ECONOMETRIC INSTITUTE

SYSTEM IDENTIFICATION AND NONLINEAR FILTERING: LIE ALGEBRAS AND A SUMMARY OF APPROXIMATION METHODS FOR NONLINEAR FILTERING PROBLEMS ARISING IN SYSTEM IDENTIFICATION

P.S. KRISHNAPRASAD, S.I. MARCUS AND M. HAZEWINKEL
AND
P.S. KRISHNAPRASAD, M. HAZEWINKEL AND B. HANZON

REPRINT SERIES no. 299

This article appeared in "IEEE Conference on Decision & Control", (1981).

ERASMUS UNIVERSITY ROTTERDAM - P.O. BOX 1738 - 3000 DR ROTTERDAM - THE NETHERLANDS

SYSTEM IDENTIFICATION AND NONLINEAR FILTERING : LIE ALGEBRAS

P.S. Krishnaprasad* Steven I. Marcus** Michiel Hazewinkel***

*Univ. of Md.
College Park, Md. 20742

**Univ. of Texas
Austin, Texas 78712

***Erasmus Univ.
Rotterdam.
The Netherlands

Abstract

This paper is continuation of our previous work ([1], [2], [3]) to understand the identification problem of linear system theory from the viewpoint of nonlinear filtering. The estimation algebra of the identification problem is a subalgebra of a current algebra. It therefore follows that the estimation algebra is embeddable as a Lie algebra of vector fields on a finite dimensional manifold. These features permit us to develop a Wei-Norman type procedure for the associated Cauchy problem and reveal a set of functionals of the observations that play the role of joint sufficient statistics for the identification problem.

1. Introduction

Consider the stochastic differential system:

$$\begin{aligned} d\theta &= 0 \\ dx_t &= A(\theta)x_t dt + b(\theta)dw_t \qquad (1) \\ dy_t &= \langle c(\theta), x_t\rangle dt + dv_t . \end{aligned}$$

Here $\{w_t\}$ and $\{v_t\}$ are independent, scalar, standard, Wiener processes, and $\{x_t\}$ is an $\mathbb{R}^n$-valued process. Assume that θ takes values in a smooth manifold $\Theta\to\mathbb{R}^N$, and the map $\theta\to\Sigma(\theta):=(A(\theta), b(\theta), c(\theta))$ in a smooth map taking values in minimal triples. By the identification problem we shall mean the nonlinear filtering problem associated with eqn. (1); i.e. the problem of recursively computing conditional expectations of the form $\pi_t(\phi)\triangleq E[\phi(x_t,\theta)|Y_t]$ where Y_t is the σ-algebra generated by the observations $\{y_s : 0\le s\le t\}$ and ϕ belongs to a suitable class of functions on $\mathbb{R}^n\times\Theta$.

The joint unnormalized conditional density $\rho\triangleq\rho(t,x,\theta)$ of x_t and θ given Y_t satisfies the stochastic partial differential equation (Stratonovitch sense)

$$d\rho = A_o\rho dt + B_o\rho dy_t \qquad (2)$$

where the operators A_o and B_o are given by

$$A_o := \frac{1}{2}\langle b(\theta), \frac{\partial}{\partial x}\rangle^2 - \langle\frac{\partial}{\partial x}, A(\theta)x\rangle - \langle c(\theta), x\rangle^2/2 \qquad (3)$$

$$B_o := \langle c(\theta), x\rangle . \qquad (4)$$

(see [4] for background).

From the Bayes formula ([5]), it follows that

$$\pi_t(\phi) = \sigma_t(\phi)/\sigma_t(1) \qquad (5)$$

where

$$\sigma_t(\phi) = \int_\Theta \int_{\mathbb{R}^n} \phi(x,\theta)\rho(t,x,\theta)|dx|.|d\theta| \qquad (6)$$

where $|dx|$ and $|d\theta|$ are fixed volume elements on $\mathbb{R}^n$ and Θ respectively. Further if $Q(t,\theta)$ denotes the unnormalized posterior density of θ given $_t$, then it satisfies the Ito equation:

$$dQ = E[\langle c(\theta), x_t\rangle|\theta, Y_t].Q(t,\theta)dy_t . \qquad (7)$$

Recent work in nonlinear filtering theory (see the proceedings [6]) shows that it is natural to look at eqn. (2) formally as a deterministic partial differential equation,

$$\frac{\partial\rho}{\partial t} = A_o\rho + \dot{y}B_o\rho . \qquad (8)$$

By the Lie algebra of the identification problem, we shall mean the operator Lie algebra $\bar{G}$ generated by A_o and B_o. For more general nonlinear filtering problems, estimation algebras analogous to G have been emphasized by Brockett and Clark [7], Brockett ([8] - [11]), Mitter ([12], [13]), Hazewinkel and Marcus [14] and others (see [6]) as being objects of central interest. In the papers ([1], [2]) the Lie algebra $\bar{G}$ is used to classify identification problems and to understand the role of certain sufficient statistics.

2. The Structure of the Estimation Algebra $\bar{G}$:

To understand the structure of the estimation algebra $\bar{G}$ it is well-worth considering an example.

Example 1:

Let $dx_t = \theta.dw_t$; $d\theta = 0$

$dy_t = x_t dt + dv_t$

Then $A_o = \frac{\theta^2}{2}\frac{\partial^2}{\partial x^2} - \frac{x^2}{2}$ and $B_o = x$, and

$\bar{G} = \{A_o, B_o\}_{L.A.}$ is spanned by the set of operators $(\frac{\theta^2}{2}\frac{\partial^2}{\partial x^2} - \frac{x^2}{2})$, $\{\theta^{2n}x\}_{n=0}^{\infty}$, $\{\theta^{2n}\frac{\partial}{\partial x}\}_{n=1}^{\infty}$ and

$\{\theta^{2n}1\}_{n=1}^{\infty}$. We then notice that,

$$\tilde{G} \subseteq \mathbb{R}[\theta^2] \otimes \{\frac{\partial^2}{\partial x^2}, x\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, x^2, x, 1\} L.A.$$

is a subalgebra of the Lie algebra obtained by tensoring the polynomial ring $\mathbb{R}[\theta^2]$ with a 6 dimensional Lie algebra.//

The general situation is very much as in this example. Consider the vector space (over the reals) of operators spanned by the set,

$$S := \{\frac{\partial^2}{\partial x_i \partial x_j}, \; x_i \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}, \; x_i x_i, x_j, 1\} \tag{7}$$

$$i = 1,2,..,n, \quad j=1,2,..,n$$

This space of operators has the structure of a Lie algebra henceforth denoted as $\tilde{G}_o$ (of dimension $3n^2+2n+1$) under operator commutation (the commutation rules being $[\frac{\partial^2}{\partial x_i \partial x_j}, x_k] = \delta_{jk}\frac{\partial}{\partial x_i} + \delta_{ik}\frac{\partial}{\partial x_j}$ etc., where δ_{jk} denotes the Kronecker symbol). For each choice $\theta \in \Theta$, A_o and B_o take values in $\tilde{G}_o$. It follows that in general A_o and B_o are smooth maps from Θ into $\tilde{G}_o$. So let us consider the space of smooth maps $C^{\infty}(\Theta;\tilde{G}_o)$. This space can be given the structure of a Lie algebra (over the reals) in the following way: given $\phi, \psi \in C^{\infty}(\Theta;\tilde{G}_o)$, define the Lie bracket $[.,.]_c$ on $C^{\infty}(\Theta;\tilde{G}_o)$ by

$$[\phi,\psi]_c(P) = [\phi(P),\psi(P)] \tag{10}$$

for every $P \in \Theta$. Here the bracket on the right hand side of eqn. (10) is in $\tilde{G}_o$. We denote as $\tilde{G}_c$ the Lie algebra $(C^{\infty}(\Theta;\tilde{G}_o);[.,.]_c)$. Whenever the dimension of Θ is greater than zero, $\tilde{G}_o$ is infinite dimensional and is an example of a current algebra. Current algebras play a fundamental role in the physics of Yang-Mills fields where they occur as Lie algebras of gauge transformations [15]. Elsewhere in mathematics they are studied under the guise of local Lie algebras ([16] [18]). The following is immediate.

Proposition 1:

The Lie algebra $\tilde{G}$ of operators generated by

$$A_o = \frac{1}{2}\langle b(\theta), \frac{\partial}{\partial x}\rangle^2 - \langle\frac{\partial}{\partial x}, A(\theta)x\rangle - \langle c(\theta), x\rangle^2/_2$$

and $B_o = \langle c(\theta), x\rangle$, is a subalgebra of the current algebra $C^{\infty}(\Theta;\tilde{G}_o)$.

3. Representation Questions:

In [3] we observe that $\tilde{G}$ admits a faithful representation as a Lie algebra of vector fields on a finite dimensional manifold. Specifically, consider the system of equations,

$$d\theta = 0$$
$$dz = [A(\theta)-Pc(\theta)c^T(\theta)]zdt + Pc(\theta)dy_t$$
$$\frac{dP}{dt} = A(\theta)P + PA^T(\theta) + b(\theta)b^T(\theta) - Pc(\theta)c^T(\theta)P$$
$$ds = \frac{1}{2}\langle c(\theta),z\rangle^2 dt - \langle c(\theta),z\rangle dy_t \tag{11}$$

The system of equations (11) evolves on the product manifold $\Theta \times \mathbb{R}^{n(n+3)/2+1}$. Associate with eqn. (11) the pair of vector fields (first order differential operators),

$$a_0^* = \langle(A(\theta)-Pc(\theta)c^T(\theta))z, \partial/\partial z\rangle$$
$$+tr((A(\theta)P+PA^T(\theta)+b(\theta)b^T(\theta)-Pc(\theta)c^T(\theta)P).\partial/\partial P)$$
$$+ 1/2\langle c(\theta),z\rangle^2 \partial/\partial s$$

and

$$b_o^* = \langle P(\theta), \partial/\partial z\rangle - \langle c(\theta),z\rangle\, \partial/\partial s. \tag{12}$$

(Here $\partial/\partial P = [\partial/\partial P_{ij}] = (\partial/\partial P)^T$ = nxn symmetric matrix of differential operators). Consider the Lie algebra of vector fields generated by a_0^* and b_0^*. Since a_0^* and b_0^* are vertical vector fields with respect to the fibering $\Theta \times \mathbb{R}^{n(n+3)/2+1} \to \Theta$, so is every vector field in this Lie algebra. One of the main results in [3] is the following:

Theorem 1: The map

$$\phi_k : \tilde{G}_c \to \mathcal{U}\Theta \times \mathbb{R}^{n(n+3)/2+1}$$

defined by

$$\phi_k(A_o) = a_0^* \; ; \; \phi_k(B_0) = b_0^*$$

is a faithful representation of the Lie algebra of the identification problem as a Lie algebra of (vertical) vector fields on a finite dimensional manifold fibered over Θ.

Example 2:

To illustrate Theorem 1, consider the Lie algebra of example 1. The embedding equations (11) take the form

$$d\theta = 0$$
$$dp = (\theta^2 - p^2)dt$$
$$dz = -pzdt + pdy_t$$
$$ds = z^2/2dt - zdy_t.$$

Then

$$\Phi_k(B_0) = \Phi_k(x) = b_0^* = p\partial/\partial z + (-z)\partial/\partial s$$

The induced maps on Lie brackets are given by

$$\Phi_k(\theta^{2k}\partial/\partial x) = \theta^{2k}\partial/\partial z \quad k = 0,1,2,\ldots$$

$$\Phi_k(\theta^{2k}x) = \theta^{2k}(p\partial/\partial z - z\partial/\partial s) \quad k = 1,2,\ldots$$

$$\Phi_k(\theta^{2k}1) = \theta^{2k}\partial/\partial s \quad k = 1,2,\ldots //$$

The embedding equations have the following statistical interpretation. Assume that the initial condition for (12) is of the form,

$$\rho_0(x,\theta) = (2\pi\det\Sigma(\theta))^{-n/2}\exp(-\langle x-\mu(\theta), \frac{\Sigma^{-1}(\theta)}{2}\cdot(x-\mu(\theta))\rangle)\cdot Q_0(\theta) \quad (13)$$

where $\theta\to(\mu(\theta)\Sigma(\theta),Q_0(\theta))$ is a smooth map, $\Sigma(\theta)>0$ $\theta\in\Theta$ and $Q_0(\theta)>0$ for $\theta\in\Theta$. Suppose eqn. (11) is initialized at,

$$(\theta_0,z_0,P_0,s_0)=(\theta_0,\mu(\theta_0),\Sigma(\theta_0),-\log(Q_0(\theta)) \quad (14)$$

Append to the system (11) an output equation,

$$\overline{Q}_t = e^{-s_t}. \quad (15)$$

Now if (11) is solved with initial condition (14), one can show by differentiating $\overline{Q}_t$ that $\overline{Q}_t$ satisfies eqn. (7). In other words, the system (11)-(15) with initial condition (14) is a finite dimensional recursive estimation for the posterior density $Q(t,\theta_0)$. We have thus verified the homomorphism principle of Brockett [8]: that finite dimensional recursive estimators must involve Lie algebras of vector fields that are homomorphic images of the Lie algebra of operators associated with the unnormalized conditional density equation.

4. A Sobolev Lie Group Associated to $\tilde{G}$:

It has been remarked elsewhere ([8], [13], [21], [22] and [3]) that the Cauchy problem associated with (8) may be viewed as a problem of integrating a Lie algebra representation. In this connection one should be interested whether there is an appropriate topological group associated with $\tilde{G}$. We have the following general procedure.

Let M be a compact Riemannian manifold of dimension d. Let L be a Lie algebra of dimension $n<\infty$. We can always view L as a subalgebra of the general linear Lie algebra $g\ell(m;\mathbb{R})$, $m>n$ (Ado's theorem).

Assumption:

Let $G=\{\exp(L)\}_G\subset g\ell(m;\mathbb{R})$ be the smallest Lie group containing the exponentials of elements of L. We assume that G is a closed subset of $g\ell(m;\mathbb{R})$.

Define,

$$\mathcal{R} = C^\infty(M;g\ell(m,\mathbb{R}))$$

$$\mathcal{L} = C^\infty(M;L)$$

$$\mathcal{G} = C^\infty(M;G).$$

Clearly $\mathcal{R}$ is an algebra under pointwise multiplication and

$$\mathcal{L}\subset\mathcal{R},\ \mathcal{G}\subset\mathcal{R}.$$

Let $\{(U_\alpha,\varphi_\alpha)\}$ be a C^∞ atlas for M. Then for $f_1,f_2\in\mathcal{R}$, define

$$\|f_1-f_2\|_k=[\int_{\varphi_\alpha(U_\alpha)} d\,\mathrm{vol}\sum_{\ell=0}^{k}|D^\ell(f_1-f_2)\varphi_\alpha^{-1}|^2]^{1/2} \quad (16)$$

where

$$|f|^2 = \mathrm{tr}\,(f'f). \quad (17)$$

(Here $k=d/2+s$. $s>0$). Let $\mathcal{R}_k$ be the completion of $\mathcal{R}$ and $\mathcal{G}_k$ the completion of $\mathcal{G}$ in the norm $\|\cdot\|_k$. ($\mathcal{G}_k$ is closed in $\mathcal{R}_k$). By the Sobolev theorem $\mathcal{R}_k$ is a Banach algebra and the group operation

$$\mathcal{G}_k\times\mathcal{G}_k\to\mathcal{G}_k$$

$$(f_1,f_2)\to f_1f_2$$

when $(f_1f_2)(m) = f_1(m)f_2(m)$ is continuous. Thus $\mathcal{G}_k$ is a topological group.

Proceeding as before, one can given a Sobolev completion of $\mathcal{L}$ to obtain $\mathcal{L}_k$ an infinite dimensional Lie algebra where once again by the Sobolev theorem the bracket operation

$$[\cdot,\cdot]:\mathcal{L}_k\times\mathcal{L}_k\to\mathcal{L}_k$$

$$(f_1,f_2)\to[f_1,f_2]$$

with $[f_1,f_2](m) = [f_1(m),f_2(m)]$ is continuous. Now for a small enough neighborhood $V(0)$ of $0\in\mathcal{L}_k$ one can define

$$\exp: V(0)\to\mathcal{G}_k$$

$$\xi\to\exp(\xi)$$

by pointwise exponentiation. This permits us to provide a Lie group structure on $\mathcal{G}_k$ with $\mathcal{L}_k$ canonically identified as the Lie algebra of $\mathcal{G}_k$.

The procedure outlined above appears to play a significant role in several contexts (the index theorem, Yang-Mills fields [24] [25] [26] [27].

For our purposes L will be identified with a faithful matrix representation of G_0. Thus we associate with the identification problem a Sobolev Lie group which is a subgroup of $\mathcal{G}_k$ corresponding to $\tilde{G}_0$.

Remark:

One of the important differences between the problem of filtering and the problems of Yang-Mills theories is that in the latter case there are natural norms for Sobolev completion. This follows from the fact that in Yang-Mills theories, the algebra L is compact (semi-simple) and one has the Killing form to work with. In filtering problems $\bar{G}_0$ is never compact.

Remark:

We would like to acknowledge here that Prof. Sanjoy Mitter was kind enough to acquaint one of us (P.S.K) with the work of P.K. Mitter.

5. The Integration Problem & Sufficient Statistics

In [3] we look for a representation of the form,

$$\rho(t,x,\theta)=\exp(g_1(t,\theta)A1)\ldots\exp(g_n(t,\theta)A^n)\rho_0 \qquad (18)$$

for the solution to the eqdation (8). In the case of example (1) this takes the form

$$\rho(t,x,\theta)=\exp(g_1(t,\theta)\cdot(\frac{\theta^2}{2}\frac{\theta^2}{\theta_x}-\frac{x^2}{2}))\cdot$$
$$\cdot\exp(g_2(t,\theta)\cdot\theta^2\frac{\partial}{\partial x})$$
$$\cdot\exp(g_3(t,\theta)x)\cdot\exp(g_4(t,\theta)\cdot 1)\rho_0 \qquad (19)$$

Differentiating and substituting in (8) we get,

$$\frac{\partial g_1}{\partial t}(t,\theta) = 1$$

$$\frac{\partial g_2}{\partial t}(t,\theta) = \cosh(g_1\cdot\theta)\dot{y} \qquad (20)$$

$$\frac{\partial g_3}{\partial t}(t,\theta) = -\frac{1}{\theta}\sinh(g_1\cdot\theta)\dot{y}$$

$$\frac{\partial g_4}{\partial t}(t,\theta) = \frac{\partial g_3}{\partial t}(t,\theta)g_2(t,\theta).$$

and $g_i(0,\theta) = 0$ for $i = 1,2,3,4,\theta\epsilon\Theta$. The above first-order partial differential equations may be easily solved by quadrature and one has the representation,

$$\rho(t,x,\theta) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sinh(|\theta|t)\cdot t|\theta|)}} \exp(-\frac{1}{2}\coth^2(\left|\frac{x}{\theta}\right|^2+z)\cdot\exp(\frac{xz}{\sqrt{|\theta|\sinh(|\theta|t)}})\cdot\exp(g_4(t,\theta)\theta^2)\cdot$$
$$\cdot\exp(g_2(t,\theta)\sqrt{|\theta|}z)\cdot\rho_0(g_3(t,\theta)\theta^2\sqrt{|\theta|}z,\theta)dz \qquad (21)$$

where $\rho_0(\cdot,\theta)\epsilon L_2(\mathbb{R})$ for every $\theta\epsilon\Theta$ and is smooth in θ. Further $\Theta\subset\mathbb{R}$ is a bounded set and $\mathcal{O}$/ closure Θ.

In equation (21) the g_i's should be viewed as canonical coordinates of the second kind on the corresponding SobolevLie group. Now expand g_2 and g_3 to obtain

$$g_2(t,\theta) = \sum_{k=0}^{\infty}\theta^{2k}\int_0^t \frac{\sigma^{2k}}{(2k)!}\dot{y}_\sigma d\sigma \quad k=1,2,\ldots$$
$$g_3(t,\theta) = -\sum_{k=0}^{\infty}\theta^{2k}\int_0^t \frac{\sigma^{2k+1}}{(2k+1)!}\dot{y}_\sigma d\sigma \quad k=1,2,\ldots \qquad (22)$$

It follows that all the "information" contained by the observations $\{y_\sigma : 0\le\sigma\le t\}$ about the joint unnormalized conditional density is contained in the sequence

$$T\underline{\Delta}\{\int \frac{\sigma^k}{k!}\dot{y}_\sigma d\sigma;\ k=0,1,2,\ldots\} \qquad (23)$$

Thus T is nothing but a joint sufficient statistic for the identification problem.

Acknowledgements

Partial support for this work was provided by the Air Force of Scientific Research under grant AFOSR79-0025 and by the Department of Energy under Contract DEAC01-80-RA50420-A001. One of us (P.S.K) would also like to acknowledge the hospitality of Erasmus University during a recent visit.

References

[1] Krishnaprasad, P.S. and S.I. Marcus (1981a). Some nonlinear filtering problems arising in recursive identification. In M. Hazewinkel and J.C. Willems (Ed.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, Reidel, Dordrecht.

[2] Krishnaprasad, P.S. and S.I. Marcus (1981b). Identification and tracking: a class of nonlinear filtering problems. In Proc. 1981 Joint Automatic Control Conference, Charlottesville.

[3] Krishnaprasad, P.S. and S.I. Marcus, (1982). On the Lie algebra of the identification problem. IFAC Symposium on Digital Control. New Delhi, India.

[4] Davis, M. and S.I. Marcus (1981). An introduction to nonlinear filtering. In M. Hazewinkel and J.C. Willems (Ed.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, Reidel, Dordrecht.

[5] Kallianpur, G. (1980). Stochastic Filtering Theory. Springer-Verlag, New York.

[6] Hazewinkel, M. and J.C. Willems (Ed.)(1981). Stochastic Systems: The Mathematics of Filtering and Identification and Applications. Reidel, Dordrecht.

[7] Brockett, R.W. and J.M.C. Clark (1978). The geometry of the conditional density equation. In O.L.R. Jacobs (Ed.), Analysis and Optimization of Stochastic Systems, Academic Press, New York, pp. 299-309.

[8] Brockett, R.W. (1978). Remarks on finite

dimensional nonlinear estimation. In C. Lobry (Ed.), Analyse des Systems, Bordeaux. (1980) Asterisque, 75, 76.

[9] Brockett, R.W. (1979). Classification and equivalence in estimation theory. In Proc. 18th IEEE Conf. on Decision and Control, Ft. Lauderdale. 172-174.

[10] Brockett, R.W. (1980). Estimation theory and the representation of Lie algebras. In Proc. 19th IEEE Conf. on Decision and Control, Albuquerque.

[11] Brockett, R.W. (1981). Nonlinear systems and nonlinear estimation theory. In M. Hazewinkel and J.C. Willems (Ed.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications. Reidel, Dordrecht.

[12] Mitter, S.K. (1978). Modeling for stochastic systems and quantum fields. In Proc. 17th Conf. on Decision and Control, San Diego.

[13] Mitter, S.K. (1980). On the analogy between mathematical problems of nonlinear filtering and quantum physics. Richerche di Automatica, 10, 163-216.

[14] Hazewinkel, M. and S.I. Marcus (1980). On Lie algebras and finite dimensional filtering. Submitted to Stochastics.

[15] Daniel, M. and C.M. Viallet (1980). The geometrical setting of guage theories of the Yang-Mills type. Reviews of Modern Physics, 52.

[16] Hermann, R. (1973). Topics in the Mathematics of Quantum Mechanics: Interdisciplinary Mathematics vol. VI. Math Sci Press, Brookline.

[17] Davies, E.B. (1980). One Parameter Semigroups. Academic Press, London.

[18] Kirillov, A.A. (1976). Local Lie Algebras. Russian Math. Surveys. 31:4, 1976, pp. 55-75.

[19] Benes, V.E. (1981). Exact finite dimensional filters for certain diffusions with nonlinear drift. Stochastics, 5, 65-92.

[20] Liu, C.-H. and S.I. Marcus (1980). The Lie algebraic structure of a class of finite dimensional filters. In C.I. Byrnes and C.F. Martin (Ed.), Lectures in Applied Mathematics, Vol. 18, American Math. Soc., Providence, pp. 277-297.

[21] Ocone, D. (1980a). Topics in Nonlinear Filtering Theory. Ph.D. Thesis, M.I.T., Cambridge, Massachusetts.

[22] Ocone, D. (1980b). Nonlinear filtering problems with finite dimensional Lie algebras. In Proc. 1980 Joint Automatic Control Conference, San Francisco.

[23] Ocone, D. (1981). Finite dimensional estimation algebras in nonlinear filtering. In M. Hazewinkel and J.C. Willems (Ed.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, Reidel, Dordrecht.

[24] Palais, R.S.. Seminar on the Atiyah-Singer Index Theorem, Annals of Mathematics Studies, No. 57, Princeton University Press, Princeton 1965, (chs. 4,8,9,10).

[25] Narasimhan, M.S. and T.R. Ramadas, "Geometry of SU(2) Gauge fields", Communications in Math. Physics, 67, (179), pp. 21-36.

[26] Mitter, P.K. and C.M. Viallet, "On the Bundle of Connections and the Gauge Orbit Manifold in Yang-Mills Theory", Communications in Math. Physics 79, (1981), 457-472.

[27] Mitter, P.K., "Geometry of the space of Gauge Orbits and the Yang-Mills Dynamical System", in Recent Developments in Gauge Theories (Cargese School) eds: G.T. Hooft et al, Plenum Press, 1980.

APPROXIMATION METHODS FOR NONLINEAR FILTERING PROBLEMS ARISING IN SYSTEM IDENTIFICATION

Summary

P.S. Krishnaprasad   Michiel Hazewinkel*   Ben Hanzon*

University of Maryland
College Park, MD 20742

*Erasmus University
Rotterdam, The Netherlands

Abstract: In this paper we investigate various approximate methods for computing the conditional density of a parameter. These techniques are related to the structure of certain Lie algebras of operators with the identification problem.

Summary

Consider the stochastic differential system:

$$d\theta=0$$

$$dx_t = A(\theta)x_t dt + b(\theta)dw_t \tag{1}$$

$$dy_t = \langle c(\theta), x_t\rangle dt + dv_t.$$

Here $\{w_t\}$ and $\{v_t\}$ are independant, scalar, standard Wiener processes and $\{x_t\}$ is an $\mathbb{R}^n$-valued process. We let θ take values in a smooth manifold Θ $\mathbb{R}^n$. Assume that the map

$$\theta \to \Sigma(\theta):=(A(\theta), b(\theta), c(\theta)) \tag{2}$$

is sufficiently smooth and takes values in the space of minimal triples.

Define two differential operators,

$$A_o := \tfrac{1}{2}\langle b(\theta), \partial/\partial x\rangle^2 - \langle\partial/\partial x, A(\theta)x\rangle - \langle c(\theta)\, x,\rangle^2/2 \tag{3}$$

$$B_o := \langle c(\theta), x\rangle. \tag{4}$$

The problem is to devise approximate finite dimensional, recursive techniques for calculating the conditional density of the parameter θ given Y_t = the σ-algebra generated by the observations $\{y_s: 0\le s\le t\}$. The general formulas are known:

$$Q(t,\theta)= \frac{\int \rho(t,x,\theta)|dx|}{\int\int \rho(t,x,\theta)|dx|\cdot|d\theta|} \tag{5}$$

where $|dx|$ and $|d\theta|$ are fixed Riemannian volume elements on $\mathbb{R}$ and Θ and

$$\rho(t,x,\theta)= e^{\langle c(\theta),x\rangle y_t}\psi(t,x,\theta) \tag{6}$$

and

$$\frac{\partial\psi}{\partial t} = \{\mathcal{L}_o + y_t\mathcal{L}_1 + \frac{y_t^2}{2}\mathcal{L}_2 + \mathcal{L}_3\}\psi \tag{7}$$

where

$$\mathcal{L}_o := A_o$$

$$\mathcal{L}_1 := \langle c(\theta),b(\theta)\rangle\langle b(\theta),\partial/\partial x\rangle - \langle c(\theta),A(\theta)x\rangle$$

$$\mathcal{L}_3 := \langle c(\theta),\ b(\theta)\rangle^2$$

$$\mathcal{L}_4 = -\mathrm{tr}\ (A(\theta)). \tag{8}$$

Let $Q(t,\theta)= e^{-S(t,\theta)}$. In this pair we consider approximations related to

(a) local series approximations

$$S(t,\theta)= \sum_{i=0}^{\infty} a_i(t)\ \theta^{[i]}$$

(b) Gaussian initial conditions:

$$\rho(0,.,\theta) \text{ Gaussian for } \theta\in\Theta$$

Both these approximations are connected to the following algebraic objects.

(a) A sequence of Lie algebras $\{G^{(k)}\}_{k=0}^{\infty}$

where $\tilde{G}^{(0)} := \{A_o, B_o\}_{L.A.}$

$$\tilde{G}^{(1)} := \left\{\begin{bmatrix} A_o & 0 \\ \frac{\partial A_o}{\partial\theta} & A_o\end{bmatrix}, \begin{bmatrix} B_o & 0 \\ \frac{\partial B_o}{\partial\theta} & B_o\end{bmatrix}\right\}_{L.A.}$$

(b) Finite dimensional quotients of $\tilde{G}^{(0)}$ in one-to-one correspondence with rings that are quotients of $\mathbb{R}[\theta]$.

Our results use the fact that $\tilde{G}^{(0)}$ is a subalgebra of a current algebra ([1],[2]).

References

[1] P.S. Krishnaprasad and S.I. Marcus: "On the Lie Algebra of the identification problem", IFAC Symposium on Digital Control, New Delhi, Jan. 1982.

[2] P.S. Krishnaprasad, S.I. Marcus and M. Hazewinkel, "System identification and nonlinear filtering: Lie Algebras", Proc. 20th IEEE Conference on Decision and Control. San Diego, 1981.

Proc. IEEE Conf. on Decision and Control, San Diego, Dec. 1981.