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AN EIGENVALUE PROBLEM RELATED TO CELL GROWTH

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An eigenvalue problem related to cell growth*)

by

H.J.A.M. Heijmans

ABSTRACT

In this paper, the eigenvalues of the operator corresponding to the partial differential equation, which describes the evolution of a population reproducing by simple fission , are investigated. This is done by transforming the eigenvalue problem to an integral equation. The theory concerning positive operators on a Banach space appears to be very useful.

KEY WORDS & PHRASES: *population density, cone in a Banach space, positive operator,* u_0*-positive operator, non-support cperator*

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*) This report will be submitted for publication elsewhere.

INTRODUCTION

In this paper, we study the eigenvalue problem

$$\frac{d}{dx}(g(x)n(x)) = -\lambda n(x) - \mu(x)n(x) - b(x)n(x) + 4b(2x)n(2x), \qquad \tfrac{1}{2}a < x < 1$$

(0.1) $\quad$ (where one should read $b(2x)n(2x) = 0$, $x \geq \frac{1}{2}$)

$$n(\tfrac{1}{2}a) = 0$$

n is summable.

The study of this eigenvalue problem can be seen as part of a bigger project, namely the investigation of the partial differential equation

(0.2)
$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(g(x)n(t,x)) = -\mu(x)n(t,x) - b(x)n(t,x) + 4b(2x)n(t,2x), \qquad \tfrac{1}{2}a < x < 1,\ t > 0$$

$$n(0,x) = \phi(x)$$

$$n(t,\tfrac{1}{2}a) = 0,$$

which describes the dynamics of a population, the members of which reproduce by simple fission into two equal parts (for instance algae or cells). Here t is the time, x stands for the weight of an individual, n is the population density as a function of t and x, μ is the death-rate, g is the growth-rate (of an individual) and b is the rate at which individuals divide.

This evolution equation which originally has been derived by SINKO and STREIFER ([9]; see also BELL and ANDERSON [1]) will be studied in a forthcoming paper [2]. The present paper is entirely concerned with the investigation of the eigenvalue problem (0.1). Our main conclusion will be that (0.1) has a dominant real eigenvalue λ_0 with corresponding positive eigenvector n_0. In [2], this conclusion will be used to prove that the solution of the linear evolution problem (0.2) behaves, under the extra assumption on the growth-rate $g(2x) < 2g(x)$, asymptotically for $t \to \infty$ as

$$n(t,x) \sim Ce^{\lambda_0 t}\, n_0(x),$$

where C is a constant depending on the initial condition ϕ only. λ_0 may be interpreted as the Malthusian parameter (intrinsic rate of natural increase) for this model and n_0 as the stable weight distribution.

The organization of this paper is as follows. In section one, the eigenvalueproblem (0.1) and the properties of the functions μ,b, and g are described in more detail. In section two we shall reduce the problem to an integral equation, by means of some elementary transformations.

In section three some results from the theory of positive operators are presented, and in section four and five these results will be used to prove the existence of a dominant eigenvalue (i.e. an eigenvalue with largest real part). The eigenvector corresponding to this dominant eigenvalue will appear to be positive.

The position of the remaining elements of the spectrum will be investigated in section six, and here the characteristic equation which provides us with a tool to compute the eigenvalues explicitly, will be derived.

In section seven, finally, the adjoint equation is studied.

SECTION ONE: ASSUMPTIONS ON b,g AND μ

In this section we will specify what values b,g and μ can take, and we will derive the boundary conditions for the eigenvalue problem (0.1).

We make the following assumptions concerning g:

(1.1)
1^{o} g is continuous on the interval $[\frac{1}{2}a,1]$
2^{o} $g(x) > 0, \quad \frac{1}{2}a \le x \le 1$.

This last assumption is very essential. In fact, the whole theory developed in this paper would not work anymore if g(x) would become zero or negative for some values of x. The assumption that g is continuous has been made for convenience. It makes things easier to work with.

For μ we only assume that it is summable, i.e.

$$\mu \in L_1[\tfrac{1}{2}a,1] .$$

b is supposed to satisfy the following conditions:

(1.2)

1. b is summable on $[\frac{1}{2}a, 1-\varepsilon]$ for all $\varepsilon > 0$,

2. $b(x) = 0, \quad \frac{1}{2}a \le x \le a$

$b(x) > 0, \quad a < x < 1$,

3. $\lim_{\varepsilon \downarrow 0} \int_a^{1-\varepsilon} b(x)dx = \infty.$

Condition 2. tells us that the minimum weight at which individuals can divide is a. This is described by the boundary condition

$$(1.3) \qquad n(\tfrac{1}{2}a) = 0.$$

As a consequence of 3., individuals have to divide before they can reach $x = 1$. Indeed we have $n(1) = 0$, for a solution of (0.1) as we show now. Let

$$(1.4) \qquad E(x) := \exp\left(-\int_{\frac{1}{2}a}^{x} \frac{b(\xi)+\mu(\xi)}{g(\xi)}\, d\xi\right), \qquad \tfrac{1}{2}a \le x \le 1 ,$$

then $\qquad E(1) = 0$.

$$(1.5) \qquad G(x) := \int_{\frac{1}{2}a}^{x} \frac{d\xi}{g(\xi)}, \qquad \tfrac{1}{2}a \le x \le 1.$$

Then the solution of (0.1) on $[\frac{1}{2},1]$ is given by

$$(1.6) \qquad n(x) = A.\, \frac{E(x)}{g(x)}\, e^{-\lambda G(x)} ,$$

where A is some constant.
This implies among others

$$(1.7) \qquad n(1) = 0$$

if n is a solution of (0.1).

The assumptions on the functions b, g and μ are very natural, taking their biological interpretation into account.

The interpretation of n indicates that the solutions of (0.1) have to be summable. There is no justification for working in the space of continuous functions. However, in section two, it will appear that all solutions of (0.1) are continuous.

In case that $a > 0$ the problem can be solved in a finite number of steps, whereas this is impossible when $a = 0$. With this in mind we have thought it interesting to study both cases, although the case $a = 0$ is less relevant from a biological point of view.

SECTION TWO: REDUCTION TO THE INTEGRAL EQUATION.

An abstract way of writing equation (0.1) is:

$$(2.1) \qquad An = \lambda n, \quad n \in D(A) ,$$

where A is the closed linear operator given by

$$(2.2) \qquad (An)(x) = -\frac{d}{dx}(g(x)n(x)) - \mu(x)n(x) - b(x)n(x) + 4b(2x)n(2x) ,$$

with domain

$$(2.3) \text{ and } \quad \begin{aligned} D(A) = \{n \in L_1[\tfrac{1}{2}a,1] \mid \ & \frac{d}{dx}(g(x)n(x)) \text{ is defined a.e., } n(\tfrac{1}{2}a) = 0, \\ \forall_{x \in [\frac{1}{2}a,1]}(-\frac{d}{dx}(g(x)n(x)) - \mu(x)n(x) - b(x)n(x) \ + & \\ + 4b(2x)n(2x) \in L_1[\tfrac{1}{2}a,1]\}. & \end{aligned}$$

Equation (0.1) can be put into a more tractable form by means of the transformation

$$(2.4) \qquad v(x) = \frac{g(x)}{E(x)} e^{\lambda G(x)} n(x), \quad \tfrac{1}{2}a \le x \le 1 ,$$

where E and G are given by (1.4) and (1.5). Substitution of this expression in (0.1) yields

$$(2.5) \qquad \begin{aligned} & \frac{dv}{dx} = k_\lambda(x)v(2x), \quad \tfrac{1}{2}a < x < 1 \\ & (\text{where by definition } k_\lambda(x)v(2x) = 0, \text{ if } x > \tfrac{1}{2}) \\ & v(\tfrac{1}{2}a) = 0 , \end{aligned}$$

(2.6) where $k_\lambda(x) := 4b(2x)\,\frac{E(2x)}{E(x)}\,\frac{1}{g(2x)}\,e^{-\lambda(G(2x)-G(x))}$

satisfies the following conditions:

(2.7) $k_\lambda \in L_1[\frac{1}{2}a,\frac{1}{2}]$

(2.8) $k_\lambda(x) \geq 0,\ \frac{1}{2}a \leq x \leq \frac{1}{2},\ \lambda \in \mathbb{R}$

(2.9) $k_\lambda(\frac{1}{2}a) = 0$

(2.10) $k_\lambda(x) = k(x)e^{-\lambda r(x)}$,

where k does not depend on λ and satisfies (2.7), (2.8) and (2.9). $r(x) = G(2x) - G(x)$ is continuous and positive, except at $x = 0$, for the case that $a = 0$.

From (2.5) one sees immediately that $v(x)$ = constant for $\frac{1}{2} \leq x \leq 1$. This fact together with (2.4) and the condition that n has to be summable yields

(2.11) $v \in L_1[\frac{1}{2}a,1]$.

Integration of equation (2.5) on both sides and substitution of the boundary condition $v(\frac{1}{2}a) = 0$ gives us

(2.12) $$v(x) = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)v(2\xi)\,d\xi, \quad \frac{1}{2}a \leq x \leq 1.$$

Thus the eigenvalueproblem (2.1) (or equivalently (0.1)) has been reduced to the integral equation (2.12). It is well-known that quite often integral equations are not unpleasant to deal with, because the corresponding integral operator is compact.

In (2.11) we have already mentioned that v has to be an L_1-function. If $n \in D(A)$, and v is given by (2.4), one can easily see that $\Psi_x k_\lambda(x)v(2x) \in L_1[\frac{1}{2}a,\frac{1}{2}]$; as a consequence, we find that v is continuous, if v is a solution of the integral equation (2.12). This permits, us to study equation (2.12) in the space of continuous functions. Moreover, we have that the corresponding n is continuous as well.

Let the Banach-space X_0 be defined by

$$X_0 = \{\phi \in C[\tfrac{1}{2}a,1] \mid \phi(\tfrac{1}{2}a) = 0\} \tag{2.13}$$

with the usual supremum - norm.
The integral operator corresponding to equation (2.12) is given by

$$(T_\lambda\phi)(x) = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)\phi(2\xi)d\xi, \quad \phi \in X_0 . \tag{2.14}$$

The following result follows immediately from the Arzêla - Ascoli - theorem. (See [11]).

THEOREM 2.1. T_λ *is a bounded, linear, compact operator on* X_0 *for all* $\lambda \in \mathbb{C}$.

For an operator L we denote by $\sigma(L)$ resp. $P\sigma(L)$ the spectrum of L, respectively the point spectrum of L. The spectral radius is denoted by $r(L)$.

As we have seen, there is a correspondence between the operators A and T_λ. We can formulate this in the following way.
Let Σ be defined by

$$\Sigma := \{\lambda \in \mathbb{C} \mid 1 \in P\sigma(T_\lambda)\} . \tag{2.15}$$

THEOREM 2.2.

$$\sigma(A) = P\sigma(A) = \Sigma. \tag{2.16}$$

PROOF. We have seen that $An = \lambda n$, for $n \in D(A)$ if and only if $T_\lambda v = v$, where $v \in X_0$ is given by (2.4). This implies that $\Sigma = P\sigma(A)$. Now suppose that $\lambda \notin P\sigma(A)$, and $\psi \in L_1[\frac{1}{2}a,1]$. We are going to construct a solution $\bar{n} \in C[\frac{1}{2}a,1]$ of the inhomogeneous equation $A\bar{n} - \lambda\bar{n} = \psi$. We do not demand that $\bar{n}(\frac{1}{2}a) = 0$.
Let

$$R_\lambda(x) := E(x)e^{-\lambda G(x)}, \quad \tfrac{1}{2}a \le x \le 1 .$$

$$\bar{n}(x) = \frac{R_\lambda(x)}{g(x)} \cdot \{1 - \int_{\frac{1}{2}}^{x} \frac{\psi(\xi)}{R_\lambda(\xi)}\, d\xi\}, \qquad \tfrac{1}{2} \le x \le 1 \quad .$$

Suppose that we have computed $\bar{n}$ on the interval $[2^{-p},1]$, $p \ge 1$.
Then the solution on $[2^{-p-1},2^{-p}]$ is given by

$$\bar{n}(x) = \frac{R_\lambda(x)}{g(x)}\{A_p + \int_{x}^{2^{-p}} \frac{\psi(\xi)-4b(2\xi)\bar{n}(2\xi)}{R_\lambda(\xi)}\, d\xi\} \quad ,$$

where

$$\bar{n}(2^{-p}) = \frac{R_\lambda(2^{-p})}{g(2^{-p})} \cdot A_p \, .$$

The constructed solution $\bar{n}$ is continuous on all intervals $[2^{-p},1]$, $p \ge 0$. In case that $a = 0$, it can be shown by a straightforward computation that $\lim_{x\downarrow 0} \bar{n}(x)$ exists, and is finite, which proves that the solution is continuous on the interval $[0,1]$.The basic idea behind this computation is that the length of the successive intervals, on which the solution $\bar{n}$ is computed, reduces each time by a factor two.

In case we set $\psi = 0$, we find a solution of the homogeneous equation, which we denote with n_h.

$$An_h - \lambda n_h = 0, \qquad n_h \in C[\tfrac{1}{2}a,1] \, .$$

Because $\lambda \notin P\sigma(A)$, we have $n_h(\frac{1}{2}a) \neq 0$, and therefore $n_h \notin D(A)$, and the equation above is only formally right.
Let

$$\gamma := -\frac{\bar{n}(\frac{1}{2}a)}{n_h(\frac{1}{2}a)} \quad \text{and} \quad \tilde{n} := \bar{n} + \gamma n_h \quad .$$

Then $\tilde{n} \in C[\frac{1}{2}a,1]$, $\tilde{n}(\frac{1}{2}a) = 0$, $\tilde{n} \in D(A)$, and $\tilde{n}$ is a solution of the inhomogeneous eqaution $An - \lambda n = \psi$. Now we have proved that the range of $A - \lambda I$ is the whole space $L_1[\frac{1}{2}a,1]$. As a consequence of the closed - graph - theorem (See TAYLOR and LAY [11], theor. IV.5.8.) we have $\lambda \notin \sigma(A)$. □

We shall end this section by showing that all elements of $\sigma(A)$ are isolated. To do this we need a theorem, proved by S. STEINBERG [10].

THEOREM 2.3. *Let* E *be a Banach space and* $K(\lambda)$ *an analytic family of compact operators, defined on a domain* Ω. *Let* $S(\lambda) = I - K(\lambda)$. *If* $S(\lambda)$ *is invertible for some* $\lambda_0 \in \Omega$, *then* $S^{-1}(\lambda)$ *exists for all* $\lambda \in \Omega\backslash\Lambda$ *where* Λ *is a discrete subset of* Ω.

In our case, one sees immediately that T_λ is an analytic family of compact operators defined on the whole complex space $\mathbb{C}$. Furthermore, in section six, we shall prove that $S_\lambda = I - T_\lambda$, is invertible for all λ in a right-half-plane. Consequently, a combination of theorem 2.3. and theorem 2.2. yields:

THEOREM 2.4. $\sigma(A)$ *consists of isolated points which are eigenvalues.*

It will turn out that the dominant eigenvalue of A, i.e. the eigenvalue with largest real part, is algebraically simple, and that the corresponding eigenvector is positive. In terms of the integral operator T_λ, this means that we must investigate the following "positive eigenvalue problem":

$$(2.17)\qquad \begin{aligned} &T_\lambda\phi = \phi, \quad \phi \in X_0 \\ &\phi(x) \geq 0, \quad \tfrac{1}{2}a \leq x \leq 1. \end{aligned}$$

For doing this, we need some theory concerning positive operators.

SECTION THREE: POSITIVE OPERATORS

In this section we shall present some results concerning positive operators, emphasizing the existence and uniqueness of positive eigenvectors.

With X we denote an arbitrary Banach space, while X^* stands for the dual space.

Let T: X $\to$ X be bounded linear operator. With $T^*: X^* \to X^*$ we denote the adjoint operator.

DEFINITION. A subset $K \subset X$ is called a cone if

a) K is closed.

b) $\alpha\phi + \beta\psi \in K$ if $\phi,\psi \in K$ and $\alpha,\beta \geq 0$.

c) $K \cap (-K) = \{0\}$.

All basic theory concerning cones and positive operators can be found in the monograph of KRASNOSELSKII [5].

The cone K is called reproducing if $K - K = X$, where $K - K := \{\phi-\psi \mid \phi,\psi \in K\}$. We say that K is total if $\overline{K - K} = X$. K^* is by definition the subset of X^* consisting of all positive functionals on K, i.e. $F \in K^*$ if and only if $F \in X^*$ and $F(\phi) \geq 0$, for all $\phi \in K$. An element $\phi \in K$ is called non-support if $F \in K^*$, $F \neq 0$ implies that $F(\phi) > 0$. (See lemma 5.2. for an example). The subset of K consisting of non-support elements is denoted by Q_K.

The positive functional $F \in K^*$ is said to be strictly positive if $F(\phi) > 0$, for all $\phi \in K$ satisfying $\phi \neq 0$.

DEFINITION. Let $T: X \to X$ be a bounded, linear operator, then T is called positive (with respect to the cone K; also K-positive) if $T\phi \in K$ for all $\phi \in K$. Notation $T \geq 0$.

The first instigation for generalizing the Frobenius theory (of non-negative matrices) to the case of positive operators on a Banach space was given in 1948 by KREIN and RUTMAN in their famous paper [6]. That paper gives a.o. (partial) answers to two fundamental questions.

(1) Does the positive eigenvalue problem $T\phi = \lambda\phi$ have a solution $\phi \in K$, $\phi \neq 0$?
(2) If so, is this solution unique?

The theorems that we need for answering these two questions are just generalizations of their results.

DEFINITION. Let $T: X \to X$ be a positive operator with respect to the cone K and let u_0 be some fixed non-zero element of K. Then the operator T is called u_0-positive if for every non-zero $\phi \in K$ some positive numbers α, β and a positive integer n can be found such that $\alpha u_0 \leq T^n\phi \leq \beta u_0$.

THEOREM 3.1. *Let the cone* K *be reproducing and let* $T: X \to X$ *be positive and compact; suppose further that* T *is* u_0*-positive for some* $u_0 \in K$: (a) *then there exists a* $\phi_0 \in K\backslash\{0\}$ *such that* $T\phi_0 = \lambda_0\phi_0$, *where* $\lambda_0 = r(T)$ *is an algebraically simple eigenvalue.* ϕ_0 *is the only positive eigenvector of* T.

(b) *There is a strictly positive eigenfunctional* $F_0 \in K^*\backslash\{0\}$ *such that* $T^*F_0 = \lambda_0 F_0$.

PROOF. (a) See KRASNOSELSKII [5], section 2.3.
(b) In [6], KREIN and RUTMAN have proved the existence of a positive eigenfunctional $F_0 \in K^*\backslash\{0\}$, such that $T^*F_0 = \lambda_0 F_0$. We only have to prove that F_0 is strictly positive. Suppose $F(\phi) = 0$, for some $\phi \in K\backslash\{0\}$. $\alpha u_0 \leq T^n\phi \leq \beta u_0$ for some $n \in \mathbb{N}$ and $\alpha,\beta > 0$. Therefore $\alpha F_0(u_0) \leq F_0(T^n\phi) = \lambda_0^n F_0(\phi) \leq \beta F_0(u_0)$. Consequently $F_0(u_0) = 0$, which implies that $F_0(\psi) = 0$, for all $\psi \in K$. Here we have used: $\alpha' u_0 \leq T^m\psi \leq \beta' u_0$. Using the fact that K is reproducing, we find that $F_0 = 0$, which is a contradiction. □

Theorem 3.1. in this form, will appear not to be suitable for our purposes, since the requirement that the cone K has to be reproducing, happens to be too strong. Therefore we shall weaken this condition.

DEFINITION. Let the operator T be positive with respect to the cone K. We say that K is T-reproducing if for all $\phi \in X$ there exist ϕ_1, $\phi_2 \in K$ such that $T\phi = \phi_1 - \phi_2$.

THEOREM 3.2. *If in theorem* 3.1. *the condition* "K *is reproducing*" *is replaced by* "K *is* T-*reproducing*", *then the conclusions remain valid.*

PROOF. Follows immediately from the proof of theorem 3.1. (a) which can be found in [5], section 2.3.

We need another result, due to SAWASHIMA ([8]). She introduced the notion of a non-support operator which is in fact a generalization of the notion of an indecomposable, positive matrix.

DEFINITION. A bounded, positive operator $T: X \to X$ is called non-support with respect to K, if for all $\phi \in K$, $\phi \neq 0$ and $F \in K^*$, $F \neq 0$, there exists an integer p such that for all $n \geq p$ we have $F(T^n\phi) > 0$.

THEOREM 3.3. *Let the cone* K *be total and Let* T *be non-support with respect to* K; *suppose that* $\lambda_0 = r(T)$ *is a pole of the resolvent* $R(\lambda,T)$, *then*

(a) λ_0 *is an algebraically simple eigenvalue of* T.

(b) *There exists an eigenvector* $\phi_0 \in K$ *such that* $T\phi_0 = \lambda_0\phi_0$. *Furthermore* $\phi_0 \in Q_K$, *i.e.* ϕ_0 *is non-support.*

(c) *There exists a strictly positive eigenfunctional* $F_0 \in K^*$ *such that* $T^*F_0 = \lambda_0 F_0$.

(d) ϕ_0 *is the only positive eigenvector of* T.

PROOF. (a), (b) and (c) were proved by SAWASHIMA in [8]. To prove (d) we assume that there exists a $\lambda_1 \neq \lambda_0$ and $\phi \in K\backslash\{0\}$ such that $T\phi = \lambda_1\phi$. Using the non-supportness of T, we have $F_0(T^p\phi) > 0$ for some integer p. Clearly

$$0 < F_0(T^p\phi) = F_0(\lambda_1^p\phi) = \lambda_1^p F_0(\phi) = T^{*p}F_0(\phi) = \lambda_0^p F_0(\phi).$$

Hence $\lambda_0^p = \lambda_1^p$. Since $\lambda_0 \neq \lambda_1$ and both values are positive, this is a contradiction. □

SECTION FOUR: THE CASE a > 0

In section 2 we have introduced a family of compact operators T_λ, where $\lambda \in \mathbb{C}$. Here we shall make clear that for all real λ the operator T_λ is positive with respect to some suitable cone. We assume during this and the following section that λ is real unless otherwise stated.

DEFINITION. Let the cones K_0, $K_m \subseteq X_0$ be defined by

$$K_0 = \{\phi \in X_0 | \phi(x) \geq 0,\ \tfrac{1}{2}a \leq x \leq 1\},$$

$$K_m = \{\phi \in X_0 | \phi(x) \geq 0,\ \tfrac{1}{2}a \leq x \leq 1 \text{ and } \phi \text{ is non-decreasing}\}.$$

Immediately it follows that $K_m \subseteq K_0$.

THEOREM 4.1.

(a) K_0 *is reproducing.*

(b) $T_\lambda K_0 \subseteq K_m$.

(c) K_m *is* T_λ*-reproducing.*

(d) T_λ *is positive with respect to both cones* K_0 *and* K_m.

PROOF. (a), (b) and (d) are straightforward. We shall only prove (c). Suppose $\phi \in X_0$; because of (a) we have $\phi = \phi_1 - \phi_2$, where $\phi_1, \phi_2 \in K_0$. Hence $T_\lambda\phi = T_\lambda\phi_1 - T_\lambda\phi_2$. Using (b) we have $T_\lambda\phi_1, T_\lambda\phi_2 \in K_m$. □

REMARK. $T_\lambda K_0 \subset K_m$ implies among others, that, if T_λ has an eigenvector $\phi \in K_0$, then also $\phi \in K_m$.

The Riesz-representation theorem tells us what the dual cone K_0^* looks like.

THEOREM 4.2.

(a) $F \in K_0^*$ *if and only if* F *is given by* $F(\phi) = F_\mu(\phi) = \int_{[\frac{1}{2}a,1]} \phi d\mu$, $\phi \in X_0$, *for some positive Borel-measure* μ *on* $[\frac{1}{2}a,1]$.

(b) $F = F_\mu \in K_0^*$ *is not identically zero iff* μ *is not identically zero, i.e.* $\int_{(\frac{1}{2}a,1]} d\mu \neq 0$.

PROOF.

(a) See RUDIN [7], theorem 2.14.

(b) In order that F is not identically zero, it is not sufficient that $\int_{[\frac{1}{2}a,1]} d\mu \neq 0$, because $\phi(\frac{1}{2}a) = 0$, for all $\phi \in X_0$. □

As we have already mentioned, we shall make a distinction between two cases, namely $a > 0$ and $a = 0$. In the rest of this section, we shall deal with the case $a > 0$. Let $\lambda \in \mathbb{R}$ be fixed.
Let $u_0 \in K_m$ be defined by

$$(4.3) \qquad u_0(x) := \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)d\xi, \quad x \in [\tfrac{1}{2}a,1].$$

THEOREM 4.3. T_λ *is* u_0*-positive with respect to the cone* K_m.

PROOF. We shall use a result which was proved by KRASNOSELSKII ([5], theorem 2.2) which says the following: suppose that for all $\phi \in K_m$ there exist integers n and m, and positive numbers α, β such that $\alpha u_0 \leq T_\lambda^n\phi$ and $T_\lambda^m\phi \leq \beta u_0$, then T_λ is u_0-positive. Now let $\phi \in K_m\backslash\{0\}$.
We have

$$\phi(1).u_0(x) - (T_\lambda\phi)(x) = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)\{\phi(1) - \phi(2\xi)\}d\xi ,$$

which implies that $\phi(1).u_0 - T_\lambda\phi \in K_m$, because $\phi(1) - \phi(2\xi) \geq 0$, for all $\frac{1}{2}a \leq \xi \leq \frac{1}{2}$. This means that $T_\lambda\phi \leq \phi(1).u_0$, and $\phi(1) > 0$, because $\phi \not\equiv 0$. A straightforward computation shows that $T_\lambda^n\phi \in K_m$ and $(T_\lambda^n\phi)(x) > 0$, for all $2^{-n} \leq x \leq 1$. If n is such that $2^{-n} \leq \frac{1}{2}a$, then we have $T_\lambda^n\phi \in K_m$ and $(T_\lambda^n\phi)(x) > 0$, $\frac{1}{2}a \leq x \leq 1$.
Therefore

$$(T_\lambda^{n+1}\phi)(x) - (T_\lambda^n\phi)(a).u_0(x) =$$

$$= \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi).\{(T_\lambda^n\phi)(2\xi) - (T_\lambda^n\phi)(a)\}\, d\xi \in K_m,$$

because $(T_\lambda^n\phi)(2\xi) - (T_\lambda^n\phi)(a) \geq 0$, for $\frac{1}{2}a \leq \xi \leq \frac{1}{2}$. This, together with the result of KRASNOSELSKII, proves the theorem. □

Using the fact that the cone K_m is T_λ-reproducing (theorem 4.1-c) and theorem 3.2, we have the following. There exists a $\phi_\lambda \in K_m$ and a strictly positive eigenfunctional $F_\lambda \in K_m^*$ such that

$$T_\lambda\phi_\lambda = r_\lambda\phi_\lambda \quad , \tag{4.4.}$$

$$T_\lambda^* F_\lambda = r_\lambda F_\lambda \quad , \tag{4.5.}$$

where $r_\lambda = r(T_\lambda)$ is an algebraically simple eigenvalue. Furthermore ϕ_λ is the only positive eigenvector of T_λ with respect to K_m. A more extensive study of equation (4.5.) is made in section 7.

As we have seen in section two, we are only interested in positive eigenvectors of T_λ corresponding to the eigenvalue 1. Therefore we have to look for those values $\lambda \in \mathbb{R}$ satisfying $r(T_\lambda) = 1$.

<u>THEOREM 4.4</u>. $\lambda \in \mathbb{R}$ *is uniquely determined by the condition* $r(T_\lambda) = 1$.

<u>PROOF</u>. Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_2 > \lambda_1$. Let $\phi \in K_0$.

$$(T_{\lambda_1}\phi)(x) = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_{\lambda_1}(\xi)\phi(2\xi)d\xi = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k(\xi)e^{-\lambda_1 r(\xi)}\phi(2\xi)d\xi \quad ,$$

where we have used (2.12.).
Since $r(\xi) = G(2\xi) - G(\xi) \geq \delta$ for some $\delta > 0$ (Here we have explicitly used that $a > 0$),

$$(T_{\lambda_1}\phi)(x) = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} e^{(\lambda_2-\lambda_1)r(\xi)}k(\xi)e^{-\lambda_2 r(\xi)}\phi(2\xi)d\xi$$

$$\geq e^{(\lambda_2-\lambda_1)\delta} \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_{\lambda_2}(\xi)\phi(2\xi)d\xi =: (\eta+1)(T_{\lambda_2}\phi)(x) ,$$

where

$$\eta := e^{(\lambda_2-\lambda_1)\delta}-1 > 0 .$$

Let ϕ_{λ_2} be the positive eigenvector of T_{λ_2} corresponding to the eigenvalue $r_{\lambda_2} = \hat{r}(T_{\lambda_2})$. Thus $T_{\lambda_2}\phi_{\lambda_2} = r_{\lambda_2}\phi_{\lambda_2}$. Then

$$T_{\lambda_1}\phi_{\lambda_2} \geq (1+\eta)T_{\lambda_2}\phi_{\lambda_2} = (1+\eta)r_{\lambda_2}\phi_{\lambda_2} .$$

A straightforward computation shows that

$$T^n_{\lambda_1}\phi_{\lambda_2} \geq (1+\eta)^n r^n_{\lambda_2}\phi_{\lambda_2} .$$

This implies: $\| T^n_{\lambda_1} \| \geq (1+\eta)^n r^n_{\lambda_2}$. Hence

$$r_{\lambda_1} \geq (1+\eta)r_{\lambda_2} .$$

This implies that $r(T_\lambda)$ is strictly monotone decreasing in λ. Furthermore, using that $\eta = e^{(\lambda_2-\lambda_1)\delta}-1$, one sees immediately

$$\lim_{\lambda\to-\infty} r(T_\lambda) = +\infty ,$$

$$\lim_{\lambda\to+\infty} r(T_\lambda) = 0 .$$

This makes the proof complete. □

Now we have proved that there exists a unique $\lambda_0 \in \mathbb{R}$, a unique $\phi_0 \in K_m$, and a unique, strictly positive functional F_0 such that

$$T_{\lambda_0}\phi_0 = \phi_0 \;,$$

$$T^*_{\lambda_0}F_0 = F_0 \;,$$

and the eigenvalue 1 of T_λ is algebraïcally simple.

<u>REMARK</u>. There is a more elegant and transparant way to obtain the results of this section. The basic idea is to study the integral equation (2.12.) on the subinterval $[a,1]$.

$$(\tilde{T}_\lambda\tilde{\phi})(x) = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)\tilde{\phi}(2\xi)d\xi, \qquad \tilde{\phi} \in C[a,1] \quad (*)$$

The values of $T_\lambda\phi$, for $\phi \in X_0$, on the interval $[\frac{1}{2}a,a]$ are completely determined by the values of

$$\tilde{\phi} := \phi\big|_{[a,1]} \in C[a,1] \,.$$

Suppose $\tilde{\phi} \in C[a,1]$ is a solution of $\tilde{T}_\lambda\tilde{\phi} = \tilde{\phi}$, where $\tilde{T}_\lambda$ is given by (*), and let the extension ϕ of $\tilde{\phi}$ on $[\frac{1}{2}a,1]$ be defined by

$$\begin{cases} \phi(x) = \tilde{\phi}(x), & a \le x \le 1 \\ \phi(x) = \displaystyle\int_{\frac{1}{2}a}^{x} k_\lambda(\xi)\tilde{\phi}(2\xi)d\xi, & \frac{1}{2}a \le x \le a \,. \end{cases}$$

Then $\phi \in X_0$ and ϕ is a solution of the original integral equation (2.12.). The advantage of this method is, that it permits us to work in the cone $\tilde{K} = \{\tilde{\phi} \in C[a,1] \mid \tilde{\phi}(x) \ge 0\}$, which has non-empty interior $\mathring{\tilde{K}}$. The operator $\tilde{T}_\lambda$ is strongly-positive with respect to $\mathring{\tilde{K}}$, i.e. for all $\phi \in \tilde{K}$ there exists on integer $n = n(\phi)$ such that $\tilde{T}^n_\lambda\phi \in \mathring{\tilde{K}}$. Now the unicity of the positive eigenvector is given by theorem 6.3. of KREIN and RUTMAN. However this approach fails in the case that $a = 0$, and for that reason, we have chosen a different road.

SECTION FIVE: THE CASE $a = 0$

In this section we are going to deal with the case that $a = 0$. There is an important distinction between this case and the former one. If a is non-zero, then the problem can be solved in a finite number of steps; this can not be done if $a = 0$. As a consequence the methods used in section four, have to be adapted.

Let $\lambda \in R$ be fixed.

THEOREM 5.1. *The operator* T_λ *is non-support with respect to the cone* K_0.

PROOF. Let $\phi \in K_0$, $\phi \neq 0$, and $F \in K_0^*$, $F \neq 0$. Following theorem 4.2. there exists a positive Borel measume μ on $[0,1]$ such that

$$\int_{(0,1]} d\mu \neq 0, \text{ and } F(\psi) = F_\mu(\psi) = \int_{[0,1]} \psi d\mu, \text{ for all } \psi \in X_0.$$

Hence there exists an $\alpha > 0$ such that for all ε satisfying

$$0 < \varepsilon < \alpha \text{ one has: } \int_{(\alpha-\varepsilon,\alpha+\varepsilon)} d\mu > 0 .$$

Let p be an integer such that $2^{-p} < \alpha$. Then for all $n \geq p$ we have

$$(T_\lambda^n \phi)(\alpha) > 0 .$$

Hence

$$F(T_\lambda^n \phi) = F_\mu(T_\lambda^n \phi) = \int_{[0,1]} (T_\lambda^n \phi) d\mu \geq \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} (T^n \phi) d\mu > 0 \text{ if } n \geq p. \qquad \square$$

Since T_λ is compact, all eigenvalues are poles of the resolvent. Furthermore K_0 is reproducing (and hence total) as we have seen in theorem 4.1. Therefore we can apply theorem 3.3. There exist an eigenvector $\phi_\lambda \in K_0$ (and following the remark on p.24. $\phi_\lambda \in K_m$) and a positive eigenfunctional $F_\lambda \in K_0^*$ such that

$$T_\lambda \phi_\lambda = r_\lambda \phi_\lambda ,$$

$$T_\lambda^* F_\lambda = r_\lambda F_\lambda ,$$

where $r_\lambda = r(T_\lambda)$ is an algebraically simple eigenvalue, $\phi_\lambda \in Q_{K_0}$, and ϕ_λ is the only positive eigenvector belonging to T_λ, and F_λ is strictly positive.

As in section four it remains to prove that $\lambda \in \mathbb{R}$ is uniquely determined by the condition $r(T_\lambda) = 1$. Note that we cannot apply theorem 4.4., because the proof of that theorem explicitly makes use of the fact that a is non-zero. We need the following lemma.

LEMMA 5.2. *Suppose* $\phi \in K_0$. *Then* $\phi \in Q_{K_0}$ *iff* $\phi(x) > 0$ *for all* $x \in (0,1]$.

PROOF.

(i) Let $\phi \in Q_{K_0}$ and suppose $\phi(\alpha) = 0$, for some $\alpha \in (0,1]$. Let the positive non-zero Borel measure μ on $(0,1]$ be given by:

for all $V \subset [0,1]$: $\mu(V) = 0$, if $\alpha \notin V$.
$\mu(V) = 1$, if $\alpha \in V$.

Then

$$F_\mu(\phi) = \int_{[0,1]} \phi d\mu = \phi(\alpha) = 0 \text{ and } F_\mu \neq 0.$$

This is a contradiction.

(ii) Let $\phi \in K_0$ and $\phi(x) > 0$, for all $x \in (0,1]$. Suppose $F = F_\mu \in K_0^* \backslash \{0\}$; then the positive Borel measure μ is not identically zero, i.e. $\int_{(0,1]} d\mu > 0$ which means that for some $\alpha > 0$, and for $\varepsilon > 0$ sufficiently small we have $\int_{(\alpha-\varepsilon,\alpha+\varepsilon)} d\mu > 0$. Using $\phi(\alpha) > 0$ we find

$$\int_{(0,1]} \phi d\mu = F_\mu(\phi) \geq \int_{(\alpha-\varepsilon,\alpha+\varepsilon)} \phi d\mu > 0. \quad \square$$

THEOREM 5.3. *The number* $\lambda \in \mathbb{R}$ *is uniquely determined by the condition* $r(T_\lambda) = 1$.

PROOF. Let $\lambda_1 < \lambda_2$ and let ϕ_{λ_i}, F_{λ_i}, $i = 1,2$, be the positive eigenvector and eigenfunctional of T_{λ_i} and $T^*_{\lambda_i}$:

$$T_{\lambda_i}\phi_{\lambda_i} = r_{\lambda_i}\phi_{\lambda_i}, \quad i = 1,2,$$
$$T^*_{\lambda_i}F_{\lambda_i} = r_{\lambda_i}F_{\lambda_i}, \quad i = 1,2.$$

Then

$$r_{\lambda_2} = \frac{(T^*_{\lambda_2}F_{\lambda_2})(\phi_{\lambda_1})}{F_{\lambda_2}(\phi_{\lambda_1})} = \frac{F_{\lambda_2}(T_{\lambda_2}\phi_{\lambda_1})}{F_{\lambda_2}(\phi_{\lambda_1})} =$$

$$= \frac{F_{\lambda_2}(T_{\lambda_1}\phi_{\lambda_1})}{F_{\lambda_2}(\phi_{\lambda_1})} - \frac{F_{\lambda_2}((T_{\lambda_1} - T_{\lambda_2})\phi_{\lambda_1})}{F_{\lambda_2}(\phi_{\lambda_1})} =: r_{\lambda_1} - \Delta .$$

$$((T_{\lambda_1} - T_{\lambda_2})\phi_{\lambda_1})(x) = \int_0^{\min(\frac{1}{2},x)} \{k_{\lambda_1}(\xi) - k_{\lambda_2}(\xi)\}\phi_{\lambda_1}(2\xi)d\xi > 0 ,$$

for all $x > 0$, which means that $(T_{\lambda_1} - T_{\lambda_2})\phi_{\lambda_1} \in Q_{K_0}$. Here we have used lemma 5.2. This and the strict positivity of F_{λ_2} imply that $\Delta > 0$. Hence $r_{\lambda_1} > r_{\lambda_2}$ which implies that $r(T_\lambda)$ is strictly monotone decreasing in λ. Now let $\lambda \in \mathbb{R}$: there exists a $\phi_\lambda \in K_m$ such that $T_\lambda\phi_\lambda = r_\lambda\phi_\lambda$ and $\|\phi_\lambda\| = 1$. Clearly $(T_\lambda\phi_\lambda)(1) = \|T_\lambda\phi_\lambda\| = r_\lambda\phi_\lambda(1) = r_\lambda\|\phi_\lambda\| = r_\lambda = \int_0^{\frac{1}{2}} k_\lambda(\xi)\phi(2\xi)d\xi$, where we have used that for any vector $\Psi \in K_m$ we have $\|\Psi\| = \Psi(1)$. One sees immediately that $\phi_\lambda(x)$ is constant for all $x \in [\frac{1}{2},1]$. It follows that

$$\int_{\frac{1}{4}}^{\frac{1}{2}} k_\lambda(\xi)d\xi \le r_\lambda \le \int_0^{\frac{1}{2}} k_\lambda(\xi)d\xi .$$

Using (2.12.) we find

$$\lim_{\lambda\to-\infty} r(T_\lambda) = \infty ,$$

$$\lim_{\lambda\to+\infty} r(T_\lambda) = 0 .$$

This completes the proof. □

Now we have proved the existence and uniqueness of $\lambda_0 \in \mathbb{R}$, $\phi_0 \in K_m$ and a strictly positive functional F_0 such that

$$T_{\lambda_0}\phi_0 = \phi_0 \ ,$$

$$T^*_{\lambda_0}F_0 = F_0 \ ,$$

and the eigenvalue 1 of T_{λ_0} is algebraically simple.

The remaining part of this section is valid both for the cases $a > 0$ and $a = 0$.

Let n_0 be defined by

$$(5.1.) \qquad n_0(x) := \frac{E(x)}{g(x)} e^{-\lambda_0 G(x)} \phi_0(x) \ .$$

Then we have the following results:

$n_0(x) \geq 0, \quad \frac{1}{2}a \leq x \leq 1.$
n_0 is continuous.
$An_0 - \lambda_0 n_0 = 0.$
n_0 is the only positive eigenvector of A.

Furthermore we have:

<u>THEOREM 5.4.</u> *The eigenvalue* $\lambda_0 \in P\sigma(A)$ *is algebraically simple.*

<u>PROOF.</u> The geometric simplicity of the eigenvalue $\lambda_0 \in P\sigma(A)$ follows directly from the geometric simplicity of the eigenvalue $1 \in P\sigma(T_{\lambda_0})$. Now suppose that $(A-\lambda_0)^2 n = 0$ and $(A-\lambda_0)n \neq 0$ for some $n \in D(A^2)$. Defining $\bar{n} := (A-\lambda_0)n$ we have $(A-\lambda_0)\bar{n} = 0$ and $\bar{n} \neq 0$. Hence $\bar{n}(x) = \alpha n_0(x)$ for some $\alpha \in \mathbb{C}$, which we assume to be 1 (without loss of generality). $(A-\lambda_0)n = n_0$ can be reduced to $(T_{\lambda_0}-1)v = \psi$ where v is given by (2.4)
and

$$\psi(x) = \int_{\frac{1}{2}a}^{x} \frac{n_0(\xi)}{E(\xi)} e^{\lambda_0 G(\xi)} d\xi.$$

Using the Fredholm alternative, we find that this equation is solvable iff $F_0(\psi) = 0$ where F_0 is the strictly positive eigenfunctional satisfying $T^*_{\lambda_0} F_0 = F_0$. Using the fact that $\psi \in K_m$, we find a contradiction. □

In the forthcoming section we shall make clear why this eigenvalue λ_0 is so important.

SECTION SIX: ON THE POSITION AND COMPUTATION OF THE EIGENVALUES

In the former two sections we have seen that the operator A has exactly one positive eigenvector corresponding to an eigenvalue $\lambda_0 \in \mathbb{R}$. (See corollary 5.4.). Now we shall prove that λ_0 is the principal value of A, i.e. the eigenvalue with the largest real part. We need the following elementary lemma.

<u>LEMMA 6.1</u>. *Suppose* $a < b$, *and let* $f \in L_1[a,b]$ *be a complex-valued function. Then we have:* $|\int_a^b f(x)dx| = \int_a^b |f(x)| dx$ *if and only if there exists a constant* $\alpha \in \mathbb{C}$, *with* $|\alpha| = 1$, *such that* $|f(x)| = \alpha f(x)$ *a.e. on* $[a,b]$.

<u>PROOF</u>. Let $z := \int_a^b f(x)dx$ and define $\alpha \in \mathbb{C}$ such that $\alpha z = |z|$. Clearly $|\alpha| = 1$. Putting $u(x) := \mathrm{Re}\{\alpha f(x)\}$ we have $u(x) \le |\alpha f(x)| = |f(x)|$ and the inequality is strict for all $x \in V$, where the subset $V \subset [a,b]$ is defined by: $x \in V$ iff $\mathrm{Im}\{\alpha f(x)\} \neq 0$. Hence $u(x) < |\alpha f(x)| = |f(x)|$, for $x \in V$ and $\int_a^b u(x)dx < \int_a^b |f(x)| dx$ iff $\mu(V) > 0$, where $\mu(V)$ is the measure of the set V.

$$\left|\int_a^b f(x)dx\right| = |z| = \alpha z = \int_a^b \alpha f(x)dx = \mathrm{Re}\left\{\int_a^b \alpha f(x)dx\right\} =$$

$$= \int_a^b \mathrm{Re}\{\alpha f(x)\}dx = \int_a^b u(x)dx.$$

Consequently $|\int_a^b f(x)dx| < \int_a^b |f(x)| dx$ iff $\mu(V) > 0$. In other words: $|\int_a^b f(x)dx| = \int_a^b |f(x)| dx$ iff $u(x) = \alpha f(x)$ a.e., which is the same as

$|f(x)| = \alpha f(x)$ a.e. $\square$

THEOREM 6.2. *If* $\lambda \in P\sigma(A)$ *and* $\lambda \neq \lambda_0$ *then* $\mathrm{Re}\lambda < \lambda_0$.

PROOF. (i) Suppose $\mathrm{Re}\lambda > \lambda_0$ and $\lambda \in \sigma(A)$. Then $1 \in P\sigma(T_\lambda)$ which implies that $T_\lambda \phi = \phi$ for some $\phi \in X_0$.
In other words

$$\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)\phi(2\xi)d\xi = \phi(x).$$

Using (2.10) we arrive at

$$\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k(\xi)e^{-\lambda r(\xi)}\phi(2\xi)d\xi = \phi(x).$$

Taking absolute values on both sides, we find
$\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k(\xi)e^{-\mathrm{Re}\lambda . r(\xi)}|\phi(2\xi)|d\xi \geq |\phi(x)|$, which can be written as: $T_{\mathrm{Re}\lambda}\ |\phi| \geq |\phi|$ (with respect to K_0) where $|\phi| \in X_0$ is defined by $|\phi|\ (x) := |\phi(x)|$. Using theorem 6.2. of KREIN and RUTMAN (See [6]) we obtain $T_{\mathrm{Re}\lambda}\psi = \rho\psi$ for some $\psi \in K_0/\{0\}$ and $\rho \geq 1$. Consequently $r(T_{\mathrm{Re}\lambda}) \geq 1$. On the other hand, theorem 4.4. and theorem 5.3. state that $r(T_{\mathrm{Re}\lambda}) < 1$ both for the cases $a > 0$ and $a = 0$. Now we have proved that $\lambda \in \sigma(A)$ implies that $\mathrm{Re}\lambda \leq \lambda_0$.

(ii) Now suppose that $\lambda = \lambda_0 + i\eta$ and $\lambda \in \sigma(A)$. This implies that $T_\lambda\ \psi = \psi$ for some $\psi \in X_0$ and as in (a) we deduce $T_{\mathrm{Re}\lambda}\ |\psi| \geq |\psi|$, i.e. $T_{\lambda_0}|\psi| \geq |\psi|$. Suppose that $T_{\lambda_0}\ |\psi| \neq |\psi|$. This yields $T_{\lambda_0}\ |\psi| \in K_0/\{0\}$. Let F_0 be the strictly positive eigenfunctional satisfying $T^*_{\lambda_0}F_0 = F_0$. Then $0 < F_0(T_{\lambda_0}\ |\psi| - |\psi|) = (T^*_{\lambda_0}F_0)(|\psi|) - F_0(|\psi|) = 0$, which is a contradiction. Consequently $T_{\lambda_0}\ |\psi| = |\psi|$, which means, by the simplicity of the eigenvalue 1 of T_{λ_0} : $|\psi| = \gamma\phi_0$, for some constant $\gamma \in \mathbb{C}$, which we may assume to be one, without loss of generality. As a consequence $|\psi(x)| = \phi_0(x)e^{i\alpha(x)}$, where $\alpha(x) \in \mathbb{R}$, $x \in [\frac{1}{2}a,1]$. Using $|T_\lambda\psi| = |\psi| = T_{\mathrm{Re}\lambda}\ |\psi| = T_{\lambda_0}\phi_0$, we find

$$\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_{\lambda_0}(\xi)\phi_0(2\xi)d\xi = \left|\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)\psi(2\xi)d\xi\right| =$$

$$\left|\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} e^{-i\eta r(\xi)} k_{\lambda_0}(\xi)\,\phi_0\,(2\xi)e^{i\alpha(2\xi)}d\xi\right| .$$

Using lemma 6.1. we obtain $\alpha(2\xi) - \eta\, r(\xi) = C$ where C is a constant. Hence $\alpha(x) = C + \eta\; r(\frac{1}{2}x)$. Inserting this in

$$\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)\psi(2\xi)d\xi = \psi(x) =$$

$$= \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} e^{-i\eta r(\xi)} k_{\lambda_0}(\xi)\phi_0(2\xi)e^{i\alpha(2\xi)}d\xi = \phi_0(x)e^{i\alpha(x)},$$

we obtain

$$e^{iC}\int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_{\lambda_0}(\xi)\phi_0(2\xi)d\xi = \phi_0(x)e^{iC + i\eta r(\frac{1}{2}x)},$$

which implies

$$\phi_0(x) = \phi_0(x)e^{i\eta r(\frac{1}{2}x)} \quad \text{a.e.}$$

As a consequence $\eta = 0$, which implies that $\lambda = \lambda_0$. □

In section two we noticed that all elements of $\sigma(A)$ are isolated. Now we are going to show that in every vertical strip $s \le \mathrm{Re}\lambda \le t$, there are only finitely many of them.

Let the Banach space X be the space of all continuous functions on $[\frac{1}{2}a,1]$ with the supnorm. Clearly X_0 is a closed subspace of X. For every $\lambda \in \mathbb{C}$ the operator $T_\lambda\colon X_0 \to X_0$ can be extended to the larger space X. This extension is also denoted by the symbol T_λ.

(6.1) $$(T_\lambda\phi)(x) = \int_{\frac{1}{2}a}^{\min(\frac{1}{2},x)} k_\lambda(\xi)\phi(2\xi)d\xi,\ \phi \in X .$$

One sees immidiately: $T_\lambda X \subset X_0$. As a consequence $T_\lambda\phi = \phi$, $\phi \in X$, implies that $\phi \in X_0$. Using theorem 2.2, we have

(6.2.) $$\lambda \in \sigma(A) \iff 1 \in P\sigma(T_\lambda|_{X_0}) \iff 1 \in P\sigma(T_\lambda) \quad ,$$

where $T_\lambda|_{X_0}$ denotes the restriction of T_λ: $X \to X$ to the subspace X_0.

(6.3) Let $e_1 \in X$ be defined by: $e_1(x) = 1$, $\frac{1}{2}a \le x \le 1$.
T_λ: $X \to X$ can be decomposed in the following way:

Let $\phi \in X$:

(6.4) $$(T_\lambda\phi)(x) = \int_{\frac{1}{2}a}^{\frac{1}{2}} k_\lambda(\xi)\phi(2\xi)d\xi - \int_{\min(\frac{1}{2},x)}^{\frac{1}{2}} k_\lambda(\xi)\phi(2\xi)d\xi = H_\lambda(\phi)e_1 + N_\lambda\phi \quad ,$$

where H_λ is a bounded linear functional.

(6.5) $$H_\lambda(\phi) := \int_{\frac{1}{2}a}^{\frac{1}{2}} k_\lambda(\xi)\phi(2\xi)d\xi \ .$$

and N_λ is a bounded linear operator on X.

(6.6) $$(N_\lambda\phi)(x) := -\int_{\min(\frac{1}{2},x)}^{\frac{1}{2}} k_\lambda(\xi)\phi(2\xi)d\xi \ .$$

The reason that we have embedded X_0 in the larger space X might be clear now: X is invariant under N_λ, but X_0 isn't. Again we make a distinction between the cases $a > 0$ and $a = 0$.

I. $a > 0$

THEOREM 6.3. *The operator* N_λ *is compact and nilpotent, for all* $\lambda \in \mathbb{C}$, *i.e.* $N_\lambda^p = 0$ *for some* $p \in \mathbb{N}$, *where* p *does not depend on* λ.

PROOF. Compactness is trivial. Let $p \in \mathbb{N}$ be such that $2^{-p+1} \le a < 2^{-p+2}$. Then we have $N_\lambda^{p-1} \neq 0$ and $N_\lambda^p = 0$. To see this, we observe that for all $\phi \in X$

$$(N_\lambda \phi)(x) = 0, \qquad x \geq \tfrac{1}{2}$$
$$(N_\lambda^2 \phi)(x) = 0, \qquad x \geq \tfrac{1}{4}$$
$$\vdots \qquad\qquad \vdots$$
$$(N_\lambda^p)(x) = 0, \qquad x \geq \tfrac{1}{2}a. \quad \square$$

Substitution of $T_\lambda \phi$ in (6.4) gives us

(6.7) $$T_\lambda^2 \phi = H_\lambda(T_\lambda \phi)e_1 + N_\lambda(T_\lambda \phi) = H_\lambda(T_\lambda \phi)e_1 + H_\lambda(\phi)N_\lambda e_1 + N_\lambda^2 \phi .$$

DEFINITION.

(6.8) $$e_j := N_\lambda e_{j-1} \quad , \; j = 2,\ldots, p .$$

REMARK.

(6.9) $$N_\lambda e_p = N_\lambda^p e_1 = 0 .$$

LEMMA 6.4. $e_1,\ldots,e_p$ *are linearly independent in* X. *Furthermore* $R(T_\lambda^p) \subset$ *span* $\langle e_1,\ldots,e_p\rangle$, *where span* $\langle e_1,\ldots,e_p\rangle$ *is the subspace of* X *spanned by the functions* $e_1,\ldots,e_p$.

PROOF.

$$e_2(x) = (N_\lambda e_1)(x) \neq 0, \quad \text{if } x < \tfrac{1}{2}.$$

A straightforward computation shows that for all i, with $1 \leq i \leq p$, we have

$$e_i(x) \neq 0 \text{ if } x < 2^{-i+1} .$$

Now suppose that for certain $\alpha_i \in \mathbb{C}$, $i = 1,\ldots,p$,

$$\alpha_1 e_1 + \ldots\ldots + \alpha_p e_p = 0,$$

Then

$$N^{p-1}(\alpha_1 e_1 + \ldots + \alpha_p e_p) = \alpha_1 e_p = 0,$$

wich implies that $\alpha_1 = 0$. Likewise we find that $\alpha_i = 0$ for all $i = 2,\dots,p$. This proves the linear independence of $e_1,\dots,e_p$. A computation similar to (6.7) yields

$$T_\lambda^p\phi = H_\lambda(T_\lambda^{p-1}\phi)e_1 + H_\lambda(T_\lambda^{p-2}\phi)e_2 + \dots + H_\lambda(\phi)e_p \tag{6.10}$$

for all $\phi \in X$, where we have used that $N_\lambda^p = 0$. This completes the proof. □

Defining

$$f_j := H_\lambda(e_j), \quad j = 1,\dots,p, \tag{6.11}$$

we have

$$T_\lambda e_j = H_\lambda(e_j)e_1 + N_\lambda e_j = f_j e_1 + e_{j+1}, \ j = 1,\dots,p, \text{ where } e_{p+1} := 0. \tag{6.12}$$

REMARK.

One should keep in mind that e_j and f_n both depend on λ.

Now suppose that $\lambda \in \sigma(A)$. This implies that $1 \in P\sigma(T_\lambda)$. Therefore $T_\lambda\phi = \phi$ for some $\phi \in X$, $\phi \neq 0$. Consequently $T_\lambda^p\phi = \phi$. In other words $\phi \in (T_\lambda^p) \subset$ span $< e_1,\dots,e_p >$. Hence we can write $\phi = \phi_1 e_1 + \dots + \phi_p e_p$. Using (6.12) we find

$$\sum_{i=1}^p \phi_i e_i = \phi = T_\lambda\phi = \sum_{i=1}^p \phi_i T_\lambda e_i = \sum_{i=1}^p \phi_i(f_i e_1 + e_{i+1}).$$

Using the linear independence of the functions e_i we conclude

$$\phi_1 = \phi_1 f_1 + \dots + \phi_p f_p,$$

$$\phi_1 = \phi_2 = \dots = \phi_p.$$

$\phi \neq 0$ implies $\phi_1 \neq 0$ and therefore $f_1 + \dots + f_p = 1$. Furthermore $f_p = H_\lambda(e_p) = 0$. Now we have proved:

THEOREM 6.5. $\lambda \in \sigma(A)$ *if and only if* $H_\lambda(e_1 + \dots + e_{p-1}) = 1$.

THEOREM 6.6. *Suppose* $s < t$. *In the vertical strip* $s \le \mathrm{Re}\lambda \le t$, *there are only finitely many points of* $\sigma(A)$.

PROOF. Suppose $\lambda \in \sigma(A)$. Following theorem 6.5, we conclude that $H_\lambda(e_1 + \dots + e_{p-1}) = 1$.

$$H_\lambda(e_1) = \int_{\frac{1}{2}a}^{\frac{1}{2}} k_\lambda(\xi)d\xi = \int_{\frac{1}{2}a}^{\frac{1}{2}} k(\xi)e^{-\lambda r(\xi)}d\xi,$$

where we have used (2.12). Moreover

$$r(\xi) = G(2\xi) - G(\xi) = \int_{\xi}^{2\xi} \frac{d\tau}{g(\tau)} \geq \delta\xi ,$$

where

$$\delta := \inf_{\eta\in[\frac{1}{2}a,1]} \frac{1}{g(\eta)}.$$

From considerations similar to those which are used to prove the well-known RIEMANN-LEBESGUE lemma, it follows that

$$\lim_{\mathrm{Im}\lambda\to\pm\infty} H_\lambda(e_1) = 0, \text{ uniformly in } s \leq \mathrm{Re}\lambda \leq t.$$

Using the same argements for $i > 1$, we find

$$\lim_{\mathrm{Im}\lambda\to\pm\infty} H_\lambda(e_1 + \ldots + e_{p-1}) = 0, \text{ uniformly in } s \leq \mathrm{Re}\lambda \leq t.$$

This together with the fact that all elements of $\sigma(A)$ are isolated (see th. 2.4.), proves the theorem. □

II. $a = 0$

In this situation, the proof of theorem 6.6. follows the same lines, although we have to pay more attention to some details.

Let H_λ and N_λ be defined by (6.5) and (6.6) where $\frac{1}{2}a$ is replaced by 0. Again we have

(6.13) $\qquad T_\lambda\phi = H_1(\phi)e_1 + N_\lambda\phi, \ \phi \in X.$

Let e_j be defined by (6.8) for all $j \geq 1$.

THEOREM 6.7. N_λ *is compact and quasinilpotent.*

PROOF. The proof that N_λ is compact is trivial. Now suppose that $\mu \in P\sigma(N_\lambda)$; hence there exists a $\psi \in X\backslash\{0\}$ such that $N_\lambda\psi = \mu\,\psi$. Consequently $N_\lambda^k\psi = \mu^k\psi$, for all $k \geq 1$. Observing that $(N_\lambda^k\psi)(x) = 0$, for $x \geq 2^{-k}$ we conclude that $\mu = 0$. As a consequence $\sigma(N_\lambda) = \{0\}$, which proves the theorem. □

LEMMA 6.8. $\eta_\lambda := \Sigma_{k=1}^{\infty} e_k \in X$, *and* $\| \eta_\lambda \|$ *is uniformly bounded in every vertical strip* $s \leq \mathrm{Re}\lambda \leq t$.

PROOF. It suffices to prove that $\Sigma_{j=1}^{\infty} \| e_j \| < \infty$. We have $\| e_1 \| = 1$; suppose $s \leq \mathrm{Re}\lambda \leq t$.

$$|e_2(x)| \leq \int_{\min(\frac{1}{2},x)}^{\frac{1}{2}} |k_\lambda(\xi)|\,d\xi < \int_0^{\frac{1}{2}} |k_\lambda(\xi)|\,d\xi < \infty,$$

where we have used (2.7). This yields

$$e_2(x) = 0, \quad x \geq \tfrac{1}{2},$$

$$|e_2(x)| \leq M, \quad x \leq \tfrac{1}{2},$$

where

$$M := \max_{s\leq \mathrm{Re}\lambda\leq t} \left(\int_0^{\frac{1}{2}} |k_\lambda(\xi)|\,d\xi \right).$$

$$|e_3(x)| \leq \int_0^{\frac{1}{4}} |k_\lambda(\xi)|M\,d\xi \leq \tfrac{1}{4} L M,$$

where

$$(6.14) \qquad L := \max\{|k_\lambda(\xi)| \mid 0 \leq \xi \leq \tfrac{1}{4},\ s \leq \mathrm{Re}\lambda \leq t\}.$$

By induction we find that

$$\|e_k\| \leq \frac{1}{4} \cdot \frac{1}{8} \cdots \frac{1}{2^{k-1}} L^{k-2}.M,$$

wich completes the proof.

THEOREM 6.9. $T_\lambda\phi = \phi$ *is solvable if and only if* $H_\lambda(\eta_\lambda) = 1$. *In that case*

$\phi = H_\lambda(\phi)\eta_\lambda$.

PROOF.

(i) Suppose $T_\lambda\phi = \phi$. Inserting (6.13) we obtain $N_\lambda\phi = \phi - H_\lambda(\phi)e_1$. If we put $\hat{\phi} := H_\lambda(\phi)\eta_\lambda$ then $N_\lambda(\phi - \hat{\phi}) = \phi - H_\lambda(\phi)e_1 - H_\lambda(\phi)N_\lambda\eta_\lambda =$ $= \phi - H_\lambda(\phi)e_1 - H_\lambda(\phi)(e_2 + e_3 + \ldots.) = \phi - \hat{\phi}$. Now the quasinilpotence of N_λ implies that $\phi - \hat{\phi} = 0$ and therefore $\phi = H_\lambda(\phi)\eta_\lambda$. Consequently $H_\lambda(\phi) = H_\lambda(\phi)H_\lambda(\eta_\lambda)$. Moreover $H_\lambda(\phi) \neq 0$ because $\phi \neq 0$ and thus $H_\lambda(\eta_\lambda) = 1$.

(ii) Suppose $H_\lambda(\eta_\lambda) = 1$. Putting $\phi := \alpha\eta_\lambda$ (where α is to be determined), we obtain $T_\lambda\phi = \alpha T_\lambda\eta_\lambda = \alpha H_\lambda(\eta_\lambda)e_1 + \alpha N_\lambda\eta_\lambda = \alpha\eta_\lambda = \phi$. As a consequence $H_\lambda(\phi) = \alpha H_\lambda(\eta_\lambda) = \alpha$. From this we conclude that $\phi = H_\lambda(\phi)\eta_\lambda$. □

Now suppose that $s, t \in \mathbb{R}$ and $s \le t$. According to lemma 6.8. there exists a constant $M_1 > 0$ such that $\|\eta_\lambda\| \le M_1$ for all λ in the vertical strip $s \le \mathrm{Re}\lambda \le t$. We have

$$H_\lambda(\eta_\lambda) = \int_0^1 k_\lambda(\xi)\eta_\lambda(2\xi)d\xi =$$

$$= \int_0^\varepsilon k_\lambda(\xi)\eta_\lambda(2\xi)d\xi + \int_\varepsilon^1 k_\lambda(\xi)\eta_\lambda(2\xi)d\xi$$

$$\left| \int_0^\varepsilon k_\lambda(\xi)\eta_\lambda(2\xi)d\xi \right| \le M_1 \int_0^\varepsilon | k_\lambda(\xi) | d\xi \le LM_1\varepsilon ,$$

where L is defined by (6.14). We choose $\varepsilon < \frac{1}{4}$ such that $\varepsilon LM_1 \le \frac{1}{2}$.Hence

$$| H_\lambda(\eta_\lambda) | \le \frac{1}{2} + \left| \int_\varepsilon^{\frac{1}{2}} k_\lambda(\xi)\eta_\lambda(2\xi)d\xi \right|$$

for all λ satisfying $s \le \mathrm{Re}\lambda \le t$. There exists a $j_0 \in \mathbb{N}$ such that $j > j_0$ implies $e_j(x) = 0$ if $x \ge \varepsilon$.

This yields

$$| H_\lambda(\eta_\lambda) | \le \frac{1}{2} + \sum_{j=1}^{j_0} \left| \int_\varepsilon^{\frac{1}{2}} k_\lambda(\xi)e_j(2\xi)d\xi \right| .$$

In the proof of theorem 6.6. we have seen that $\lim_{\mathrm{Im}\lambda\to\pm\infty} H_\lambda(e_1 + \ldots + e_p) = 0$, uniformly in the vertical strip $s \le \mathrm{Re}_\lambda \le t$.Similarly we have

$$\lim_{\mathrm{Im}\lambda\to\pm\infty}\left(\sum_{j=1}^{j_0}\left|\int_{\varepsilon}^{\frac{1}{2}} k_\lambda(\xi)e_j(2\xi)d\xi\right|\right) = 0$$

uniformly is the vertical strip $s \le \mathrm{Re}\lambda \le t$. As a consequence, there exists a $\Lambda > 0$ such that for all λ satisfying $s \le \mathrm{Re}\lambda \le t$ and $|\mathrm{Im}\,\lambda| \ge \Lambda$ we have

$$\sum_{j=1}^{j_0}\left|\int_{\varepsilon}^{\frac{1}{2}} k_\lambda(\xi)e_j(2\xi)d\xi\right| \le \frac{1}{4}.$$

For these values of λ we obtain $|H_\lambda(\eta_\lambda)| \le \frac{3}{4}$ and by theorem 6.9. this implies $\lambda \notin \sigma(A)$. Now we have proved:

THEOREM 6.10. *Suppose* $a = 0$. *In every vertical strip* $s \le \mathrm{Re}\lambda \le t$, *there are only finitely many points of* $\sigma(A)$.

EXAMPLE. Suppose $a \ge \frac{1}{2}$. Then the characteristic equation looks as follows:

$$\int_{\frac{1}{2}a}^{\frac{1}{2}} k_\lambda(\xi)d\xi = 1.$$

The value of the parameter λ_0, especially the sign of λ_0 appears to be very important. In fact the asymptotic behavior for $t \to \infty$ of the solution of the time-dependent equation (0.2) is completely determined by the value of λ_0, as will be proved in [2]. Therefore we shall deduce two equations from which, in some practical cases the sign of λ_0 can be computed.

We have:

$$\frac{d}{dx}(g(x)n_0(x)) = -\lambda_0 n_0(x) - \mu(x)n_0(x) - b(x)n_0(x) + 4b(2x)n_0(2x), \quad \tfrac{1}{2}a \le x \le 1.$$

Integration along the interval $[\frac{1}{2}a,1]$ gives us

$$\text{(6.15)} \qquad \lambda_0 = \frac{\int_{\frac{1}{2}a}^{1}(b(x) - \mu(x))n_0(x)dx}{\int_{\frac{1}{2}a}^{1} n_0(x)dx}.$$

Another similar equation can be derived, if we first multiply with x, and then integrate along the interval $[\frac{1}{2}a,1]$. In that case we find

$$(6.16)\qquad \lambda_0 = \frac{\int_{\frac{1}{2}a}^{1} (g(x) - x\mu(x))n_0(x)dx}{\int_{\frac{1}{2}a}^{1} xn_0(x)dx}.$$

SECTION SEVEN: THE ADJOINT OPERATOR

In section four and five we have seen that there exists a positive functional F_0, satisfying

$$(7.1)\qquad T^*_{\lambda_0}F_0 = F_0$$

where $F_0 \in K^*_m$ if $a > 0$ and $F_0 \in K^*_0$ if $a = 0$. Here $T^*_{\lambda_0} : M \to M$ where M denotes the space of all Borel-measures on $[\frac{1}{2}a,1]$.

<u>REMARK.</u> The fact that $\phi(\frac{1}{2}a) = 0$ for $\phi \in X_0$ implies that we can restrict ourselves to the space of Borelmeasures on $(\frac{1}{2}a,1]$.

Consequently F_0 is represented by some measure $\mu_0 \in M$.

$$F_0(\phi) = F_{\mu_0}(\phi) = \int_{[\frac{1}{2}a,1]} \phi d\mu_0, \; \phi \in X_0.$$

With M_0^+ we denote the subset of M consisting of all positive Borelmeasures; i.e. $\mu \in M_0^+$ implies

$$\mu((x,y]) \geq 0 \text{ for } \tfrac{1}{2}a \leq x \leq y \leq 1.$$

Let $\qquad M_m^+ := \{\mu \in M \mid \mu((x,1]) \geq 0, \quad \text{if } \frac{1}{2}a \leq x \leq 1\}.$

Then $\quad M_0^+ \subseteq M_m^+.$

Furthermore

$$K_0^* = M_0^+ \quad \text{and } K_m^* = M_m^+.$$

Let NBV $[\frac{1}{2}a,1]$ be all bounded -variation- functions f which are normalized by the condition: $f(1) = 0$.

DEFINITION For $\mu \in M$, the function $\overline{\mu} \in$ NBV $[\frac{1}{2}a,1]$ is defined by $\overline{\mu}(x) := \mu((x,1])$.

If $\mu \in M_m^+$, then $\overline{\mu}(x) \geq 0$, for all $x \in [\frac{1}{2}a,1]$. For the adjoint operator $T_\lambda^* : M \to M$ one can deduce the following explicit expression.

Let $\mu \in M$ and $T_\lambda^* \mu = \nu$, then we have

(7.2) $$\overline{\nu}(x) = \frac{1}{2} \int_{\max(x,a)}^{1} k_{\overline{\lambda}}(\tfrac{1}{2}\xi)\, \overline{\mu}(\tfrac{1}{2}\xi)d\xi.$$

Now (7.1) can be rewritten as $T_{\lambda_0}^* \mu_0 = \mu_0$, or equivalently

(7.3) $$\overline{\mu}_0(x) = \frac{1}{2} \int_{\max(x,a)}^{1} k_{\lambda_0}(\tfrac{1}{2}\xi)\, \overline{\mu}_0(\tfrac{1}{2}\xi)d\xi$$

where $\mu_0 \in M_m^+$, if $a > 0$ and $\mu_0 \in M_0^+$ if $a = 0$. From (7.3) one sees that $\mu_0 \in M_m^+$ implies that $\mu_0 \in M_0^+$. (This can also be proved without using this explicit expression). Furthermore $\overline{\mu}_0 \in C[\frac{1}{2}a,1]$.Let

(7.4) $$n_0^*(x) := \frac{e^{\lambda_0 G(x)}}{E(x)}\, \overline{\mu}_0(x).$$

Then $n_0^* \in C[\frac{1}{2}a,1]$. (A straightforward computation shows that n_0^* has a removable singularity for $x = 1$) Furthermore $n_0^*(x) > 0$ if $x < 1$. The adjoint of A, $A^* : D(A^*) \to L_\infty[\frac{1}{2}a,1]$ is given by

$$(A^*m)(x) = g(x)\frac{dm}{dx} - \mu(x)m(x) - b(x)m(x) + 2b(x)m(\tfrac{1}{2}x)$$
$$(\text{where } m(\tfrac{1}{2}x) = 0,\ x < a)$$

for $m \in D(A^*) = \{m \in L_\infty[\frac{1}{2}a,1] \mid \frac{dm}{dx}$ is defined a.e.,

$$m(1) = 0, \text{ and } \psi_{x\in[\frac{1}{2}a,1]}\ g(x)\frac{dm}{dx} - \mu(x)m(x) - b(x)m(x) +$$
$$+ 2b(x)m(\tfrac{1}{2}x) \in L_\infty[\tfrac{1}{2}a,1]\}.$$

THEOREM 7.1. $n_0^* \in D(A^*)$ *and* n_0^* *is the unique solution of the equation* $A^*m = \lambda_0 m$.

The proof of this theorem is straightforward.

LEMMA 7.2. $\lambda I - A$ *is a Fredholm-operator with index* 0 *for all* $\lambda \in \mathbb{C}$.

PROOF. Suppose $\lambda \in \mathbb{C}\backslash\sigma(A)$ and let $R_\lambda := (A-\lambda I)^{-1}$. From the construction of R_λ in the proof of theorem 2.2 it is clear that R_λ as an operator from $L_1[\frac{1}{2}a,1]$ to $C[\frac{1}{2}a,1]$ is bounded, which implies that R_λ as an operator from $L_1[\frac{1}{2}a,1]$ to $L_p[\frac{1}{2}a,1]$ is bounded, for all p. Let p be fixed, $p \neq 1$, $p \neq \infty$. Let $\{\phi_n\}_{n=1}^\infty$ be a bounded sequence in $L_1[\frac{1}{2}a,1]$, then we have that $\{R_\lambda\phi_n\}_{n=1}^\infty$ is a bounded sequence in $L_p[\frac{1}{2}a,1]$, and as a consequence of the theorem of Alaoglu it has a weakly convergent subsequence which we denote with $\{R_\lambda\phi_{n_k}\}_{k=1}^\infty$.

$$R_\lambda\phi_{n_k} \to \psi_{lim}, \quad k \to \infty, \text{ weakly in } L_p.$$

Let q be given by $\frac{1}{p} + \frac{1}{q} = 1$, then we have $L_\infty[\frac{1}{2}a,1] \subset L_q[\frac{1}{2}a,1]$, because we are working on a finite interval. Consequently $R_\lambda\phi_{n_k} \to \psi_{lim}$, $k \to \infty$, weakly in L_1. Now we have proved that $R_\lambda: L_1[\frac{1}{2}a,1] \to L_1[\frac{1}{2}a,1]$ is weakly compact. Using corollary V.2.4 of GOLDBERG [3], we find the result. □

Now using this lemma, the algebraic simplicity of the eigenvalue λ_0, and a result of KAASHOEK ([4], theorem 4.3) we may give the following decomposition of the space $L_1[\frac{1}{2}a,1]$:

$$(7.5) \qquad L_1[\tfrac{1}{2}a,1] = \mathrm{Ker}(A-\lambda_0 I)\oplus\mathrm{Ran}(A-\lambda_0 I),$$

where $\mathrm{Ker}(A-\lambda_0 I)$ is the nullspace of $A - \lambda_0 I$ and $\mathrm{Ran}(A-\lambda_0 I)$ is the range of $A - \lambda_0 I$.

Let P be the orthogonal projection on $\mathrm{Ker}(A-\lambda_0 I)$, then

$$(7.6) \qquad Pn = \frac{\langle n_0^*, n\rangle}{\langle n_0^*, n\rangle}\, n_0 ,$$

where

$$\langle n_0^*, n\rangle = \int_{\frac{1}{2}a}^{1} n_0^*(x)n(x)dx, \qquad n \in L_1[\tfrac{1}{2}a,1].$$

REMARK. $\langle n_0^*, n\rangle > 0$ if $n(x) \geq 0$, a.e. and $n \neq 0$.

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