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A.M. COHEN

EXCEPTIONAL PRESENTATIONS OF THREE GENERALIZED
HEXAGONS OF ORDER 2

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Exceptional presentations of three generalized hexagons of order 2 *)

by

Arjeh M. Cohen

ABSTRACT

Exceptional presentations of the generalized hexagon of order (2,1) on 21 points in the complex projective plane and of the dual of the classical generalized hexagon of order (2,2) on 63 points in the quaternionic projective plane are known. In this note, a third presentation of this kind is described, namely that of the (unique) generalized hexagon of order (2,8) on 819 points in the octonionic projective plane.

The construction employed leads to an embedding of the finite group of Lie type ${}^3D_4(2)$ in the Lie group of type $F_4(\mathbb{R})$.

KEY WORDS & PHRASES: *generalized hexagons*, ${}^3D_4(2)$, *octonionic projective plane*

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*) This report will be submitted for publication elsewhere.

1. PECULIARITIES OF THE OCTONIONS

Let $\mathbb{O}$ be the real division algebra of the *octonions* (also called *octaves* or *Cayley numbers*). Choose an $\mathbb{R}$-basis $e_0 = 1, e_1, \ldots, e_7$ such that multiplication in $\mathbb{O}$ is determined by the rules

(1) $$e_i^2 = -1 \qquad (i = 1,2,\ldots,7)$$

and

(2) $e_i e_j = e_k$ whenever (ijk) is one of the 3-cycles $(1+r, 2+r, 4+r)$, where i, j, k, r run through the integers modulo 7 and take their values in $\{1,2,\ldots,7\}$.

The anti-automorphism $x \mapsto \bar{x}$ of order 2 defined by

(3) $$\bar{x} = \xi_0 - \sum_{i=1}^{7} \xi_i e_i \qquad \text{whenever } x = \sum_{i=0}^{7} \xi_i e_i \in \mathbb{O}$$

is called *conjugation*. The real part $\mathrm{Re}(x)$ of an element x of $\mathbb{O}$, is given by

(4) $$\mathrm{Re}(x) = \tfrac{1}{2}(x+\bar{x}).$$

We recall that $\mathbb{O}$ is nonassociative and satisfies the following equations for $x,y,z \in \mathbb{O}$:

(5) $$x(yx) = (xy)x \qquad \text{(hence also denoted by } xyx)$$

(6) $$x(y\bar{x}) = (xy)\bar{x} \qquad \text{(hence also denoted by } xy\bar{x})$$

(7) $$(zxz)y = z(x(zy)) \quad \text{and} \quad y(zxz) = ((yz)x)z$$

(8) $$(zx)(yz) = z(xy)z$$

(9) $$x(xy) = x^2 y \quad \text{and} \quad \bar{x}(xy) = (\bar{x}x)y$$

For more details and an excellent introduction, the reader is referred to [7] or [8].

We shall need a particular element of $\mathbb{O}$:

$$\alpha = \frac{1}{2}\sum_{i=0}^{7} e_i \tag{10}$$

and a particular subset of $\mathbb{O}$:

$$Q = \{-e_i, e_i \mid i = 0,1,\ldots,7\}. \tag{11}$$

The following relations hold for $d \in Q$:

$$\left.\begin{aligned} \alpha d\alpha + \alpha &= -2d\bar{\alpha}d \\ \alpha d\alpha - \alpha &= -2\bar{d} \end{aligned}\right\} \text{ if } \operatorname{Re} \alpha d = \frac{1}{2}$$

(12)

$$\left.\begin{aligned} \alpha d\alpha + \alpha &= -2\bar{d} \\ \alpha d\alpha - \alpha &= 2d\bar{\alpha}d \end{aligned}\right\} \text{ if } \operatorname{Re} \alpha d = -\frac{1}{2}$$

The stabilizer in Aut $\mathbb{O}$ of Q is denoted by C. This group is known (see [4], [7]):

$$C = \langle (1234567),(124)(365),\delta^{-}_{\{1,2,4,7\}}(12)(46)\rangle, \tag{13}$$

where a permutation π stands for the $\mathbb{R}$-linear transformation induced on $\mathbb{O}$ by the permutation $e_i \mapsto e_{\pi(i)}$ $(i = 0,1,\ldots,7)$ and δ^{-}_K for $K \subseteq \{0,1,\ldots,7\}$ stands for the $\mathbb{R}$-linear map sending e_i to $-e_i$ whenever $i \in K$ and fixing e_i if $i \notin K$ $(i = 0,1,\ldots,7)$.

C is a nonsplit extension of a (diagonal) group of order 2^3 by $PS\ell_2(7)$. Thus C has order $2^6.3.7$. The stabilizer in C of α, denoted by C_0, is a non-abelian group of order 21:

$$C_0 = \langle (1234567),(124)(365)\rangle. \tag{14}$$

The set $(Q\alpha)Q$ will be used frequently in the sequel; we record some of its properties here:

(15) $$(Q\alpha)Q = C(\alpha) \cup C(-\alpha) = \{\frac{1}{2}\sum_{i=0}^{7} \varepsilon_i e_i \mid \varepsilon_i \in \{-1,1\};\ \prod_{i=0}^{7} \varepsilon_i = 1\}.$$

Let $c,d,e,f \in Q$. Then

(16) $$(c\alpha)d = (e\alpha)f \Rightarrow c = \pm e \text{ and } d = \pm f$$

(17) $$c(d\alpha) \in Q\alpha$$

(18) $$(\alpha c)(cd) \in (Q\alpha)d$$

(19) $$(Q\alpha)d = Q(\alpha d)$$

By use of C_0 and C, the verification of these equalities may be reduced to a slight amount of work. Details are omitted here.

2. THE EXCEPTIONAL JORDAN ALGEBRA $\mathbb{J}_3(\mathbb{F})$

Let $\mathbb{F}$ be a division subalgebra of $\mathbb{O}$. The *exceptional Jordan algebra* $\mathbb{J}_3(\mathbb{F})$ is defined on the set of 3×3 hermitian matrices (with respect to conjugation) over $\mathbb{F}$. Its multiplication is given by

(20) $$A \circ B = \frac{1}{2}(AB + BA) \qquad (A,B \in \mathbb{J}_3(\mathbb{F}))$$

where AB stands for the usual matrix product of A and B. Note that $A^2 = A \circ A$. Let I be the 3×3 identity matrix. Multiplication is commutative, but non-associative, (see [8]). The Jordan algebra $\mathbb{J}_3(\mathbb{F})$ has a natural inner product $(.,.)$ given by

(21) $$(A,B) = \mathrm{Re\ Trace}(AB) \qquad (A,B \in \mathbb{J}_3(\mathbb{F})).$$

Aut $\mathbb{J}_3(\mathbb{F})$, the automorphism group of $\mathbb{J}_3(\mathbb{F})$, preserves this inner product.

Let $P(\mathbb{F})$ be the set of idempotents in $\mathbb{J}_3(\mathbb{F})$ having trace 1. Then $P(\mathbb{F})$ together with $\{\{A \in P(\mathbb{F}) \mid A \circ B = 0\} \mid B \in P(\mathbb{F})\}$ for the collection of lines, is a projective plane over $\mathbb{F}$. For $p \in P(\mathbb{F})$, let $\sigma_p\colon \mathbb{J}_3(\mathbb{F}) \to \mathbb{J}_3(\mathbb{F})$ be the map given by

(22) $\sigma_p(A) = ((1-2p)A)(1-2p) \qquad (A \in \mathbb{J}_3(\mathbb{F}))$.

Then

(23) $\sigma_p \in \mathrm{Aut}\,\mathbb{J}_3(\mathbb{F})$ and $\sigma_p^2 = 1$

if $p \in P(\mathbb{F}')$ for $\mathbb{F}'$ a commutative subfield of $\mathbb{F}$.

This observation follows from (5.4) of [8] since any two maximal commutative subfields of $\mathbb{O}$ are in the same Aut $\mathbb{O}$ - orbit. For $\tau \in \mathrm{Aut}(\mathbb{F})$, let $\hat{\tau}$ denote the automorphism of $\mathbb{J}_3(\mathbb{F})$ given by

(24) $\hat{\tau}(A) = (\tau a_{ij})_{1\le i,j\le 3}$ if $A = (a_{ij})_{1\le i,j\le 3} \in \mathbb{J}_3(\mathbb{F})$.

For π a permutation of 1, 2, 3, denote by $\tilde{\pi}$ the automorphism of $\mathbb{J}_3(\mathbb{F})$ given by

(25) $\tilde{\pi}(A) = (a_{\pi(i)\pi(j)})_{1\le i,j\le 3}$ if $A = (a_{ij})_{1\le i,j\le 3} \in \mathbb{J}_3(\mathbb{F})$.

In fact, $\hat{\tau}$ and $\tilde{\pi}$ as above preserve matrix multiplication.

If $p \in P(\mathbb{F}')$ for $\mathbb{F}'$ a commutative subfield of $\mathbb{F}$ and if ϕ is an automorphism of $\mathbb{J}_3(\mathbb{O})$ preserving matrix multiplication, then

(26) $\phi\sigma_p\phi^{-1} = \sigma_{\phi p}$

In fact, σ_p, when defined, is on $P(\mathbb{F})$ a homology with center p and axis $\{A \in P(\mathbb{F}) \mid A\circ p = 0\}$. This explains the importance of $\phi\sigma_p\phi^{-1}$; it is a homology with center ϕp. As $\mathrm{Aut}(\mathbb{J}_3(\mathbb{F}))$ is transitive on $P(\mathbb{F})$ (see [8] in case $\mathbb{F} = \mathbb{O}$), this yields that for any $q \in P(\mathbb{F})$ there is a 'canonical' homology $\sigma_q \in \mathrm{Aut}(\mathbb{J}_3(\mathbb{F}))$ with center q and axis $\{A \in P(\mathbb{F}) \mid A\circ q = 0\}$.

3. THE GENERALIZED HEXAGON OF ORDER (2,8)

For $\pi = (123)$; $i \in \{1,2,3\}$; $x \in (Q\alpha)Q$; $c,d,e \in Q$ (see (11)) discern the following elements of $\mathbb{J}_3(\mathbb{O})$ in $P(\mathbb{O})$.

(27) $$p(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(28) $$p(i) = \tilde{\pi}^{i-1} p(1)$$

(29) $$p(1,e) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & e \\ 0 & \bar{e} & 1 \end{pmatrix}$$

(30) $$p(i,e) = \tilde{\pi}^{i-1} p(1,e)$$

(31) $$p(1,x,e) = \frac{1}{4} \begin{pmatrix} 2 & \overline{ex} & x \\ ex & 1 & e \\ x & \bar{e} & 1 \end{pmatrix}$$

(32) $$p(i,x,e) = \tilde{\pi}^{i-1} p(1,x,e).$$

Recall from (10) that $\alpha = 1/2\ \Sigma_{i=0}^{7} e_i$. Consider the following subsets of $P(\mathbb{O})$:

(33) $$L_0 = \{p(i) \mid i = 1,2,3\}$$

(34) $$L_1 = \{p(i,e) \mid i = 1,2,3;\ e \in Q\}$$

(35) $$L_2 = \{p(i,x,e) \mid i = 1,2,3;\ e \in Q;\ x \in (Q\alpha)e\}$$

The set $H = L_0 \cup L_1 \cup L_2$ has $3(1+16+16^2) = 819$ elements. H is turned into a graph $(H,\sim)$ by requiring:

(36) $$p \sim q \iff (p,q) = 0 \qquad (p,q \in H)$$

Note that L_0 is maximal clique of $(H,\sim)$. Let $\underline{\underline{h}}$ be the set of all maximal cliques of H; its members are called *lines*. Denote by D the subgroup

(37) $$D = \langle \sigma_p, \hat{\tau}, \tilde{\pi} \mid p \in \{p(1,\alpha,1)\} \cup L_0 \cup L_1;\ \tau \in C_0;\ \pi \in Sym(3) \rangle$$

of Aut $\mathbb{J}_3(\mathbb{O})$. The following result comprises the presentation we are after.

<u>THEOREM</u>. *Let* $(H,\sim)$, $\underline{\underline{h}}$, D *and* $(.,.)$ *be as described. Then*

(i) D *stabilizes* H *and its restriction to* H *is faithful. Its image is a transitive group of automorphisms of* $(H,\sim)$.

(ii) *For any two distinct points* p, q *of* H *the inner product* (p,q) *is one of* $0, \frac{1}{4}, \frac{1}{2}$.

(iii) $(H,\underline{\underline{h}})$ *is the classical generalized hexagon of order* $(2,8)$ *and* $D \cong \mathrm{Aut}({}^3D_4(2))$, *an extension of* ${}^3D_4(2)$ *by an element of order* 3.

<u>PROOF</u>.

(i) First, we establish the hardest part of the proof, namely the verification that D stabilizes H. Frequent use is made of the formulae (5), (6), ...,(19).

It is immediate from the construction of H that H is invariant under $\hat{\tau}$ for $\tau \in C_0$ and under $\widetilde{(123)}$. Let $\pi = (23)$. Then for any $e \in Q$ and $x \in (Q\alpha)Q$:

$$\tilde{\pi}p(1,x,e) = p(1,ex,\bar{e}), \tag{38}$$

while $x \in (Q\alpha)e$ implies $(ex)\bar{e} \in Q\alpha$, so that $p(1,x,e) \in H$ leads to $p(1,ex,\bar{e}) \in H$. Moreover,

$$\tilde{\pi}p(2,x,e) = p(3,ex,\bar{e}) \tag{39}$$

As above, $p(2,x,e) \in H$ implies $(ex)\bar{e} \in Q\alpha$ so that $\pi p(2,x,e) \in H$. It readily follows that $\tilde{\pi}L_2$ is contained in H, and $\tilde{\pi}H = H$. Since (123) and (23) generate Sym(3), this settles that $\tilde{\pi}H = H$ for any $\pi \in \mathrm{Sym}(3)$.

Let $q = p(1)$. For $e \in Q$ and $x \in (Q\alpha)e$, we have

$$\begin{cases} \sigma_q\ p(i) = p(i) & (i = 1,2,3) \\ \sigma_q\ p(1,e) = p(1,e) & \\ \sigma_q\ p(i,e) = p(i,-e) & (i = 2,3) \\ \sigma_q\ p(1,x,e) = p(1,-x,e) & \\ \sigma_q\ p(2,x,e) = p(2,-x,-e) & \\ \sigma_q\ p(3,x,e) = p(3,x,-e) & \end{cases} \tag{40}$$

Thus $\sigma_{p(1)}H = H$. In view of (26), it follows that $\sigma_q H = H$ for any $q \in L_0$.

Next, let $q = p(1,e)$ for some $e \in Q$. For any $d \in Q$ and $x \in (Q\alpha)d$, we have

$$
(41) \qquad \begin{cases}
\sigma_q\ p(1) = p(1) \\
\sigma_q\ p(2) = p(3) \\
\sigma_q\ p(1,d) = p(1,e\bar{d}e) \\
\sigma_q\ p(2,d) = p(3,\overline{ed}) \\
\sigma_q\ p(1,x,d) = p(1,-\bar{e}(dx),e\bar{d}e) \\
\sigma_q\ p(2,x,d) = p(3,-e(dx)e,-\overline{ed})
\end{cases}
$$

From (5),...,(9) and (17),(18),(19), it is readily deduced that $\sigma_{p(1,e)}H = H$. Applying $(\widehat{123})$, we get $\sigma_q H = H$ for any $q \in L_1$.

Finally, let $q = p(1,\alpha,1)$. For $d \in Q$, we have

$$
(42) \qquad \begin{cases}
\sigma_q\ p(1) = p(1,1) \\
\sigma_q\ p(2) = p(1,\alpha,-1) \\
\sigma_q\ p(3) = p(1,-\alpha,-1) \\
\sigma_q\ p(1,d) = \begin{cases} p(1) & \text{if } d = 1 \\ p(1,-1) & \text{if } d = -1 \\ p(1,d\alpha,-1) & \text{if } d \in Q\backslash\{\pm 1\} \end{cases} \\
\sigma_q\ p(2,d) = \begin{cases} p(2,-d\alpha d,-d) & \text{if } \mathrm{Re}\ \alpha\bar{d} = \frac{1}{2} \\ p(3,-\alpha\bar{d},-\bar{d}) & \text{if } \mathrm{Re}\ \alpha\bar{d} = -\frac{1}{2} \end{cases}
\end{cases}
$$

This shows that $\sigma_q(L_0 \cup L_1) \subseteq H$.

In verifying that $\sigma_q\ L_2 \subseteq H$, we may restrict considerations to $\sigma_q p$ for $p = p(i,y,d)$ with

(i) $i \in \{1,2\}$ (as $(\widehat{23})\sigma_q = \sigma_q(\widehat{23})$).

(ii) $d = 1,-1,e_1,-e_1$ (as C_0 acts on Q with these octonions as orbit representatives and $\tilde{\tau}\sigma_q = \sigma_q\tilde{\tau}$ for $\tau \in C_0$).

(iii) $y = (cx)d$ with $c = 1,-1,e_1,-e_1,e_3,-e_3$ (as $\tau = (124)(365)$ acts on Q and stabilizes $\{\pm e_0, \pm e_1, \alpha\}$ pointwise).

(iv) Moreover, if $d = \pm 1$, we may take $c \in \{1,-1,e_1,-e_1\}$ (for then all of C_0 stabilizes d).

Thus we need to check whether $\sigma_q p \in H$ for 40 particular points p. This is done in the table. It should be remarked that (still) many equalities listed are superfluous. For instance, $\sigma_q p(2,-e_1\alpha e_1,-e_1) = p(2,e_1)$ by (42).

Thus $\sigma_{p(1,\alpha,1)}$ stabilizes H. The conclusion is that the generators of D, and hence D itself, too, stabilize H. Thus the restriction of D to H is a permutation group of H. Since H contains an $\mathbb{R}$-basis of $\mathbb{J}_3(\mathbb{O})$, this restriction is faithful and D may be viewed as a group of permutations on H. Since D consists of automorphisms of $\mathbb{J}_3(\mathbb{O})$, it preserves the inner product $(.,.)$ and therefore adjacency $\sim$ in H. We obtain that D is a subgroup of $\mathrm{Aut}(H,\sim)$. From (38),...,(42) and the table it is readily seen that D is transitive on H. This proves (i).

(ii) Since D is transitive the claim need only be checked for pairs p, q where $p = p(1)$ and $q \in H\backslash\{p\}$. Thus the proof amounts to the observation that the 1,1-coefficient of any matrix $q \in H\backslash\{p(1)\}$ is one of 0, $\frac{1}{4}$, $\frac{1}{2}$.

(iii) To establish that $(H,\underline{\underline{h}})$ is a generalized hexagon of order (2,8) one need only show (see [5]) that $(H,\sim)$ is a distance-regular graph with intersection array (18,16,16;1,1,9) in the terminology of [1]. From the preceding formulae, it is easily obtained that the stabilizer in D of p(1) has orbits $\{p(2),p(3)\} \cup \{p(1,e) \mid e \in Q\}$, $\{p(i,e),\ p(1,(c\alpha)e,e) \mid e \in Q\}$, and $\{p(i,(c\alpha)e,e) \mid i = 2,3;\ c,e \in Q\}$. The proof is omitted, but it is noted that $\sigma_{p(1,\alpha,1)}\sigma_{p(1,d)}\sigma_{p(1,\alpha,1)}\sigma_{p(1,e)}\sigma_{p(1,\alpha,1)}$ fixes p(1) for all $d,e \in Q\backslash\{\pm 1\}$. This implies that D is distance-transitive on H and that $p,q \in H$ are of distance 1 (2,3 resp.) iff $(p,q) = 0$ ($\frac{1}{2}$, $\frac{1}{4}$ resp.). Distance-transitivity of D accounts for distance-regularity of $(H,\sim)$. It is now straightforward to compute the actual intersection array.

By [6], the generalized hexagon $(H,\underline{\underline{h}})$ is the unique one of order (2,8), i.e. the classical one associated with the group ${}^3D_4(2)$.

As to D, so far we have that D and ${}^3D_4(2)$ are subgroups of $\mathrm{Aut}(H,\sim)$ which are distance-transitive. But $\sigma_{p(1)}$ fixes all vertices of H adjacent to p(1), and leaves invariant all lines containing a point adjacent to p(1) (cf. (40)), so corresponds to the unique central involution of ${}^3D_4(2)$ associated with p(1) (cf. [6], [9]). Therefore, D contains $\{\phi\sigma_{p(1)}\phi^{-1} \mid \phi \in D\}$ which by transitivity of D is the set of all central involutions in ${}^3D_4(2)$. By simplicity of ${}^3D_4(2)$ these involutions generate all of ${}^3D_4(2)$, so that ${}^3D_4(2)$ is contained in D.

TABLE

Images of σ_q for $q = p(1,\alpha,1)$ on 40 points.

p_i	$\sigma_q p_1$	$\sigma_q p_2$
$p(i,\alpha,1)$	$p(1,\alpha,1)$	$p(3,-1)$
$p(i,-\alpha,-1)$	$p(3)$	$p(2,1)$
$p(i,\alpha e_1,e_1)$	$p(3,-e_1\alpha,1)$	$p(2,e_1\alpha,1)$
$p(i,-\alpha e_1,-e_1)$	$p(3,-e_1\alpha e_1,-e_1)$	$p(3,e_1\alpha,-1)$
$p(i,-\alpha,1)$	$p(1,-\alpha,1)$	$p(3,-\alpha,1)$
$p(i,\alpha,-1)$	$p(2)$	$p(2,\alpha,-1)$
$p(i,-\alpha e_1,e_1)$	$p(2,e_1\alpha,-1)$	$p(1,e_1\alpha e_1,e_1)$
$p(i,\alpha e_1,-e_1)$	$p(2,\alpha e_1,-e_1)$	$p(1,\alpha e_1,-e_1)$
$p(i,e_1\alpha,1)$	$p(1,-e_1\alpha,1)$	$p(2,\alpha e_1,e_1)$
$p(i,-e_1\alpha,-1)$	$p(1,-e_1)$	$p(3,e_1\alpha e_1,e_1)$
$p(i,e_1\alpha e_1,e_1)$	$p(2,-\alpha e_1,e_1)$	$p(3,e_1)$
$p(i,-e_1\alpha e_1,-e_1)$	$p(3,-e_1\alpha,-1)$	$p(2,e_1)$
$p(i,-e_1\alpha,1)$	$p(1,e_1\alpha,1)$	$p(1,e_1\alpha e_1,e_1)$
$p(i,e_1\alpha,-1)$	$p(1,e_1)$	$p(1,-\alpha e_1,e_1)$
$p(i,-e_1\alpha e_1,e_1)$	$p(3,-e_1\alpha e_1,e_1)$	$p(3,\alpha e_1,-e_1)$
$p(i,e_1\alpha e_1,-e_1)$	$p(2,-e_1\alpha,1)$	$p(2,e_1\alpha e_1,-e_1)$
$p(i,(e_3\alpha)e_1,e_1)$	$p(3,-(e_1\alpha)e_6,e_6)$	$p(1,-(e_6\alpha)e_4,-e_4)$
$p(i,-(e_3\alpha)e_1,-e_1)$	$p(2,(e_7\alpha)e_5,e_5)$	$p(1,-(e_7\alpha)e_4,e_4)$
$p(i,-(e_3\alpha)e_1,e_1)$	$p(2,(e_2\alpha)e_6,e_6)$	$p(2,-(e_5\alpha)e_2,e_2)$
$p(i,(e_3\alpha)e_1,-e_1)$	$p(3,(e_1\alpha)e_5,e_5)$	$p(3,(e_4\alpha)e_2,e_2)$

Standard permutation representation theoretic arguments yield that $Aut(H,\sim)$ is a subgroup of $Aut({}^3D_4(2))$. On the other hand, $(H,\sim)$ is isomorphic to the graph whose vertex set is the conjugacy class of central involutions and in which two vertices are adjacent whenever they commute. Therefore $Aut(H,\sim) = Aut({}^3D_4(2))$, up to isomorphism, and ${}^3D_4(2)$ is a normal subgroup of D.

Now $\tau = (235)(476)$ regarded as an element of C (cf. [13]) induces an automorphism $\hat{\tau}$ of $(H,\sim)$ that fixes $p(1)$ and three of the nine lines through p(1). Since the stabilizer of p(1) in ${}^3D_4(2)$ induces $PSL_2(8)$ in its natural action on these 9 lines, $\hat{\tau}$ is not contained in ${}^3D_4(2)$. This yields that $|D|$ is a multiple of $|{}^3D_4(2)|.3$.

We finish by showing that Aut(H,~) has exactly this order, thus establishing $D = \mathrm{Aut}(H,\sim) = \langle {}^3D_4(2), \hat{\tau} \rangle$ of order $2^{12}.3^5.7^2.13$.

Write $G = \mathrm{Aut}(H,\sim)$ and let G_0 be the stabilizer in G of p(1). If $\sigma \in G_0$ stabilizes each of the 18 points of H adjacent to p(1). Then $\sigma \in \{1, \sigma_{p(1)}\}$ by the arguments of [6].

Suppose now that σ is an element of G_0 of prime order $r \neq 2,3,7$. Then σ acts nontrivially on the 9 lines through p(1), so that r = 5. Since there is no point in H all whose neighbours are fixed by σ (by the same reasoning as for p(1)), the set of fixed points under σ is a generalized hexagon of order (2,4). Since there are no such geometries, this leads to a contradiction.

So far, we have that G_0 has order $2^a.3^b.7^c$ for $a,b,c \in \mathbb{N}$ and that the kernel of the action of G_0 on the neighbours of p(1) has order 2.

The kernel of the action of G_0 on the 9 lines through p(1) has order a divisor of $2.2^9 = 2^{10}$. But the order is not 2^{10}, for there is no involutory automorphism fixing all neighbours of p(1) except for two (collinear) points, as the labels, in the terminology of [6], of the points at distance 3 of p(1) in (H,~) all have the same parity. It follows that the order of the kernel is at most 2^9. Next consider the action of G_0 on the 9 lines through p(1). We already know that $\langle D \cap G_0, \hat{\tau} \rangle$ induces a group isomorphic to $P\Gamma L_2(8)$ on these lines. This is a maximal subgroup of Sym(9). As G_0 does not have elements of order 5, it follows that $|G_0|$ divides $2^9|P\Gamma L_2(8)|$ and that $|G|$ divides $|{}^3D_4(2)|.3$. This ends the proof of the theorem. □

4. TWO MORE GENERALIZED HEXAGONS OF ORDER 2

Let $Q_1 = \{1,-1,e_1,-e_1\}$ and $Q_2 = \{1,-1\}$. For j = 1,2, write

$$L_1^j = \{p(i,e) \mid i = 1,2,3;\ e \in Q_j\},$$

$$L_2^j = \{p(i,x,e) \mid i = 1,2,3;\ e \in Q_j;\ x \in (Q_j\alpha)e\},$$

and define $H_j = L_0 \cup L_1^j \cup L_2^j$.

Then H_j is a subset of H and it is straightforward to see that the subgraph $(H_j,\sim)$ of $(H,\sim)$ is the point graph of a generalized hexagon of order $(2,3-j)$. In fact, $(H_1,\sim)$ is the dual of the classical generalized hexagon associated with $G_2(2)$ (see [4]) and $(H_2,\sim)$ is the unique generalized hexagon associated with $PSL_2(7)$ (whose line graph is the Heawood graph on 14 points).

As $H_1 \subseteq \mathbb{J}_3(\mathbb{R}(\alpha,e_1))$, the orthogonality preserving map $v \mapsto vv^*$ (where v^* is the usual conjugate transpose of v) from $(\mathbb{R}(\alpha,e_1))^3$ to $\mathbb{J}_3(\mathbb{R}(\alpha,e_1))$ exhibits the above presentation of the dual classical generalized hexagon as a well known one on the 'root system' of the quaternionic reflection group W(Q) studied in [3] (note that $\mathbb{R}(\alpha,e_1)$ is indeed a quaternion division algebra). Similarly, $H_2 \subseteq \mathbb{J}_3(\mathbb{R}(\alpha))$ corresponds to the root system of the complex reflection group $W(J_3(4))$ studied in [4].

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