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FACTORING MULTIVARIATE INTEGRAL POLYNOMIALS, II

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Factoring multivariate integral polynomials, II *)

by

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ABSTRACT

We show that the problem of factoring multivariate integral polynomials can be reduced in polynomial-time to the univariate case. Our reduction makes use of lattice techniques as introduced in [3].

KEY WORDS & PHRASES: *polynomial algorithm, polynomial factorization*

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*) This report will be submitted for publication elsewhere.

1. Introduction.

In [5] we presented a polynomial-time algorithm to factor polynomials in $\mathbb{Z}[X, Y]$, and we pointed out how to generalize the algorithm to $\mathbb{Z}[X_1, X_2, \ldots, X_t]$ for $t \geq 3$. A nice feature of this algorithm is that it doesn't depend on the polynomial-time algorithm to factor in $\mathbb{Z}[X]$ (cf. [3]). Instead of working out the details of this direct approach for $t \geq 3$ (this will be done for $\mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$ in a forthcoming paper [6]), we here simplify the method from [5] somewhat, which results in a polynomial-time reduction from factoring in $\mathbb{Z}[X_1, X_2, \ldots, X_t]$ to factoring in $\mathbb{Z}[X]$. This reduction is similar to the reduction from $\mathbb{F}_q[X_1, X_2, \ldots, X_t]$ to $\mathbb{F}_q[X, Y]$ that was given in [4].

An outline of our reduction is as follows. First we evaluate the polynomial $f \in \mathbb{Z}[X_1, X_2, \ldots, X_t]$ in a suitably chosen integer point $(X_2 = s_2, X_3 = s_3, \ldots, X_t = s_t)$, to obtain a polynomial $\tilde{f} \in \mathbb{Z}[X_1]$. Using the algorithm from [3] we then compute an irreducible factor $\tilde{h} \in \mathbb{Z}[X_1]$ of $\tilde{f}$. Next we construct an integral lattice containing the factor h_0 of f that corresponds to $\tilde{h}$, and we prove that h_0 is the shortest vector in this lattice. As usual, this enables us to compute h_0 by means of the so-called *basis reduction algorithm* (cf. [3: Section 1]; in the sequel we will assume the reader to be familiar with this basis reduction algorithm and its properties).

2. Factoring multivariate integral polynomials.

Let $f \in \mathbb{Z}[X_1, X_2, \ldots, X_t]$ be the polynomial to be factored, with the number of variables $t \geq 2$. By $\delta_i f = n_i$ we denote the degree of f in X_i. We often use n instead of n_1. We put $N_i = \prod_{k=i}^{t}(n_i+1)$, and $N = N_1$. The *content* $\mathrm{cont}(f) \in \mathbb{Z}[X_2, X_3, \ldots, X_t]$ of f is defined as the greatest common divisor of the coefficients of f with respect to X_1; we say that f is *primitive* if $\mathrm{cont}(f) = 1$.

Without loss of generality we may assume that $2 \leq n_i \leq n_{i+1}$ for $1 \leq i < t$, and that the gcd of the integer coefficients of f equals one.

We present an algorithm to factor f into its irreducible factors in $\mathbb{Z}[X_1, X_2, \ldots, X_t]$ that is polynomial-time in N and the size of the integer coefficients of f.

Let $s_2, s_3, \ldots, s_t \in \mathbb{Z}_{>0}$ be a $(t-1)$-tuple of integers. For $g \in \mathbb{Z}[X_1, X_2, \ldots, X_t]$ we denote by $\tilde{g}_j$ the polynomial g modulo$((X_2-s_2), (X_3-s_3), \ldots, (X_j-s_j)) \in \mathbb{Z}[X_1, X_{j+1}, X_{j+2}, \ldots, X_t]$; i.e. $\tilde{g}_j$ is g with s_i substituted for X_i for $i = 2, 3, \ldots, j$. Notice that $\tilde{g}_1 = g$, and that $\tilde{g}_j = \tilde{g}_{j-1}$modulo (X_j-s_j). We put $\tilde{g} = \tilde{g}_t$.

Suppose that an irreducible, primitive factor $\tilde{h} \in \mathbb{Z}[X_1]$ of $\tilde{f}$ is given such that

(2.1) $\quad \tilde{h}^2$ doesn't divide $\tilde{f}$ in $\mathbb{Z}[X_1]$, and $\delta_1 \tilde{h} > 0$.

This condition implies that there exists an irreducible factor $h_0 \in \mathbb{Z}[X_1, X_2, \ldots, X_t]$ of f such that $\tilde{h}$ divides $\tilde{h}_0$ in $\mathbb{Z}[X_1]$, and that this polynomial h_0 is unique up to sign.

(2.2) Let m be an integer with $\delta_1 \tilde{h} \leq m < n$. We define L as the collection of polynomials g in $\mathbb{Z}[X_1, X_2, \ldots, X_t]$ such that

(i) $\quad \delta_1 g \leq m$, and $\delta_i g \leq n_i$ for $2 \leq i \leq t$,

(ii) $\quad \tilde{h}$ divides $\tilde{g}$ in $\mathbb{Z}[X_1]$.

This is a subset of the $(m+1)N_2$-dimensional real vector space $\mathbb{R} + \mathbb{R}X_t + \ldots +$

$\mathbb{R}X_1^m X_2^{n_2} \ldots X_t^{n_t}$. We put $M = (m+1)N_2$. This vector space can be identified with $\mathbb{R}^M$ by identifying the polynomial $\Sigma_{i=0}^{m} \Sigma_{j=0}^{n_2} \ldots \Sigma_{k=0}^{n_t} a_{ij\ldots k} X_1^i X_2^j \ldots X_t^k \in \mathbb{R}[X_1, X_2, \ldots, X_t]$ with the M-dimensional vector $(a_{00\ldots0}, a_{00\ldots1}, \ldots, a_{mn_2\ldots n_t})$. The collection L is a lattice in $\mathbb{Z}^M$ of rank $M - \delta_1\tilde{h}$, and a basis for L is given by

$$\{X_1^i \Pi_{j=2}^t (X_j - s_j)^{i_j} :\quad 0 \le i \le m,\ 0 \le i_j \le n_j \quad \text{for} \quad 2 \le j \le t, \quad \text{and}$$
$$(i_2, i_3, \ldots, i_t) \ne (0, 0, \ldots, 0)\}$$
$$\cup \{\tilde{h} X_1^{i-\delta_1\tilde{h}} :\quad \delta_1\tilde{h} \le i \le m\}$$

(cf. [4: (3.2)]).

We define the *length* $|g|$ of the vector associated with the polynomial g as the ordinary Euclidean length of this vector. The *height* g_{max} is defined as the largest absolute value of any of the integer coefficients of g.

(2.3) Proposition. *Suppose that* b *is a non-zero element of* L *such that*

$$(2.4) \qquad s_j \ge f_{max}^m b_{max}^n (n+m)!\, (N_2 \Pi_{i=2}^{j-1} s_i^{n_i})^{n+m}$$

for $2 \le j \le t$. *Then* $\gcd(f,b) \ne 1$ *in* $\mathbb{Z}[X_1, X_2, \ldots, X_t]$.*)

Proof. Suppose on the contrary that $\gcd(f,b) = 1$. This implies that the resultant $R = R(f,b) \in \mathbb{Z}[X_2, X_3, \ldots, X_t]$ of f and b (with respect to the variable X_1) is unequal to zero.

We derive an upper bound for $(\tilde{R}_j)_{max}$. Because $\tilde{R}_j$ is the resultant of $\tilde{f}_j$ and $\tilde{b}_j$ we have

$$(2.5) \qquad (\tilde{R}_j)_{max} \le (\tilde{f}_j)_{max}^m (\tilde{b}_j)_{max}^n (n+m)!\, N_{j+1}^{n+m-2}$$

*) Here, and in the sequel, f_{max}^m denotes $(f_{max})^m$.

as is easily verified. Because $\tilde{b}_j = \tilde{b}_{j-1}$ modulo $(X_j - s_j)$, we have

$$(\tilde{b}_j)_{max} \le (\tilde{b}_{j-1})_{max} (n_j+1)\, s_j^{n_j},$$

so that

$$(2.6) \qquad (\tilde{b}_j)_{max} \le b_{max} \prod_{i=2}^{j} (n_i+1) s_i^{n_i},$$

and similarly

$$(2.7) \qquad (\tilde{f}_j)_{max} \le f_{max} \prod_{i=2}^{j} (n_i+1) s_i^{n_i}.$$

Combining (2.5), (2.6), and (2.7), we obtain

$$(2.8) \qquad (\tilde{R}_j)_{max} < f_{max}^m\, b_{max}^n\, (n+m)!\, \left(N_2 \prod_{i=2}^{j} s_i^{n_i}\right)^{n+m},$$

for $1 \le j < t$.

Because $\tilde{h}$ divides both $\tilde{f}$ and $\tilde{b}$ ((2.2)(ii)), we have that $\tilde{R} = 0$. But also $R \ne 0$, so there must be an index j with $2 \le j \le t$ such that s_j is a zero of $\tilde{R}_{j-1}$. This implies that

$$|s_j| \le (\tilde{R}_{j-1})_{max}$$

for some j with $2 \le j \le t$, which yields, combined with (2.4) and (2.8), a contradiction. We conclude that $\gcd(f,b) \ne 1$. □

<u>(2.9) Proposition</u>. *Let* $b_1, b_2, \ldots, b_M$ *be a reduced basis for* L *(cf.* [3: Section 1]*), where* L *and* M *are defined as in* (2.2)*. Suppose that*

$$(2.10) \qquad s_j \ge f_{max}^m \left((M\, 2^{M-1})^{1/2} f_{max}\right)^n (n+m)!\, \left(e^{\sum_{i=1}^{t} n_i} N_2 \prod_{i=2}^{j-1} s_i^{n_i}\right)^{n+m}$$

for $2 \le j \le t$, *and that* f *doesn't contain multiple factors. Then*

(2.11) $$(b_1)_{max} \le (M\,2^{M-1})^{\frac{1}{2}} e^{\Sigma_{i=1}^{t} n_i} f_{max}$$

and h_0 *divides* b_1, *if and only if* $\delta_1 h_0 \le m$.

<u>Proof</u>. If h_0 divides b_1, then $\delta_1 h_0 \le \delta_1 b_1 \le m$; this proves the "only if"-part.

We prove the "if"-part. Suppose that $\delta_1 h_0 \le m$. The polynomial h_0 is a divisor of f, so that

$$(h_0)_{max} \le e^{\Sigma_{i=1}^{t} n_i} f_{max}$$

according to [2]. With $\delta_1 h_0 \le m$ and $\delta_i h_i \le n_i$ for $2 \le i \le t$ we get

$$|h_0| \le M^{\frac{1}{2}} e^{\Sigma_{i=1}^{t} n_i} f_{max},$$

so that [3: (1.11)] combined with $h_0 \in L$ (this follows from $\delta_1 h_0 \le m$) yields

$$|b_1| \le (M\,2^{M-1})^{\frac{1}{2}} e^{\Sigma_{i=1}^{t} n_i} f_{max}.$$

This proves (2.11) because $(b_1)_{max} \le |b_1|$. With (2.10) and (2.3) we now have that $\gcd(f,b_1) \ne 1$. Suppose that h_0 doesn't divide $r = \gcd(f,b_1)$. Then $\tilde{h}$ divides $\tilde{f}/\tilde{r}$, so that, with

$$(f/r)_{max} \le e^{\Sigma_{i=1}^{t} n_i} f_{max},$$

and (2.10), (2.11), and (2.3), we get that $\gcd(f/r,b_1) \ne 1$. This is a contradiction with $r = \gcd(f,b_1)$, because f doesn't contain multiple factors. □

(2.12) Suppose that f doesn't contain multiple factors and that f is primitive. Let $s_2, s_3, \ldots, s_t$ and $\tilde{h}$ be chosen such that (2.10) with m replaced by $n-1$ and (2.1) are satisfied. The divisor h_0 of f can be determined in the following way.

For the values $m = \delta_1\bar{h}, \delta_1\bar{h}+1, \ldots, n-1$ in succession we apply the basis reduction algorithm (cf. [3: Section 1]) to the lattice L as defined in (2.2). We stop as soon as a vector b_1 is found satisfying (2.11). It is not difficult to see that the first vector b_1 satisfying (2.11) that we encounter, also satisfies $b_1 = \pm h_0$ (here we apply [3: (1.37)] and (2.9)). If no vector satisfying (2.11) is found, then $\delta_1 h_0 > n-1$, so that $h_0 = f$; this follows from (2.9).

(2.13) Proposition. *Assume that the conditions in* (2.12) *are satisfied. The polynomial* h_0 *can be computed in* $O((\delta_1 h_0 N_2)^4 \log B)$ *arithmetic operations on integers having binary length* $O(N \log B)$, *where*

$$\log B = O(\log f_{max} + n + \log N_2 + \Sigma_{i=2}^{t} n_i \log s_i).$$

Proof. Combining

$$|\bar{h}| \le \binom{2n}{n}^{\frac{1}{2}} |\bar{f}|$$

(cf. [7]) and (2.7), we find that

$$|\bar{h}| \le f_{max}\left((n+1)\binom{2n}{n}\right)^{\frac{1}{2}} \Pi_{i=2}^{t} (n_i+1) s_i^{n_i}.$$

The proof follows now immediately from (2.2), [3: (1.26)] and [3: (1.37)]. □

(2.14) We describe an algorithm to compute the irreducible factors of f in $\mathbb{Z}[X_1, X_2, \ldots, X_t]$. Assume that f is primitive.

First we compute the resultant $R = R(f,f') \in \mathbb{Z}[X_2, X_3, \ldots, X_t]$ of f and its derivative f' with respect to X_1, using the subresultant algorithm from [1]. We may assume that $R \ne 0$, i.e. f doesn't contain multiple factors. (In the case that $R=0$, the greatest common divisor g of f and f' is also computed by the subresultant algorithm, and the factoring algorithm can be applied to f/g.)

Next we determine $s_2, s_3, \ldots, s_t \in \mathbb{Z}$ such that $\tilde{R} \neq 0$ and such that (2.10) is satisfied with m replaced by $n-1$:

$$(2.15) \qquad s_j \geq (n N_2 2^{nN_2-1})^{n/2} (2n-1)! \, (e^{\Sigma_{i=1}^{t} n_i} f_{max} N_2 \Pi_{i=2}^{j-1} s_i^{n_i})^{2n-1}$$

for $2 \leq j \leq t$. It follows from the reasoning in the proof of (2.3) that if we take $s_j \in \mathbb{Z}_{>0}$ minimal such that (2.15) is satisfied, then $\tilde{R} \neq 0$.

By means of the algorithm from [3] we compute the irreducible and primitive factors of f of degree >0 in X_1. The condition $\tilde{R} \neq 0$ implies that (2.1) holds for every irreducible factor $\tilde{h}$ of $\tilde{f}$ thus found.

Finally, the factorization of f is determined by repeated application of the algorithm described in (2.12).

<u>(2.16) Theorem</u>. *Let* f *be a polynomial in* $\mathbb{Z}[X_1, X_2, \ldots, X_t]$ *with* $t \geq 2$, $\delta_i f = n_i$, *and* $2 \leq n = n_1 \leq n_2 \leq \ldots \leq n_t$. *The irreducible factorization of* f *can be found in* $O(n^{t-2}(N^6 + N^5 \log f_{max}))$ *arithmetic operations on integers having binary length* $O(n^{t-2}(N^3 + N^2 \log f_{max}))$, *where* $N = \Pi_{i=1}^{t}(n_i+1)$.

<u>Remark</u>. Because $n^t = O(N)$, Theorem (2.16) implies that f can be factored in time polynomial in N and $\log f_{max}$.

<u>Proof of (2.16)</u>. First assume that f is primitive. The resultant R can be computed in $O(n^{3t-1} N_2^2)$ arithmetic operations on integers having binary length $O(n^2 \log(f_{max} N_2))$ (cf. [1]).

From the choice of s_j (cf. (2.15)) we derive

$$\log s_j = O(n^2 N_2 + n \log f_{max} + \Sigma_{i=2}^{j-1} n\, n_i \log s_i)$$

for $2 \le j \le t$, so that

$$\log s_j = O((n^2 N_2 + n \log f_{max}) \Pi_{i=2}^{j-1} (1+n\, n_i)).$$

This yields

$$(2.17) \qquad \Sigma_{i=2}^{t} n_i \log s_i = O(n^{t-2}(N^2 + N \log f_{max})),$$

which gives, combined with (2.7),

$$(2.18) \qquad \log \tilde{f}_{max} = O(n^{t-2}(N^2 + N \log f_{max})).$$

The polynomial $\tilde{f}$ can be factored in $O(n^6 + n^5 \log \tilde{f}_{max})$ arithmetic operations on integers having binary length $O(n^3 + n^2 \log \tilde{f}_{max})$, according to [3: (3.6)]. With (2.18) this becomes

$$O(n^{t+3}(N^2 + N \log f_{max}))$$

arithmetic operations on integers having binary length

$$O(n^t (N^2 + N \log f_{max})).$$

According to (2.13) and (2.17), repeated application of the algorithm described in (2.12) takes

$$O(n^{t-2}(N^6 + N^5 \log f_{max}))$$

arithmetic operations on integers having binary length

$$O(n^{t-2}(N^3 + N^2 \log f_{max})).$$

The cost of applying (2.12) therefore dominates the costs of the computation of R and the factorization of $\bar{f}$.

The same estimates are valid in the case that $R = 0$. In this case we have that

$$(f/g)_{max} \le e^{\sum_{i=1}^{t} n_i} f_{max}$$

(cf. [2]), so that the same estimates as above are valid for the computation of the factorization of f/g.

Finally, we consider the case that the content of f is unequal to one. The computation of $\mathrm{cont}(f)$ can be done in $O(n\, n_2^{3t-4} N_3^2)$ arithmetic operations on integers having binary length $O(n_2^2 \log(f_{max} N_3))$ (cf. [1]). Because $\delta_i f = \delta_i \mathrm{cont}(f) + \delta_i (f/\mathrm{cont}(f))$ for $2 \le i \le t$, the proof follows by repeated application of the above reasoning. □

(2.19) Remark. As mentioned in the introduction, a somewhat more complicated but similar approach leads to an algorithm that doesn't depend on the polynomial-time algorithm for factoring in $\mathbb{Z}[X]$. Instead, it can be seen as a direct generalization of the $\mathbb{Z}[X]$-algorithm. We won't give a detailed description of this alternative method here, we only indicate the main differences.

The divisor $\bar{h} \in \mathbb{Z}[X_1]$ of $\bar{f}$ is replaced by a divisor $(\bar{h} \bmod p^k) \in (\mathbb{Z}/p^k\mathbb{Z})[X_1]$ of $(\bar{f} \bmod p^k)$, for some suitably chosen prime power p^k. Condition (2.2)(ii) is therefore replaced by the condition that $(\bar{h} \bmod p^k)$ divides $(\bar{g} \bmod p^k)$ in $(\mathbb{Z}/p^k\mathbb{Z})[X_1]$. The lattice $L \subset \mathbb{Z}^M$ now has rank M, and a basis for L is given by

$$\{p^k X_1^i:\ 0 \le i < \delta_1 \bar{h}\}$$

$$\cup \{(\bar{h} \bmod p^k) X_1^{i-\delta_1\bar{h}}:\ \delta_1\bar{h} \le i \le m\}$$

$$\cup \{X_1^i \prod_{j=2}^t (X_j - s_j)^{i_j}:\ 0 \le i \le m,\ \ 0 \le i_j \le n_j \ \text{ for } \ 2 \le j \le t, \ \text{ and }$$

$$(i_2, i_3, \ldots, i_t) \ne (0, 0, \ldots, 0)\}.$$

Again, it can be proven that, if $s_2, s_3, \ldots, s_t$ and p^k are sufficiently large, then the irreducible factor of f that corresponds to $(\bar{h} \bmod p^k)$ is the shortest vector in L. This factor can therefore be found by means of the basis reduction algorithm, and the resulting algorithm appears to be polynomial-time. For $f \in \mathbb{Z}[X,Y]$ the details are given in [5], and for $f \in \mathbb{Q}(\alpha)[X_1, X_2, \ldots, X_t]$ in [6].

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