor any quotient of W_n can be realized by vector fields with either C^∞ or formal power series coefficients on a finite dimensional manifold; this suggests that for these problems, no statistic of the conditional density can be computed with a finite dimensional recursive filter. This does not imply that there will not be appropriate approximation methods. Possibly partial homomorphisms of Lie algebras [39] of $L(\Sigma)$ into Lie algebras of vector fields will play a role here. Also "deformations of algebras" techniques [40]-[42] suggest a possible approach to approximate methods. For example, the Lie algebra of $dx_t = dw_t$, $dz_t = (x + \varepsilon x^3)dt + dv_t$ is W_1 for all $\varepsilon \neq 0$, but mod ε^n this algebra is finite dimensional for all n [43]. Finally, in Section IV we present another estimation problem with an interesting Lie algebraic structure and discuss the possible implications of this structure.

II. The Weyl Algebras W_n

The Weyl algebra W_n [22],[23, Chapter 1] is the algebra of all polynomial differential operators; i.e., $W_n = \mathbb{R}\langle x_1,\ldots,x_n; \frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\rangle$. A basis for W_n consists of all monomial expressions

$$e_{\alpha,\beta} \triangleq x^\alpha \frac{\partial^\beta}{\partial x^\beta} \triangleq x_1^{\alpha_1} \cdots x_n^{\alpha_n} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} \tag{2.1}$$

where α,β range over all multiindices $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\beta = (\beta_1,\ldots,\beta_n)$, $\alpha,\beta \in \mathbb{N} \cup \{0\}$ (the non-negative integers). W_n is a Lie algebra under the Lie bracket; as an example, we state the general formula for W_1:

$$\begin{aligned}[x^i \frac{\partial^j}{\partial x^j}, x^k \frac{\partial^\ell}{\partial x^\ell}] &= \sum_{r=1}^{j} \binom{j}{r}\binom{k}{r} r!\, x^{i+k-r} \frac{\partial^{j+\ell-r}}{\partial x^{j+\ell-r}} \\ &\quad - \sum_{s=1}^{\ell} \binom{\ell}{s}\binom{i}{s} s!\, x^{i+k-s} \frac{\partial^{j+\ell-s}}{\partial x^{j+\ell-s}}\end{aligned} \tag{2.2}$$

where $\binom{j}{r} = \frac{j!}{(j-r)!r!}$ is the binomial coefficient and we have used the convention that $\binom{j}{r} = 0$ if $r < 0$ or $j < r$. As is easily checked, the center of W_n (i.e., the ideal of all elements $Z \in W_n$ such that $[X,Z] = 0$ for all $X \in W_n$) is the one-dimensional space $\mathbb{R}\cdot 1$ with basis $\{1\}$ [22, p. 148]. We next prove the simplicity of the Lie algebra $W_n/\mathbb{R}\cdot 1$; this is of course stronger than showing that W_n is simple as an associative algebra [22, p. 148]. Our proof follows that of Avez and Heslot [24] for the Lie algebra P_n of polynomials under the Poisson bracket. A number of the following results are common to P_n and W_n, but these two Lie algebras are <u>not</u> isomorphic (this is basically because the expression in P_n corresponding to (2.2) would retain only the terms for r=1 and s=1). Hence, one must be careful in literally interpreting results proved for P_n in the context of W_n [30].

Theorem 2.1: The Lie algebra $W_n/\mathbb{R}\cdot 1$ is simple; i.e., it has no ideals other than $\{0\}$ and $W_n/\mathbb{R}\cdot 1$. Equivalently, the only ideals of W_n are $\{0\}$, $\mathbb{R}\cdot 1$, and W_n.

Proof: Suppose I is an ideal of W_n which contains a nonconstant element $X = \sum c_{\alpha\beta}\, x^\alpha \frac{\partial^\beta}{\partial x^\beta}$. Since commuting with x_i reduces β_i by 1 and commuting with $\frac{\partial}{\partial x_i}$ reduces α_i by 1, repeated commutation implies that an element of the form x_i or $\frac{\partial}{\partial x_j}$ is in I. Since every element $Y \varepsilon W_n$ can be obtained by commutation of x_i (or $\frac{\partial}{\partial x_j}$) with another element of W_n, this shows that $I = W_n$.

This theorem basically shows that if W_n occurs as the Lie algebra $L(\Sigma)$ for some estimation problem, then either the unnormalized conditional density itself is finite dimensionally computable or no statistic at all is finite dimensionally computable. The next two theorems complete the argument by showing that in fact neither W_n nor its quotients can be realized by vector fields on a finite dimensional manifold.

Let $\hat{V}_m$ be the Lie algebra of vector fields $\hat{V}_m \overset{\Delta}{=} \{\sum_{i=1}^{m} f_i(x_1,\ldots,x_m)\frac{\partial}{\partial x_i}\}$ with (formal) power series coefficients $f_i \varepsilon \mathbb{R}[[x_1,\ldots,x_m]]$, and let $V(M)$ be the Lie algebra of C^∞-vector fields on a C^∞-manifold M. The proofs of the following theorems are contained in Appendix A.

Theorem 2.2: Fix $n \neq 0$. Then there are no non-zero homomorphisms from W_n to $\hat{V}_m$ or from $W_n/\mathbb{R}\cdot 1$ to $\hat{V}_m$ for any m.

Theorem 2.3: Fix $n \neq 0$. Then there are no non-zero homomorphisms from W_n to $V(M)$ or $W_n/\mathbb{R}\cdot 1$ to $V(M)$ for any finite dimensional C^∞-manifold M.

These results suggest (assuming the appropriate analogs of the results of [6],[12]) that if a system Σ has estimation algebra $L(\Sigma) = W_n$ for some n, then neither the conditional density of x_t given z^t nor any nonzero statistic of the conditional density can be computed with a finite dimensional filter of the form (1.9) with a and b C^∞ or analytic. This is indeed the case for the cubic sensor (Example 2.1) [36] (as was mentioned before). We will give several examples of such systems, but first we present a general method for showing that $L(\Sigma) = W_n$.

Theorem 2.4: The Lie algebra W_n is generated by the elements

$x_i, \dfrac{\partial^2}{\partial x_i^2}, x_i^2 \dfrac{\partial}{\partial x_i}$, i=1,...,n; and $x_i x_{i+1}$, i=1,...,n-1.

Proof (similar to that of [24] for Poisson brackets): Let L be the Lie algebra generated by these elements. Since $[x_i^2 \frac{\partial}{\partial x_i}, x_i^k] = kx_i^{k+1}$, L contains x_i^k, $k \geq 1$. Now, $[\frac{\partial^2}{\partial x_i^2}, x_i] = \frac{\partial}{\partial x_i}$ and $[\frac{\partial}{\partial x_i}, x_i] = 1$. Also,

$$\left[\frac{\partial^2}{\partial x_i^2}, x_i^k (\frac{\partial}{\partial x_i})^\ell \right] = 2kx_i^{k-1} (\frac{\partial}{\partial x})^{\ell+1} + k(k-1)x_i^{k-2} (\frac{\partial}{\partial x})^\ell, \quad k \geq 2; \qquad (2.3)$$

with $\ell = 0$, (2.3) implies that $x_i^k \frac{\partial}{\partial x} \varepsilon L$, $k \geq 0$. Then by induction (2.3) implies that $x_i^k (\frac{\partial}{\partial x_i})^\ell \varepsilon L$ for all $k,\ell \geq 0$. Notice that $\left[x_i \frac{\partial^2}{\partial x_i^2}, x_i x_{i+1}\right] = 2x_i x_{i+1} \frac{\partial}{\partial x_i}$, and commuting this with $x_i^k (\frac{\partial}{\partial x_i})^\ell$ gives $x_{i+1} \cdot \mathbb{R}\langle x_i, \frac{\partial}{\partial x_i}\rangle \varepsilon L$. Repeated commutation with $x_{i+1}^2 \frac{\partial}{\partial x_{i+1}}$ and $(\frac{\partial}{\partial x_{i+1}})^2$ yields (as above) $\mathbb{R}\langle x_i, x_{i+1}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{i+1}}\rangle$. By induction, we have that $L = W_n$.

Theorem 2.4 provides a relatively systematic method for showing that $L(\Sigma) = W_n$ for a particular estimation problem: one need only show that by taking repeated Lie brackets of $L - \frac{1}{2}h^2$ and h, the generating elements of

W_n given in Theorem 2.4 are obtained. Notice that if n=1, the generating elements are x, $\frac{\partial^2}{\partial x^2}$, and $x^2 \frac{\partial}{\partial x}$. There is a "dual" result obtained by interchanging x_i and $\frac{\partial}{\partial x_i}$ in Theorem 2.4. Some interesting examples are the following.

<u>Example 2.1 (the cubic sensor problem [9],[25])</u>: Consider the system

$$dx_t = dw_t$$
$$dz_t = x_t^3 dt + dv_t$$

The Lie algebra $L(\Sigma)$ is generated by the operators

$$e_0 = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}x^6, \qquad e_1 = x^3.$$

We can compute a sequence of Lie brackets to obtain a sequence of elements $e_i \in L(\Sigma)$, eventually obtaining the desired generators of W_n:

$$[e_0,e_1] = 3x^2 \frac{\partial}{\partial x} + 3x \qquad \Rightarrow \qquad e_2 = x^2 \frac{\partial}{\partial x} + x$$

$$\mathrm{ad}_{e_2}^k e_1 = 3 \cdot 4 \cdots (k+2)x^{k+3} \qquad \Rightarrow \qquad x^k \in L(\Sigma),\ k \geq 3$$

(where $\mathrm{ad}_{e_2}^0 e_1 = e_1$ and $\mathrm{ad}_{e_0}^{k+1} e_1 = [e_0, \mathrm{ad}_{e_0}^k e_1]$). Combined with e_0, $x^6 \in L(\Sigma)$ implies that $e_3 = \frac{\partial^2}{\partial x^2} \in L(\Sigma)$. Continuing,

$$[e_3,e_2] = 4x \frac{\partial^2}{\partial x^2} + 4\frac{\partial}{\partial x} \qquad \Rightarrow \qquad e_4 = x\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}$$

$$[e_4,e_2] = 3x^2 \frac{\partial^2}{\partial x^2} + 6x\frac{\partial}{\partial x} + 1 \qquad \Rightarrow \qquad e_5 = 3x^2 \frac{\partial^2}{\partial x^2} + 6x\frac{\partial}{\partial x} + 1$$

$$[e_4,e_1] = 6x^3 \frac{\partial}{\partial x} + 9x^2 \qquad \Rightarrow \qquad e_6 = 2x^3 \frac{\partial}{\partial x} + 3x^2$$

$$[e_3,e_6] = 12x^2 \frac{\partial^2}{\partial x^2} + 24x\frac{\partial}{\partial x} + 6,$$

which combined with e_5 implies that $e_7 = 1$ and $e_8 = x^2 \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x}$ are in $L(\Sigma)$. A few more calculations will complete the demonstration:

$$[e_3, e_8] = 4x \frac{\partial^3}{\partial x^3} + 6 \frac{\partial^2}{\partial x^2} \quad \Rightarrow \quad e_9 = x \frac{\partial^3}{\partial x^3}$$

$$[e_1, e_8] = -6x^4 \frac{\partial}{\partial x} - 12x^3 \quad \Rightarrow \quad e_{10} = x^4 \frac{\partial}{\partial x}$$

$$[e_2, e_9] = -5x^2 \frac{\partial^3}{\partial x^3} - 9x \frac{\partial^2}{\partial x^2} \quad \Rightarrow \quad e_{11} = 5x^2 \frac{\partial^3}{\partial x^3} + 9x \frac{\partial^2}{\partial x^2}$$

$$[e_3, e_{10}] = 8x^3 \frac{\partial^2}{\partial x^2} + 12x^2 \frac{\partial}{\partial x} \quad \Rightarrow \quad e_{12} = 2x^3 \frac{\partial^2}{\partial x^2} + 3x^2 \frac{\partial}{\partial x}$$

$$[e_3, e_{12}] = 12x^2 \frac{\partial^3}{\partial x^3} + 24x \frac{\partial^2}{\partial x^2} + 6 \frac{\partial}{\partial x} \quad \Rightarrow \quad e_{13} = 2x^2 \frac{\partial^3}{\partial x^3} + 4x \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}$$

Now e_{13}, e_{11}, and e_4 are all linear combinations of the elements $x^2 \frac{\partial^3}{\partial x^3}$, $x \frac{\partial^2}{\partial x^2}$, and $\frac{\partial}{\partial x}$, and the coefficient matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 5 & 9 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

is nonsingular. It follows that $L(\Sigma)$ contains $e_{14} = \frac{\partial}{\partial x}$, $e_{15} = x \frac{\partial^2}{\partial x^2}$, and $e_{16} = x^2 \frac{\partial^3}{\partial x^3}$. Finally,

$$[e_{14}, e_1] = 3x^2 \quad \Rightarrow \quad e_{17} = x^2$$

$$[e_{14}, e_{17}] = 2x \quad \Rightarrow \quad e_{18} = x$$

which combined with e_2 gives $x^2 \frac{\partial}{\partial x} \in L$; thus by Theorem 2.4, $L(\Sigma) = W_1$.

Analogous computation of selected Lie brackets and the use of Theorem 2.4 yields similar results for the following examples.

Example 2.2: For the system

$$dx_t = x_t^3 dt + dw_t$$

$$dz_t = x_t dt + dv_t,$$

$L(\Sigma)$ is generated by $\frac{1}{2}\frac{\partial^2}{\partial x^2} - x^3\frac{\partial}{\partial x} - \frac{7}{2}x^2$ and x, and $L(\Sigma) = W_1$.

Example 2.3 (mixed linear-bilinear type): Consider the system with state equations

$$dx_t = dw_{1t}$$

$$dy_t = x_t dt + x_t dw_{2t}$$

with observations

$$dz_t = y_t dt + dv_t.$$

$L(\Sigma)$ is generated by $\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2\frac{\partial^2}{\partial y^2} - x\frac{\partial}{\partial y} - \frac{1}{2}y^2$ and y; it is shown in Appendix B that $L(\Sigma) = W_2$. The same result is obtained if the $x_t dt$ term is absent in the y equation; in that case we have a multiple Wiener integral of Brownian motion observed in Brownian motion noise.

Example 2.4: Consider the system with state equations

$$dx_t = dw_t$$

$$dy_t = x_t^2 dt$$

and observations

$$dz_{1t} = x_t dt + dv_{1t}$$

$$dz_{2t} = y_t dt + dv_{2t}.$$

$L(\Sigma)$ is generated by $\frac{1}{2}\frac{\partial^2}{\partial x^2} - x^2\frac{\partial}{\partial y} - \frac{1}{2}x^2 - \frac{1}{2}y^2$, x, and y; it is easily shown that $L(\Sigma) = W_2$. This is the example studied in [21], but here we have the additional observation z_2; the relationship between these examples will be examined in the next section.

III. Pro-Finite Dimensional Filtered Lie Algebras

A Lie algebra L is defined to be a pro-finite dimensional filtered Lie algebra if L has a decreasing sequence of ideals $L = L_{-1} \supset L_0 \supset L_1 \supset \ldots$ such that

(a) $\cap L_i = 0$

(b) L/L_i is a finite dimensional Lie algebra for all i.

The terminology is somewhat analogous to that of pro-finite groups [28]; no completeness assumptions are made, however. Notice that (a) implies that there is an injection from L to $\oplus_i L/L_i$. In the context of the estimation problem, this would correspond to $L(\Sigma)$ having an infinite number of finite dimensional quotients; if each of these can be realized with a recursively filterable statistic (a plausible conjecture), then the injectivity of the map makes it reasonable to conjecture that these statistics represent some type of power series expansion of the conditional density. Of course, in addition to those discussed in Section I, other difficult technical questions such as moment determinacy will also be relevant here, but the structure of the Lie algebra should provide some guidance as to possible successful approaches to the problem and some insight into the structure of the resulting approximations.

Example 3.1 [21]: A simple example of the class considered in [14]-[16] is given by the state equations

$$dx_t = dw_t$$
$$dy_t = x_t^2 dt$$

and the observations

$$dz_t = x_t dt + dv_t$$

with x_0 Gaussian. The computation of $\hat{x}_t$ is of course straightforward by means of the Kalman filter; however, as shown in [14]-[16], all conditional moments of y_t can also be computed recursively with finite dimensional filters. $L(\Sigma)$ is generated by $e_0 = -x^2 \frac{\partial}{\partial y} + \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} x^2$ and $e_1 = x$; as shown in [21], a basis for $L(\Sigma)$ is given by e_0 and $\{x \frac{\partial^i}{\partial y^i}, \frac{\partial}{\partial x} \frac{\partial^i}{\partial y^i}, \frac{\partial^i}{\partial y^i} ; i=0,1,2,\ldots\}$. Defining L_i to be the ideal generated by $x \frac{\partial^i}{\partial y^i}$, $i=0,1,2,\ldots$, it is easy to see that $L(\Sigma)$ is a pro-finite dimensional filtered Lie algebra, and realizations of the $L(\Sigma)/L_i$ in terms of recursively filterable statistics are given in [21]. In addition, $L(\Sigma)$ is solvable [21].

A similar analysis for systems of the form of Example 3.1, with x_t^2 replaced by a general monomial x_t^p has also been done [31]; for $p > 2$, a similar but more complex Lie algebraic structure is exhibited. It is interesting to compare Example 3.1 with Example 2.4, which is the same except for the additional observation $dz_{2t} = y_t dt + dv_{2t}$; in that case $L(\Sigma) = W_2$, so that no conditional statistic can be computed <u>exactly</u> with a finite dimensional filter. However, it is probable that, due to the additional observation, a <u>suboptimal</u> approximate filter (such as the Extended Kalman Filter) for the conditional mean of y_t will result in lower mean-square error than the <u>optimal</u> filter which computes $\hat{y}_t$ in Example 3.1. Thus some care must be taken in interpreting the Lie algebraic structure of a nonlinear estimation problem; this structure has direct implications on the <u>exact</u> computation of conditional statistics, but its implications on approximate filtering remains to be investigated.

Example 3.2 (degree increasing operators and bilinear systems): Consider a system of the form (Σ) (page 2), and suppose that f, G, and h are analytic with $f(0)=0$ and $G(0)=0$, so that the power series expansions of f and G around zero are of the form

$$f(x) = \sum_{|\alpha|\geq 1} f_\alpha x^\alpha, \qquad G(x) = \sum_{|\alpha|\geq 1} G_\alpha x^\alpha, \tag{3.1}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. It follows that

$$G(x)\,G'(x) = \sum_{|\alpha|\geq 2} \tilde{G}_\alpha(x)\, x^\alpha.$$

An example of such systems is the class of bilinear systems

$$\begin{aligned} dx_t &= Ax_t + \sum_{i=1}^{p} B_i x_t dw_t^i \\ dz_t &= Cx_t dt + dv_t \end{aligned} \tag{3.2}$$

Another example is

$$dx_t = x_t dt + \sin x_t \, dw_t$$

$$dz_t = h(x_t)dt + dv_t$$

with h analytic; in general, a wide variety of examples can be found.

Let $M = \mathbb{R}[[x_1,\ldots,x_n]]$ be the module of all (formal) power series in $x_1,\ldots,x_n$, and define the submodules

$$M_i = \{\sum a_\alpha x^\alpha \mid a_\alpha = 0 \text{ for } |\alpha| \leq i\}, \quad i=0,1,2,\ldots,$$

so that, e.g., M_0 consists of those power series with zero constant term. If Σ is a system satisfying the condition (3.1), it follows that for all i, the forward diffusion operator (1.3) satisfies

$$L M_i \subset M_i ;$$

hence

$$(L - \frac{1}{2}h^2(x))\, M_i \subset M_i$$

and of course

$$h(x)M_i \subset M_i .$$

Since the two generators of $L(\Sigma)$ thus leave M_i invariant, it is obvious that $L(\Sigma)M_i \subset M_i$; thus, each element of $L(\Sigma)$ can only increase (or leave the same) the degree of the first term in the power series expansion of an element of M. Let

$$L_i = \{X \varepsilon L(\Sigma) | XM \subset M_{i+1}\}, \quad i = -1,0,1,2,\dots.$$

Then L_i is an ideal in $L(\Sigma)$ and we have an induced representation

$$\rho_i : L/L_i \rightarrow \mathrm{End}(M/M_{i+1}).$$

Because M/M_{i+1} is finite dimensional, so is L/L_i, since ρ_i is injective (by definition of L_i). It is obvious that $\cap L_i = \{0\}$; thus $L(\Sigma)$ is a pro-finite dimensional filtered Lie algebra, with filtration L_i. One additional structural feature of this filtration is that L_0/L_i is a nilpotent Lie algebra for i=1,2,...; also, L_i/L_{i+1} is abelian for all $i \geq 0$. The nilpotency of the L_0/L_i is a property also possessed by the filtration of Example 3.1.

Since many systems can be well approximated by bilinear ones, these results may have important implications for approximate nonlinear filtering. We close this section with two interesting examples of this class; the first is a bilinear system of the form (3.2), but in which some elements of A are also unknown and must be estimated. The second is an angle modulation problem.

Example 3.3 (Bilinear system with unknown parameter): The simplest example of this type is

$$
\begin{aligned}
dx_t &= \alpha_t x_t dt + x_t dw_t \\
d\alpha_t &= 0 \\
dz_t &= x_t dt + dv_t
\end{aligned}
$$

Here both the state x_t and parameter α are to be estimated recursively. The Lie algebra $L(\Sigma)$ is generated by $\frac{1}{2} x^2 \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} + 1 - \alpha x \frac{\partial}{\partial x} - \alpha - \frac{1}{2} x^2$ and x. Both of these operators are "degree increasing" when operating on $\mathbb{R}[[x,\alpha]]$, so $L(\Sigma)$ is a pro-finite dimensional filtered Lie algebra.

Example 3.4 (Angle modulation without process noise): Consider the problem of observing

$$
\begin{aligned}
dz_{1t} &= \sin(\omega t + \theta) dt + dv_{1t} \\
dz_{2t} &= \cos(\omega t + \theta) dt + dv_{2t}
\end{aligned}
$$

where ω and θ are constant random variables to be estimated. To place this problem in the present framework, we have the three state equations

$\dot{\omega} = 0$

$\dot{\theta} = 0$

$\dot{t} = 1$

The Lie algebra $L(\Sigma)$ is generated by $e_0 = \frac{\partial}{\partial t} - \frac{1}{2}$, $e_1 = \sin(\omega t + \theta)$, and $f_1 = \cos(\omega t + \theta)$. It is easily shown that $L(\Sigma)$ has basis elements $e_0, e_i = \omega^i \sin(\omega t + \theta)$, $f_i = \omega^i \cos(\omega t + \theta)$, $i=0,1,2,\ldots.$ The nonzero commutation relations are $[e_0, e_i] = f_{i+1}$, $[e_0, f_i] = -e_{i+1}$. Hence, $L(\Sigma)$ is a pro-finite dimensional filtered Lie algebra, with filtration $\{L_i\}$, where L_i is the ideal generated by e_{i+1} and f_{i+1}, $i=0,1,2,\ldots.$ Phase-lock loops are often used for filtering problems such as this, but the form of the <u>optimal</u> estimator is unknown. This calculation suggests that an infinite number of statistics of the conditional density may be finite dimensionally computable.

IV. A Final Example

There are other filtering problems which do not fall into the above classes, but which have interesting Lie algebraic structures with possible implications for finite dimensional filtering. One example is the following.

Example 4.1: The system of this example is

$$dx_t = dw_t$$

$$dy_t = e^{x_t} dt$$

$$dz_t = x_t dt + dv_t$$

This does not quite fall into the class discussed in [14]-[16] (as does Example 3.1), since the y equation contains e^{x_t} rather than a polynomial in x_t. The conditional expectation $\hat{x}_t$ is again computed by the Kalman filter, but the computation of $\hat{y}_t$ is much more difficult. The Lie algebra $L(\Sigma)$ is generated by $\frac{1}{2}\frac{\partial^2}{\partial x^2} - e^x \frac{\partial}{\partial y} - \frac{1}{2}x^2$ and x; the structure of $L(\Sigma)$ is as follows. It has as basis the elements

$$x, \frac{\partial}{\partial x}, 1, \frac{\partial^2}{\partial x^2} - x^2; \quad E_{ijk} = x^i e^{jx} \frac{\partial^j}{\partial y^j}\frac{\partial^k}{\partial x^k}, \quad j \geq 1, \quad i,k \geq 0.$$

Let I_n $(n \geq 1)$ be the subspace spanned by E_{ijk} with $j \geq n$, and let I_n' be the subspace spanned by 1 and E_{ijk} with $j \geq n$. Then the only ideals of $L(\Sigma)$ are I_n, I_n', $\mathbb{R}\cdot 1$, and $I_1 \oplus \mathbb{R}\cdot 1 \oplus \mathbb{R}\frac{\partial}{\partial x} \oplus \mathbb{R}\cdot x$. The quotients I_n/I_{n+1} are infinite dimensional and abelian, so that $L(\Sigma)/I_n$ are successive extensions of the oscillator algebra $L(\Sigma)/I_1$ (the algebra of the linear filtering problem [6]-[9]) by infinite dimensional abelian kernels.

Also, $\bigcap_n I_n = \{0\}$. Due to this structure, it seems unlikely that there will be injections from $L(\Sigma)$ itself into $\hat{V}_m$; however, it does seem possible that the $L(\Sigma)/I_n$ are realizable as (infinite dimensional) Lie algebras of vector fields on some <u>finite</u> dimensional manifold.

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Appendix A

Proof of Theorems 2.2 and 2.3

A.1 Filtrations and Preliminary Results

Definition A.1: A Lie algebra L admits a filtration (or is a filtered Lie algebra) if there exists a sequence of subalgebras $L = L_{-1} \supset L_0 \supset L_1 \supset \ldots$ such that

$$\cap L_i = \{0\} \tag{A.1}$$

$$[L_i, L_j] \subset L_{i+j} \tag{A.2}$$

$$\dim (L_i/L_{i+1}) < \infty \ ; \quad i = -1, 0, 1, \ldots \tag{A.3}$$

Example A.1: A prime example of filtered Lie algebras are the $\hat{V}_n$. The filtration is defined as follows: L_i consists of all vector fields $\sum c_{\alpha,j} x^{\alpha} \frac{\partial}{\partial x_j}$ with $c_{\alpha,j} = 0$ for all α with $|\alpha| \leq i$, where the norm of the multiindex $(\alpha_1, \ldots, \alpha_n)$ is $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Given a filtration $L_{-1} \supset L_0 \supset L_1 \supset \ldots$ on a Lie algebra L, we define a valuation function $v\colon L \to \mathbb{N} \cup \{0,-1\} \cup \{\infty\}$ by

$$v(x) = \max \{j \,|\, x \in L_j\}.$$

Properties (A.1) and (A.2) of the filtration translate into

$$v(x) = \infty \iff x = 0 \tag{A.4}$$

$$v([x,y]) \geq v(x) + v(y), \tag{A.5}$$

and the fact that the L_i are vector spaces implies that

$$v(ax+by) \geq \min\,(v(x),v(y)); \quad x,y \in L, a,b \in \mathbb{R} \tag{A.6}$$

and

$$\begin{aligned} &v(x+y) = v(x) \text{ if } v(x) < v(y) \\ &v(ax) = v(x) \quad \text{if } a \neq 0 \end{aligned} \tag{A.7}$$

In addition, we will need the following results concerning W_1. First, we have the formula

$$[\frac{\partial^n}{\partial x^n}, x^r] - [\frac{\partial^{n-1}}{\partial x^{n-1}}, x^r \frac{\partial}{\partial x}] - r[\frac{\partial^{n-1}}{\partial x^{n-1}}, x^{r-1}] = rx^{r-1}\frac{\partial^{n-1}}{\partial x^{n-1}}; \tag{A.8}$$

this is easily proved by using (2.2) and formulas for the binomial coefficients. The following lemma, which also follows by a straightforward application of (2.2), shows that $x^k \frac{\partial^\ell}{\partial x^\ell}$ is an "approximate eigenvector" of $x^t \frac{\partial^t}{\partial x^t}$.

Lemma A.1: Let $\ell < t \leq k \neq 1$ be natural numbers. Then there are a nonzero $c \in \mathbb{R}$ and $d_1,\ldots,d_{t-1} \in \mathbb{R}$ such that

$$[x^t \frac{\partial^t}{\partial x^t}, x^k \frac{\partial^\ell}{\partial x^\ell}] = cx^k \frac{\partial^\ell}{\partial x^\ell} + \sum_{i=1}^{t-1} d_i\, x^{k+i} \frac{\partial^{\ell+i}}{\partial x^{\ell+i}}$$

The proof of the next lemma is quite involved and is contained in Section A.3.

Lemma A.2: Suppose that $W_1 = L_{-1} \supset L_0 \supset L_1 \supset \ldots$ is a sequence of subalgebras of W_1 satisfying (A.2),(A.3), $\dim(W_1/L_2) < \infty$, and either $\cap L_i = \{0\}$ or $\cap L_i = \mathbb{R}\cdot 1$. Let v be the valuation function defined by the filtration. Then $v(x^n) \to \infty$ as $n \to \infty$.

A.2 Proof of Theorem 2.2

The proof will be carried out for W_1; the proof is virtually identical for $W_1/\mathbb{R}\cdot 1$, and the result is true a fortiori for W_n, since W_n is clearly isomorphic to the subalgebra of W_n consisting of expressions in x_1 and $\frac{\partial}{\partial x_1}$ only. Suppose that there is a nonzero homomorphism ϕ from W_1 to $\hat{V}_m$. Then W_1 has a filtration defined by the subalgebras $M_i \overset{\Delta}{=} \phi^{-1}(L_i)$, where $\{L_i\}$ is the filtration on $\hat{V}_m$ defined in Example A.1; let v be the corresponding valuation function on W_1. Since $\hat{V}_m/L_2$ is finite dimensional, so is W_1/M_2; thus Lemma A.2 implies that $v(x^i) \to \infty$ as $i \to \infty$. We claim it also follows that

$$v(x^{k+i} \frac{\partial^{\ell+i}}{\partial x^{\ell+i}} \to \infty \text{ as } i \to \infty, \tag{A.9}$$

and that this will lead to a contradiction.

First notice that

$$\left[\frac{\partial^2}{\partial x^2}, x^{k+i+2}\right] = 2(k+i+2)\, x^{k+i+1} \frac{\partial}{\partial x} + (k+i+2)(k+i+1)\, x^{k+i},$$

so that from (A.5)-(A.7) and the fact that $v(X) \geq -1$ for all $X \varepsilon W_1$,

$$v\,(x^{k+i+1} \frac{\partial}{\partial x}) \geq \min\{v(x^{k+i}),\ v\,[\frac{\partial^2}{\partial x^2}, x^{k+i+2}]\} \geq \min\{v(x^{k+i}), v(x^{k+i+2})-1\}. \tag{A.10}$$

Then taking $r = k+i+1$ and $n = \ell+i+1$ in formula (A.8) and using (A.10) yields

$$v(x^{k+i}\frac{\partial^{\ell+i}}{\partial x^{\ell+i}})$$

$$\geq \min\{v[\frac{\partial^{\ell+i+1}}{\partial x^{\ell+i+1}}, x^{k+i+1}],\ v[\frac{\partial^{\ell+i}}{\partial x^{\ell+i}}, x^{k+i+1}\frac{\partial}{\partial x}],\ v[\frac{\partial^{\ell+i}}{\partial x^{\ell+i}}, x^{k+i}]\}$$

$$\geq \min\{v(x^{k+i+1})-1,\ v(x^{k+i+1}\frac{\partial}{\partial x})-1,\ v(x^{k+i})-1\}$$

$$\geq \min\{v(x^{k+i+1})-1,\ v(x^{k+i+2})-1,\ v(x^{k+i})-1\}$$

which converges to ∞ as $i\to\infty$, proving (A.9).

Now choose $t_0 \in \mathbb{N}$ such that

$$v(x^t\frac{\partial^t}{\partial x^t}) \geq 1 \quad \text{for } t \geq t_0. \tag{A.11}$$

Choose any $k_0 \geq 1$ and consider the sequence $\{v(x^{k_0+\ell}\frac{\partial^\ell}{\partial x^\ell});\ \ell=0,1,2,\ldots\}$. Then because by (A.9) this sequence converges to ∞ there is for any ℓ_0 an $\ell_1 \geq \ell_0$ such that

$$v(x^{k_0+\ell_1+i}\frac{\partial^{\ell_1+i}}{\partial x^{\ell_1+i}}) > v(x^{k_0+\ell_1}\frac{\partial^{\ell_1}}{\partial x^{\ell_1}})\ ; \quad i \geq 1 \tag{A.12}$$

Take $\ell_0 = t_0+1$, choose ℓ_1 such that (A.12) holds, and take $t = \ell_1+1$.

Then we can apply Lemma A.1 with $t = \ell_1+1$, $\ell = \ell_1$, and $k = k_0+\ell_1$ (notice that the assumptions are satisfied). We find

$$[x^t\frac{\partial^t}{\partial x^t}, x^k\frac{\partial^\ell}{\partial x^\ell}] = c\,x^k\frac{\partial^\ell}{\partial x^\ell} + \sum_{i=1}^{t-1} d_i\ x^{k+i}\frac{\partial^{\ell+i}}{\partial x^{\ell+i}}$$

Because of (A.12), we have by (A.7) that

$$v\left(c x^k \frac{\partial^\ell}{\partial x^\ell} + \sum_{i=1}^{t-1} d_i \, x^{k+i} \frac{\partial^{\ell+i}}{\partial x^{\ell+i}}\right) = v\left(x^k \frac{\partial^\ell}{\partial x^\ell}\right) . \tag{A.13}$$

But because $v(x^t \frac{\partial^t}{\partial x^t}) \geq 1$ (c.f., (A.11)) we have by (A.5) that

$$v\left(\left[x^t \frac{\partial^t}{\partial x^t}, x^k \frac{\partial^\ell}{\partial x^\ell}\right]\right) \geq 1 + v\left(x^k \frac{\partial^\ell}{\partial x^\ell}\right) .$$

Comparing this to (A.13) gives a contradiction, completing the proof of Theorem 2.2.

A.3 Proof of Lemma A.2

A.3.1 A Preliminary Reduction

Lemma A.3: Under the hypotheses of Lemma A.2, if there is an element $x^n \varepsilon W_1$, $n \geq 2$, such that $v(x^n) \geq 0$, then $v(x^m) \to \infty$ as $m \to \infty$.

Proof: Suppose we had such an element x^n. Because $\dim(W_1/L_2) < \infty$, there is an element $Y = \sum_{j=r}^{s} a_j \frac{\partial^j}{\partial x^j} \varepsilon W_1$, $a_s \neq 0$, $s \geq 2$, of valuation ≥ 2. A simple computation shows that $\mathrm{ad}^s_{x^n} Y = n^s\, s!\, a_s\, x^{s(n-1)}$, which has valuation ≥ 2 (by repeatedly using (A.5) and $v(x^n) \geq 0$). Thus we now have an element x^k, $k \geq 2$, with $v(x^k) \geq 2$. Now

$$Z = [x^2 \frac{\partial^2}{\partial x^2}, x^k] = k(k-1)x^k + 2k\, x^{k+1} \frac{\partial}{\partial x}$$

has valuation ≥ 1, and for any q, $\mathrm{ad}^p_Z x^q = c\, x^{pk+q}$, $c \neq 0$. For any $m \geq k$, there exist nonnegative integers p,q such that $m = pk+q$, so we have for m large enough:

$$\begin{aligned} v(x^m) = v(x^{pk+q}) &= v(\mathrm{ad}^p_Z x^q) \\ &\geq p \cdot v(Z) + v(x^q) \\ &\geq [\tfrac{m}{k}] - 1 \geq \tfrac{m}{k} - 2, \end{aligned}$$

where $[\frac{m}{n}]$ denotes the largest integer $\leq \frac{m}{n}$. Since k is fixed, this shows that $v(x^m) \to \infty$ as $m \to \infty$.

A.3.2 Some Combinatorial Lemmas

To prove that under the conditions of Lemma A.2 there is indeed an $n \in \mathbb{N}$, $n \geq 2$ such that $v(x^n) \geq 0$, we need some combinatorial lemmas.

Lemma A.4: Let $r,s \in \mathbb{N}$ with $r < s$, and let $a \in \mathbb{R}$. Then

$$\sum_{i=0}^{s} \binom{s}{i}(-1)^i (a+i+1)(a+i+2)\ldots(a+i+r) = 0$$

Proof: The proof is by induction on (r,s); in case $s=2$ and $r=1$, we have

$$\sum_{i=0}^{2} \binom{2}{i}(-1)^i (a+i+1)$$

$$= a\left[\sum_{i=0}^{2} \binom{2}{i}(-1)^i\right] + \binom{2}{0} - 2\binom{2}{1} + 3\binom{2}{2} = a \cdot 0 + 1 + 4 - 3 = 0.$$

Now assume by induction that the lemma has been proved for $(r-1,s-1)$. Then

$$\sum_{i=0}^{s} \binom{s}{i}(-1)^i (a+i+1)\ldots(a+i+r)$$

$$= a\left[\binom{s}{0}(a+2)\ldots(a+r) - \binom{s}{1}(a+3)\ldots(a+r+1) + \ldots\right]$$

$$+\binom{s}{0}(a+2)\ldots(a+r) - 2\binom{s}{1}(a+3)\ldots(a+r+1) + 3\binom{s}{2}(a+4)\ldots(a+r+2) - \ldots \tag{A.14}$$

Since each term in (A.14) has a product of $r-1$ elements and $\binom{s}{i} = \binom{s-1}{i-1} + \binom{s-1}{i}$, the induction hypothesis implies that the sum in the brackets is zero and the other sum is equal to

$$-\binom{s}{1}(a+3)\dots(a+r+1) + 2\binom{s}{2}(a+4)\dots(a+r+2) - 3\binom{s}{3}(a+5)\dots(a+r+3) + \dots$$

$$= -s\left[\binom{s-1}{0}(a+3)\dots(a+r+1) - \binom{s-1}{1}(a+4)\dots(a+r+2) + \binom{s-1}{2}(a+5)\dots(a+s+2) - \dots\right]$$

$$= 0$$

by the induction hypothesis, and the proof is complete.

Another lemma from the same general family is the following.

Lemma A.5: Let $s \in \mathbb{N}$, $a \in \mathbb{R}$, $k \in \mathbb{R}$. Then

$$\binom{s}{0}(a+s-1)\dots(a+1)a - \binom{s}{1}(a+s-2)\dots(a+1)\,a(a-k)$$

$$+ \binom{s}{2}(a+s-3)\dots(a+1)a(a-k)(a-k-1) - \dots$$

$$+ (-1)^{s-1}\binom{s}{s-1}a(a-k)\dots(a-k-s+2) + (-1)^{s}\binom{s}{s}(a-k)(a-k-1)\dots(a-k-s+1)$$

$$= k(k+1)\dots(k+s-1)$$

Proof: Using the fact that $\binom{s}{i} = \binom{s-1}{i-1} + \binom{s-1}{i}$ and noticing that $(a-k)$ is a factor of all terms except the first one and that a is a factor of all terms except the last one, we rewrite the sum above as

$$a\left[\binom{s-1}{0}(a+s-1)\dots(a+1) - \binom{s-1}{1}(a+s-2)\dots(a+1)(a-k)\right.$$

$$\left.+ \binom{s}{2}(a+s-3)\dots(a+1)(a-k)(a-k-1) - \dots + (-1)^{s-1}\binom{s-1}{s-1}(a-k)\dots(a-k-s+2)\right]$$

$$- (a-k)\left[\binom{s-1}{0}(a+s-2)\dots(a+1)a - \binom{s-1}{1}(a+s-3)\dots(a+1)a(a-k-1) + \dots\right.$$

$$\left.+ \binom{s-1}{s-2}(-1)^{s-2}a(a-k-1)\dots(a-k-s+2) + (-1)^{s-1}\binom{s-1}{s-1}(a-k-1)\dots(a-k-s+1)\right] \tag{A.15}$$

The lemma obviously holds for $s=1$, since $a-(a-k) = k$. Assuming the lemma is true for s-1, we can by induction write the terms in (A.15) as

$$a(k+1)\dots(k+s-1)$$

($s \to s-1$, $a \to a+1$, $k \to k+1$ with respect to the lemma as stated), and

$$(k-a)(k+1)\dots(k+s-1)$$

($s \to s-1$, $a \to a$, $k \to k+1$ with respect to the lemma as stated). Summing these gives the desired result.

A.3.3 Idea of the Proof and More Calculations

Because L/L_2 is finite dimensional, there is some nonzero linear combination $\sum a_m x^m$ of valuation ≥ 2. Then $[x\frac{\partial}{\partial x}, \sum a_m x^m] = \sum m a_m x^m$ has valuation ≥ 1. The idea is to produce enough elements of the form $\sum m^i a_m x^m$ of valuation ≥ 0 to be able to conclude (via Vandermonde matrices) that the individual components $a_m x^m$ have valuation ≥ 0, and thus that the hypothesis of Lemma A.3 is satisfied. For example,

$$[x^n \frac{\partial^n}{\partial x^n}, \sum a_m x^m] = \sum m(m-1)\dots(m-n+1)\, a_m x^m + \sum_{k=1}^{n-1} b_k\, x^{m+k} \frac{\partial^k}{\partial x^k}, \tag{A.16}$$

and brackets of the form $[x^{n+i} \frac{\partial^n}{\partial x^n}, [x^{r-i} \frac{\partial^r}{\partial x^r}, \sum a_m x^m]]$ produce similar terms. However, considerable effort is necessary (by another application of Vandermonde matrices) to eliminate unwanted terms (e.g., the final sum in (A.16)).

First, we perform some necessary calculations. For $m \geq r+n$, we shall need the sums

$$\sum_{i=0}^{r} (-1)^i \binom{r}{i} [x^{n+i} \frac{\partial^n}{\partial x^n}, [x^{r-i} \frac{\partial^r}{\partial x^r}, x^m]] \quad . \tag{A.17}$$

Now

$$[x^{r-i} \frac{\partial^r}{\partial x^r}, x^m] = \sum_{j=0}^{r-1} \binom{r}{j} \frac{m!}{(m-r+j)!} x^{m-i+j} \frac{\partial^j}{\partial x^j},$$

so (A.17) becomes

$$\sum_{j=0}^{r-1} \left\{ \sum_{i=0}^{r} (-1)^i \binom{r}{i} [x^{n+i} \frac{\partial^n}{\partial x^n}, \binom{r}{j} \frac{m!}{(m-r+j)!} x^{m-i+j} \frac{\partial^j}{\partial x^j}] \right\} . \tag{A.18}$$

The terms of the inner sum in (A.18) which are obtained by the action of $\frac{\partial^s}{\partial x^s}$, $1 \leq s \leq j$, on $x^{n+i} \frac{\partial^n}{\partial x^n}$ are of the form

$$-\binom{r}{j}\binom{j}{s} \frac{m!}{(m-r+j)!} x^{m+n+j-s} \frac{\partial^{n+j-s}}{\partial x^{n+j-s}} \left[\sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(n+i)!}{(n+i-j)!} \right];$$

this sum is zero by Lemma A.4, since $s \leq j < r$. The terms of the inner sum in (A.18) which are obtained by the action of $\frac{\partial^s}{\partial x^s}$, $i \leq s \leq j$, on $x^{m-i+j} \frac{\partial^j}{\partial x^j}$ are of the form

$$\binom{r}{j}\binom{n}{s} \frac{m!}{(m-r+j)!} x^{m+n+j-s} \frac{\partial^{n+j-s}}{\partial x^{n+j-s}} \left[\sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(m-i+j)!}{(m-i+j-s)!} \right];$$

this sum is also zero by Lemma A.4, since $s \leq j < r$. It follows that the only nonzero terms in (A.18) arise from the action of $\frac{\partial^s}{\partial x^s}$, $j+1 \leq s \leq n$, on $x^{m-i+j} \frac{\partial^j}{\partial x^j}$, so that (A.18) (and thus (A.17)) has the form

$$\sum_{k=1}^{n} b_k \, x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} \qquad (A.19)$$

The coefficients b_k remain to be calculated.

Fix a k, $1 \leq k \leq n$; the term in (A.18) which contributes to the k^{th} term in (A.19) is

$$\sum_{j=0}^{r-1} \left[\sum_{i=0}^{r} (-1)^i \binom{r}{i}\binom{r}{j} \frac{m!}{(m-r+j)!} \binom{n}{j+k} \frac{(m-i+j)!}{(m-i-k)!} x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} \right]$$

$$= \sum_{j=0}^{r-1} \binom{r}{j}\binom{n}{j+k} \frac{m!}{(m-k)!} \left[\sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(m+j-i)!}{(m-r+j)!} \frac{(m-k)!}{(m-k-i)!} \right] x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} . \qquad (A.20)$$

According to Lemma A.5, with $a \to m+j-r+1$, $s \to r$, $k \to k+j-r+1$, the inner sum is equal to

$$(k+j)(k+j-1)\ldots(k+j-r+1)$$

Thus (A.20) becomes

$$\frac{m!}{(m-k)!} \left[\sum_{j=0}^{r-1} \binom{r}{j}\binom{n}{j+k} (k+j)\ldots(k+j-r+1) \right] x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}}$$

$$= \frac{m!}{(m-k)!} \binom{n}{k} \left[\sum_{j=0}^{r-1} \binom{r}{j} \frac{(n-k)!}{(n-k-j)!} k(k-1)\ldots(k-r+j+1) \right] x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} \qquad (A.21)$$

The coefficient of k^r (the highest power of k) in the inner sum of (A.21) is equal to

$$\sum_{j=0}^{r-1} \binom{r}{j} (-1)^j = (-1)^{r+1} ;$$

we will assume that r is odd, since the proof is the same for r even. It follows that the inner sum in (A.21) is of the form

$$k^r + c_{r-1}^{(r)}(n)\, k^{r-1} + \ldots + c_1^{(r)}(n)\, k ,$$

where the $c_j(n)$ are polynomial functions of n and r. Hence (A.17) can be written as

$$\sum_{i=0}^{r} (-1)^i \binom{r}{i} [x^{n+i} \frac{\partial^n}{\partial x^n}, [x^{r-i} \frac{\partial^r}{\partial x^r}, x^m]]$$

$$= \sum_{k=1}^{n} \binom{n}{k} [k^r + c_{r-1}^{(r)}(n)\, k^{r-1} + \ldots + c_1^{(r)}(n)\, k] \frac{m!}{(m-k)!} x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} . \qquad (A.22)$$

For r=1, (A.22) becomes

$$\sum_{k=1}^{n} \binom{n}{k} k \frac{m!}{(m-k)!} x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} . \qquad (A.23)$$

Subtracting $c_1^{(2)}(n)$ times (A.23) from (A.22) for r=2 yields

$$\sum_{k=1}^{n} \binom{n}{k} k^2 \frac{m!}{(m-k)!} x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} .$$

Continuing by induction, we see that there are coefficients b(t,r,n) such that, for each $t \in \mathbb{N}$,

$$\sum_{r=1}^{t} b(t,r,n) \sum_{i=0}^{r} (-1)^i \binom{r}{i} [x^{n+i} \frac{\partial^n}{\partial x^n}, [x^{r-i} \frac{\partial^r}{\partial x^r}, x^m]]$$

$$= \sum_{k=1}^{n} k^t [\binom{n}{k} \frac{m!}{(m-k)!} x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}}] . \qquad (A.24)$$

A.3.4 Proof of Lemma A.2

According to Lemma A.3, we need only show that there is a $p \in \mathbb{N}$, $p \geq 2$, such that $v(x^p) \geq 0$. By assumption W_1/L_2 is finite dimensional; let $r = \dim(W_1/L_2)+1$. Then there is for each $u \in \mathbb{N}$ a nonzero sum of the form

$$X = \sum_{m=u}^{u+r-1} a_m x^m \tag{A.25}$$

with valuation ≥ 2. Take $u \geq 2r$, so that the calculations of the previous section are valid for all m in (A.25). Multiplying (A.24) by a_m and summing from m=u to m=u+r-1 yields the expressions

$$\sum_{k=1}^{n} k^t X(k,n);\ t=0,\ldots,r-1,\ n=1,\ldots,r \tag{A.26}$$

where

$$X(k,n) = \sum_{m=u}^{u+r-1} a_m \binom{n}{k} \frac{m!}{(m-k)!} x^{m+n-k} \frac{\partial^{n-k}}{\partial x^{n-k}} .$$

The elements (A.26) have thus been obtained from (A.25) by applying at most two brackets and taking linear combinations; therefore,

$$v\left(\sum_{k=1}^{n} k^t X(k,n)\right) \geq 0.$$

Using the nonsingularity of Vandermonde matrices, we can write the X(k,n) as linear combinations of the elements (A.26); thus

$$v(X(k,n)) \geq 0;\quad k=1,\ldots,n,\ n=1,\ldots,r.$$

Taking k=n we obtain in particular the elements

$$Y(n) = \sum_{m=u}^{u+r-1} a_m \frac{m!}{(m-n)!} x^m, \quad n=1,\ldots,r, \tag{A.27}$$

with valuation ≥ 0. It is easily shown that the coefficient matrix in (A.27) is nonsingular, implying that $v(a_m x^m) \geq 0$, $m=u,\ldots,u+r-1$, thus there is at least one m such that $v(x^m) \geq 0$ (because not all a_m are zero). This concludes the proof of Lemma A.2, thus proving Theorem 2.2.

A.4 Proof of Theorem 2.3

Suppose that ϕ: $W_1 \to V(M)$ is a nonzero homomorphism, where M is an n-dimensional C^∞ manifold. Then there is a point $m \in M$ such that the image of ϕ contains an element which gives a nonzero tangent vector at m. Let G be the Lie algebra of germs of C^∞ vector fields around m; i.e., in local coordinates centered at m, $G = \{\sum f_i(x) \frac{\partial}{\partial x_i}\}$, where f_i are germs of C^∞ functions around m. Let A be the ideal in G consisting of all elements for which the f_i are flat functions in a neighborhood of m (a function germ in n variables $x_1, \dots, x_n$ defined on a neighborhood N is flat on N if $\frac{\partial^\alpha f}{\partial x^\alpha}(x) = 0$ for all $x \in N$ and α). A is an ideal because derivatives of flat functions are flat. Restricting the vector fields of V(M) to their germs around m, we obtain a composed homomorphism of Lie algebras

$$W_1 \to V(M) \to G \to G/A \tag{A.28}$$

which is nonzero because at least one vector field in $\phi(W_1)$ was nonzero at m.

By Borel's extension lemma [29, p. 98], G/A is isomorphic to $\hat{V}_n$. Thus (A.28) gives a nonzero homomorphism from W_1 to $\hat{V}_n$. However, since the only ideals of W_1 are $\{0\}$, W_1, and $\mathbb{R}\cdot 1$, this would yield a nonzero homomorphism from W_1 or $W_1/\mathbb{R}\cdot 1$ to $\hat{V}_n$. This yields a contradiction by Theorem 2.2.

Appendix B

Calculations for Example 2.3

The Lie algebra $L(\Sigma)$ is generated by

$$e_0 = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2\frac{\partial^2}{\partial y^2} - x\frac{\partial}{\partial y} - \frac{1}{2}y^2, \qquad e_1 = y.$$

We proceed as in Example 2.1:

$[e_0,e_1] = -x + x^2\frac{\partial}{\partial y} \triangleq e_2$

$[e_2,e_1] = x^2 \triangleq e_3$

$[e_0,e_3] = 2x\frac{\partial}{\partial x} + 1 \triangleq e_4$

$[e_4,e_2] = -2x + 4x^2\frac{\partial}{\partial y}$, which combined with e_2 implies that $e_5 = x$ and $e_6 = x^2\frac{\partial}{\partial y}$ are in $L(\Sigma)$. Also,

$[e_0,e_5] = \frac{\partial}{\partial x} \triangleq e_7$

$[e_7,e_5] = 1 \triangleq e_8$, which combined with e_4 implies that $e_9 = x\frac{\partial}{\partial x} \in L(\Sigma)$.

Now,

$[e_7,e_6] = 2x\frac{\partial}{\partial y} \quad \Rightarrow \quad e_{10} = x\frac{\partial}{\partial y}$

$[e_7,e_{10}] = \frac{\partial}{\partial y} \triangleq e_{11}$

$[e_7,e_0] = -\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial y^2} \quad \Rightarrow \quad e_{12} = x\frac{\partial^2}{\partial y^2}$

$[e_7,e_{12}] = \frac{\partial^2}{\partial y^2} \triangleq e_{13}$

$[e_6,e_0] = -x^2 y - 2x\frac{\partial^2}{\partial x \partial y} - \frac{\partial}{\partial y} \quad \Rightarrow \quad e_{14} = 2x\frac{\partial^2}{\partial x \partial y} + x^2 y$

$$[e_7,e_{14}] = \frac{\partial^2}{\partial x\partial y} + 2xy \triangleq e_{15}$$

$$[e_{10},e_{15}+e_{11}] = x^3 - 2x \frac{\partial^2}{\partial y^2} \quad \Rightarrow \quad e_{16} = x^3$$

$$[e_{12},e_{15}] = 4x^2 \frac{\partial}{\partial y} - \frac{\partial^3}{\partial y^3} \quad \Rightarrow \quad e_{17} = \frac{\partial^3}{\partial y^3}$$

$$[e_{13},e_0] = -2y \frac{\partial}{\partial y} - 1 \quad \Rightarrow \quad e_{18} = y \frac{\partial}{\partial y}$$

$[e_{18},e_{14}] = yx^2 - 2x \frac{\partial^2}{\partial x\partial y}$, which combined with e_{14} implies that $e_{19} = yx^2$ and $e_{20} = x \frac{\partial^2}{\partial x\partial y}$ are in $L(\Sigma)$. Also,

$[e_{17},e_{19}] = 3x^2 \frac{\partial^2}{\partial y^2} \triangleq e_{21}$, which combined with e_0 and e_{10} implies that $e_{22} = \frac{\partial^2}{\partial x^2} - y^2 \in L(\Sigma)$. Continuing,

$[e_9,e_{22}] = -2 \frac{\partial^2}{\partial x^2}$, so that $e_{23} = \frac{\partial^2}{\partial x^2}$ and $e_{24} = y^2 \in L(\Sigma)$. Now,

$$[e_{23},e_{16}] = 3x^2 \frac{\partial}{\partial x} \quad \Rightarrow \quad e_{25} = x^2 \frac{\partial}{\partial x}$$

$$[e_{10},e_{24}] = 2xy \quad \Rightarrow \quad e_{26} = xy$$

$$[e_{23},e_{26}] = 2y \frac{\partial}{\partial x} \quad \Rightarrow \quad e_{27} = y \frac{\partial}{\partial x}$$

$$[e_{27},e_{16}] = 3yx^2 \quad \Rightarrow \quad e_{28} = yx^2$$

$$[e_{27},e_{28}] = 2y^2x \quad \Rightarrow \quad e_{29} = y^2x$$

$$[e_{27},e_{29}] = y^3 \quad \Rightarrow \quad e_{30} = y^3$$

$$[e_{13},e_{30}] = 3y^2 \frac{\partial}{\partial y} \quad \Rightarrow \quad e_{31} = y^2 \frac{\partial}{\partial y}$$

Noticing that the elements e_1, e_5, e_{13}, e_{23}, e_{25}, e_{26}, and e_{31} are precisely the generators of W_2 given in Theorem 2.4, we conclude that $L(\Sigma) = W_2$.