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Cauchy Problems with State–Dependent Time Evolution

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We consider a class of quasilinear Cauchy problems which frequently arise in the context of structured population dynamics and which have in common that they can be reduced to a semilinear problem by a time–scale argument. We prove existence and uniqueness of solutions, study positivity and regularity properties, and prove the principle of linearized stability. These abstract results are applied to a model describing the dynamics of a size-structured cell population whose individuals are subject to a nonlinear growth law.

Keywords & Phrases : quasilinear Cauchy problem, state–dependent time evolution, strongly continuous semigroup, semilinear Cauchy problem, variation–of–constants formula, global solution, nonlinear semigroup, positivity, linearized stability, structured population dynamics, size-dependent cell growth and division

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1. Statement of the problem and motivation

Many models of structured population dynamics have in common that individuals are assumed to interact through their environment: see METZ, DIEKMANN [1986]. One may think, for example of a situation where individuals (of one or more species) all consume from a common resource pool, and where the per capita growth–, reproduction–, and death–rate depend on this consumption. The number of individuals affects the food availability which again affects the population dynamical processes on the individual level.

The case where individuals produce some chemical substance which diminishes or even completely inhibits their reproduction may serve as another example.

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Often, such models can be properly described by an abstract Cauchy problem of the form

$$\frac{d}{dt}u(t) = A(E(t))u(t), \tag{1.1a}$$

$$\frac{d}{dt}E(t) = f(u(t), E(t)). \tag{1.1b}$$

Here $E(t)$ describes the environment (a scalar– or vector–valued function) and $u(t)$ describes the population distribution at time t. For every fixed t, $u(t)$ can be considered as an element of some Banach space X. For every possible environment E, the (differential) operator $A(E)$ is the infinitesimal generator of a strongly continuous semigroup of linear operators on X. Actually, the operator $A(E)$ and its domain are obtained from a careful bookkeeping of all processes on the individual level. Finally, $f : X \times \mathbb{R}^n \to \mathbb{R}^n$ is some nonlinear function describing the interaction of the individuals with the environment. Usually, the dependence of f on u is through some linear functional. From the interpretation it follows that E is positive and that u takes values in a positive cone X_+.

As a particular example of (1.1) we mention the following model. Suppose we have a size–structured population whose individuals reproduce by division into two equal halves. Suppose moreover that the cells produce some enzyme which has a restraining effect on the growth rate of individual cells and in such a manner controls the size of the whole population. This situation can be described by the following system of equations:

$$\frac{\partial}{\partial t}u(t,s) + \frac{\partial}{\partial s}(\hat{\gamma}(E(t))g(s)u(t,s)) = -\mu(s)u(t,s) - b(s)u(t,s) + 4b(2s)u(t,2s), \tag{1.2a}$$

$$\frac{d}{dt}E(t) = -\sigma E(t) + \int h(s)u(t,s)ds. \tag{1.2b}$$

Here s is size, $u(t,\cdot)$ denotes the population density, $E(t)$ is the enzyme concentration at time t, $\hat{\gamma}(E)g(s)$ is the individual growth rate of a cell with size s if the enzyme concentration is E, $\mu(s)$ is the death rate, and $b(s)$ is the division rate. Furthermore $h(s)$ denotes the production rate of the enzyme by a cell with size s, and σ its desintegration rate. At this place we do not want to go into the specific details of the model but rather postpone this until Section 6 where the model is discussed in more detail. Instead, we formulate a rather general class of abstract Cauchy problems in which (1.2) (but not (1.1)) is contained, and which is the subject of this paper.

Let X be an arbitrary Banach space, A_0 the infinitesimal generator of a C_0–semigroup of linear operators $\{T_0(t), t \geq 0\}$ on X, and $F : X \to X$ a linear or nonlinear continuous operator. Finally, let $\gamma : X \to \mathbb{R}_+$ be a continuous function. We consider the abstract Cauchy problem

$$\begin{aligned} &\frac{d}{dt}u(t) = \gamma(u(t))A_0u(t) + F(u(t)) \qquad \text{for } t \geq 0, \\ &u(0) = x, \end{aligned} \tag{P$_t$}$$

where $x \in X$. In Section 6 we will give explicit expressions of γ, A_0, and F for the model (1.2).

This type of Cauchy problems is typical for many structured population models. Indeed very often interaction takes place through the environment, and in such cases γ depends on u in a very special manner, namely

$$\gamma(u) = \hat{\gamma}(L_1(u), L_2(u), \ldots, L_m(u)),$$

where $L_1, L_2, \ldots, L_m$ are linear functionals on X and $\hat{\gamma}$ is a real function of m variables. This also holds for the model given by (1.2) as we will see in Section 6.

Nonlinear growth mechanisms have been considered before by DIEKMANN, LAUWERIER, ALDENBERG and METZ [1983], HEIJMANS [1984], KOOIJMAN and METZ [1984], MURPHY [1983] and TUCKER and ZIMMERMANN [1988]. The linear version of the model discussed here has been treated using semigroup methods by DIEKMANN, HEIJMANS and THIEME [1986], DIEKMANN and METZ [1986, Chapter II], and GREINER and NAGEL [1988].

Our interest in (P_t), or more precisely, the generalized version of (P_t) involving the duality framework of CLÉMENT et al. [1987], originates from the study of a structured model of the blood cell production system. In the future we intend to consider the aforementioned generalized version of (P_t) and apply it to our model of the blood cell production system.

In this paper we will thoroughly investigate the abstract Cauchy problem (P_t). The key idea is to reduce this quasilinear problem to a semilinear problem by a time–scale argument. A general approach towards quasilinear equations is given by KATO [1975] (see also PAZY [1983]). For our situation the direct approach presented here is more efficient and leads to a better understanding of the underlying dynamics.

Let u be a C^1–function which takes values in $D(A_0)$, the domain of A_0, and satisfies (P_t). We put

$$\tau_u(t) := \int_0^t \gamma(u(s))\,ds$$

and define

$$v(\tau) := u(t_u(\tau)) \qquad \text{for all } \tau \geq 0,$$

where $t_u(\cdot)$ is the inverse function of $\tau_u(\cdot)$. One can easily verify that the thus defined function v satisfies the semilinear equation

$$\begin{aligned} &\frac{d}{d\tau} v(\tau) = A_0 v(\tau) + B(v(\tau)) \qquad \text{for } \tau \geq o, \\ &v(0) = x, \end{aligned} \tag{P_τ}$$

where the (nonlinear) operator B given by

$$B(v) := F(v)/\gamma(v)$$

maps X into itself. One should note that the definition of v is of course implicit, since it depends on the solution itself. Nevertheless one can show that the Cauchy problems (P_t) and (P_τ) are essentially equivalent and thus one can deduce properties of solutions of (P_t) from those of (P_τ) and vice versa.

In Section 2 we will summarize a number of (more–or–less) standard results on the semilinear problem (P_τ). The relation between (P_t) to (P_τ) will be discussed in Section 3. As indicated above, one is merely interested in positivity–preserving solutions of the problem. Therefore, in Section 4, we will state some equivalent conditions on the perturbation F which guarantee that solutions of (P_t) are positivity–preserving if $\{T_0(t), t \geq 0\}$ is a positive semigroup. We shall deal with the *principle of linearized (in)stability* in Section 5. Finally, in Section 6, we shall work out example (1.2) in some detail.

2. THE SEMILINEAR EQUATION

As indicated in the introduction we want to study the nonlinear equation (P_t) by relating it to the semilinear equation (P_τ). Hence a good knowledge about this type of equations is essential. We recall the corresponding results from PAZY [1983] and BALL [1977]. As already mentioned it is possible to extend the results of this and the next sections to semilinear and quasilinear equations in the wider (more general) setting of CLÉMENT et al. [1987]. We therefore include some proofs which can easily be extended to the more general setting. Consider

$$\begin{aligned} &\frac{d}{dt}u(t) = A_0u(t) + B(u(t)), \\ &u(0) = x, \end{aligned} \tag{2.1}$$

where A_0 is the generator of a linear strongly continuous semigroup $\{T_0(t),\, t \geq 0\}$ on a Banach space X, and $B : X \to X$ is a locally Lipschitz continuous operator, that is, for $r \geq 0$ there exists a constant $C(r) \geq 0$ such that

$$\|B(x) - B(y)\| \leq C(r)\, \|x - y\|$$

for all $x, y \in X$ with $\|x\| \leq r$, $\|y\| \leq r$.

A (classical) C^1–solution of (2.1) satisfies a related integral equation, which is very often referred to as a variation–of–constants formula

$$u(t) = T_0(t)x + \int_0^t T_0(t-s)B(u(s))\, ds. \tag{2.2}$$

First we will study the existence of continuous solutions of (2.2). These are of course candidates for solutions of (2.1) as well. Indeed some results stating regularity properties of solutions of (2.2) are given in Theorem 2.2 and 2.3. For related results see also CLÉMENT et al. [1987]. It is well known that local Lipschitz continuity implies the existence of solutions of (2.2) on a maximal interval.

THEOREM 2.1. *For every $x \in X$ there exists a maximal $t_{\max}(x) > 0$ such that (2.2) has a unique continuous solution $u(\cdot;x)$ on $[0, t_{\max}(x))$. If $t_{\max}(x) < \infty$, then $\lim_{t \uparrow t_{\max}} \|u(t;x)\| = \infty$. This solution satisfies the semigroup property, that is, $u(t; u(s;x)) = u(t+s;x)$ for $t, s \geq 0$ with $t + s < t_{\max}(x)$.*

In order to find solutions of (2.1) in a "strong" sense, one has to study the regularity properties of solutions of (2.2). A first result by BALL [1977] shows that continuous solutions of (2.2) are indeed "weak solutions" of (2.1) in the following sense:

THEOREM 2.2. *Let $u(\cdot) := u(\cdot;x)$ be a local (continuous) solution of (2.2) on $[0, t_0)$ with initial value $x \in X$. Then $\langle u(\cdot), x^* \rangle$ is continuously differentiable for every $x^* \in D(A_0^*)$ and moreover*

$$\frac{d}{dt}\langle u(t), x^* \rangle = \langle u(t), A_0^* x^* \rangle + \langle B(u(t)), x^* \rangle, \qquad \text{for all } t \in [0, t_0). \tag{2.3}$$

Under appropriate conditions on B and the initial condition x a continuous solution of (2.2) is automatically a "strong" C^1–solutions.

Theorem 2.3. *Let B be continuously Fréchet differentiable. If $x \in D(A_0)$ and $u := u(\cdot;x)$ is the continuous solution of (2.2) on $[0,t_0)$ then $t \mapsto u(t)$ is continuously differentiable and*

$$\begin{aligned} &\frac{d}{dt}u(t) = A_0 u(t) + B(u(t)), \qquad \text{for all } t \in [0,t_0) \\ &u(0) = x \end{aligned} \tag{2.4}$$

holds.

For a proof see Pazy [1983, Chapter 6, Theorem 1.5].

The main assertions so far can be summarized as follows. Theorem 2.1 assures the existence of local solutions of (2.2). If the solution $u(\cdot;x)$ of (2.2) stays bounded on every open interval $[0,t_0)$ then, again by Theorem 2.1, equation (2.2) possesses a global solution, or in other words $t_{\max}(x) = \infty$. If this holds for every $x \in X$, then we can define a family of operators $\{T_B(t)\,,\, t \geq 0\}$ by

$$T_B(t)x := u(t;x) \qquad \text{for } t \in [0,\infty) \text{ and } x \in X. \tag{2.5}$$

The operators $T_B(t)$ form a strongly continuous (nonlinear) semigroup on X.

Very often one is interested in the long time behaviour of the solutions of (2.1), resp. (2.2). Thus we want to conclude this section by collecting and extending some of the results about "linearized stability" of semilinear equations.

We restrict our attention to the case where 0 is an equilibrium solution of (2.1), thus $B(0) = 0$. (In case we are interested in a nontrivial equilibrium $\overline{u}$ we consider the corresponding Cauchy problem for the function $w := u - \overline{u}$ and $\tilde{B}(w) := B(w + \bar{u}) - B(\overline{u})$ which has 0 as an equilibrium solution.) Furthermore we assume without loss of generality that $\|T_0(t)\| \leq 1$ for all $t \geq 0$.

For $x \in X$ let $u(t) := u(t;x)$ be the local solution of (2.2) defined for $t < t_{\max}(x)$. First we show, that for any given time t_0 the solution $u(t;x)$ exists up to time t_0 if x is close enough to the equilibrium 0. More precisely we have:

Proposition 2.4. *Let $x \in X$ and $u(t) := u(t;x)$ the continuous solution of (2.2).*
(i) *If $r \in \mathbb{R}_+$ and $\|x\| \leq r$, then $\|u(t)\| \leq e^{tC(r)}\|x\|$, whenever $t \leq (C(r))^{-1}\log(r \cdot \|x\|^{-1})$.*
(ii) *For all $t_0 \geq 0$ there exists $r > 0$ such that $t_{\max}(x) \geq t_0$ for all x with $\|x\| \leq r$.*

Proof. (i): Let $\|x\| \leq r$ and let $t \geq 0$ be such that $\|u(s)\| \leq r$ for all $0 \leq s \leq t$. Then

$$\|u(t)\| \leq \|x\| + \int_0^t \|B(u(s))\|\, ds \leq \|x\| + \int_0^t C(r)\,\|u(s)\|\, ds.$$

Using Gronwall's lemma we obtain $\|u(t)\| \leq e^{tC(r)}\|x\|$. Let $t = \inf\{s > 0 \mid \|u(s)\| = r\}$. Note that t may equal ∞. Then, if $t < \infty$, $r \leq e^{tC(r)}\|x\|$, hence $t \geq (C(r))^{-1}\log(r\|x\|^{-1})$. From this the result follows immediately.

(ii): Let $t_0 \geq 0$ and define r by $r := e^{-t_0 C(1)}$. For $\|x\| \leq r \leq 1$ we obtain by part (i) that $\|u(s)\| \leq e^{sC(1)}\|x\|$ whenever $s \leq (C(1))^{-1}\log(\|x\|^{-1})$. Hence $\|u(s)\| \leq e^{sC(1)}\|x\|$ for $s \leq t_0 = (C(1))^{-1}\log(r^{-1})$. This implies $t_{\max}(x) > t_0$. □

Now assume that B is Fréchet differentiable and denote its Fréchet derivative at 0 by $L := (DB)(0)$. Thus L is a bounded linear operator mapping X into X. We define

$$H(u) := B(u) - Lu. \tag{2.6}$$

Let $\{T_L(t), t \geq 0\}$ be the perturbed linear C_0–semigroup generated by $A_0 + L$. Since $\{T_0(t), t \geq 0\}$ is a contraction semigroup there exists $\omega > 0$, such that $\|T_L(t)\| \leq e^{\omega t}$ for all $t \geq 0$. We shall prove that for every $t \geq 0$, $x \mapsto u(t;x)$ is Fréchet differentiable at $x = 0$ and that

$$(D_x u(t;x))(0) = T_L(t). \tag{2.7}$$

We use the following identity which is a consequence of CLÉMENT et al. [1987, Part III, Proposition 2.5]:

$$u(t;x) - T_L(t)x = \int_0^t T_L(t-s)H(u(s;x))\,ds \qquad \text{for } t < t_{\max}(x). \tag{2.8}$$

THEOREM 2.5. *For every $t \geq 0$ the mapping $x \mapsto u(t;x)$ is Fréchet differentiable at 0 and its derivative equals $T_L(t)$.*

PROOF. Fix $t \geq 0$. For all $\varepsilon > 0$ we have to find $\delta > 0$ such that $\|u(t;x) - T_L(t)x\| \leq \varepsilon\,\|x\|$ whenever $\|x\| \leq \delta$. Choose $0 < \delta < 1$ such that

$$\delta \leq e^{-C(1)t}, \quad \text{and} \tag{$*$}$$

$$\|H(u)\| \leq \varepsilon C(1) e^{-(\omega + C(1))t}\,\|u\| \quad \text{whenever } \|u\| \leq e^{tC(1)}\delta. \tag{$**$}$$

Now fix $x \in X$ with $\|x\| < \delta$. For short we write $u(t) := u(t;x)$. By Proposition 2.4(i) we have

$$\|u(s)\| \leq e^{sC(1)}\,\|x\| \text{ if } s \leq C(1)^{-1}\log(\|x\|^{-1}),$$

hence by $(*)$

$$\|u(s)\| \leq e^{sC(1)}\,\|x\| \qquad \text{if } s \leq t.$$

Using $(**)$ we obtain

$$\|H(u(s))\| \leq \varepsilon C(1) e^{-(\omega + C(1))t} e^{sC(1)}\,\|x\| \qquad \text{if } s \leq t,$$

and thus by (2.8)

$$\begin{aligned}\|u(t) - T_L(t)x\| &\leq \int_0^t e^{\omega(t-s)} \varepsilon C(1) e^{-(\omega + C(1))t} e^{sC(1)}\,\|x\|\,ds \\ &\leq \varepsilon C(1) e^{-C(1)t}\,\|x\| \int_0^t e^{sC(1)}\,ds \leq \varepsilon\,\|x\|. \ \square\end{aligned}$$

As usual we denote by $\omega(A_0 + L)$ the growth rate (type) of the semigroup $\{T_L(t), t \geq 0\}$.

THEOREM 2.6. *Let $\omega(A_0 + L) < 0$ and $0 \leq \gamma < -\omega(A_0 + L)$. Then there exists $\delta > 0$ such that for $\|x\| \leq \delta$ we have $t_{\max}(x) = \infty$ and $\lim_{t \to \infty} e^{\gamma t}\,\|u(t;x)\| = 0$.*

PROOF. We take $\gamma = 0$. (If $\gamma \neq 0$ consider $e^{\gamma t}T_L(t)$ instead of $T_L(t)$.) Let $t_0 > 0$ such that $\|T_L(t_0)\| \leq \frac{1}{4}$. Furthermore choose $\delta_1 \in (0,1)$ such that for $\|x\| \leq \delta_1$ we have

$$t_{\max}(x) \geq t_0 \tag{$*$}$$

$$\|u(t_0;x) - T_L(t_0)x\| \leq \frac{1}{4}\|x\|. \tag{$**$}$$

Define $\delta := e^{-t_0 C(1)}\delta_1$. If $\|x\| \leq \delta_1$, then $t_{\max}(x) \geq t_0$ and

$$\|u(t_0;x)\| \leq \|u(t_0;x) - T_L(t_0)x\| + \|T_L(t_0)x\| \leq \frac{1}{2}\|x\| \leq \frac{1}{2}\delta_1.$$

Hence $t_{\max}(u(t_0;x)) \geq t_0$ and thus $t_{\max}(x) \geq 2t_0$ and

$$\|u(2t_0;x)\| \leq \left(\frac{1}{2}\right)^2 \|x\|.$$

By iteration we find that for every $n \geq 0$ we have

$$t_{\max}(x) \geq nt_0 \text{ and } \|u(nt_0;x)\| \leq 2^{-n}\|x\|.$$

In particular this implies that $t_{\max}(x) = \infty$.
Now let $\|x\| \leq \delta \leq \delta_1$. Then, as before, $t_{\max}(x) = \infty$. For every $t \geq 0$ we have $t = nt_0 + t_1$, where $t_1 \in [0, t_0)$. Thus

$$\|u(t;x)\| = \|u(nt_0; u(t_1;x))\| \leq 2^{-n}\|u(t_1;x)\|,$$

since $\|u(t_1;x)\| \leq e^{t_1 C(1)}\|x\| < e^{t_0 C(1)}\delta = \delta_1$ by Proposition 2.4(i), the choice of t_1 and the definition of δ. Consequently

$$\|u(t;x)\| \leq 2^{-n}e^{t_1 C(1)}\|x\| \quad \text{for all } t \geq 0 \text{ and all } n \in \mathbb{N},$$

hence $\lim_{t\to\infty}\|u(t;x)\| = 0$. □

The corresponding instability result is the following.

THEOREM 2.7. *Let $X = X_1 \oplus X_2$ where X_1 and X_2 are invariant under $T_L(t)$ and $\dim X_1 < \infty$. Let $T_i(t)$ denote the restriction of $T_L(t)$ to X_i $(i = 1,2)$ and A_i the corresponding generator. Finally assume that*

$$\omega(A_2) < \min\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A_1)\}$$

and that

$$0 < s(A_1) := \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A_1)\}.$$

Then there exists an an $\varepsilon > 0$, a sequence t_n in $\mathbb{R}_+$, $t_n \to \infty$, and a sequence x_n in X, $x_n \to 0$, such that $t_{\max}(x_n) > t_n$ and

$$\|u(t_n;x_n)\| \geq \varepsilon$$

for n large enough.

Proof. Let P denote the projection on X_1 and let $Q = \mathrm{I} - \mathrm{P}$ be the projection on X_2. Without loss of generality we may assume that

$$\|x\| = \|Px\| + \|Qx\|, \text{ for } x \in X.$$

(Otherwise one can consider the equivalent norm $\|x\|_e := \|Px\| + \|Qx\|$.)

Furthermore we suppose that

$$\min\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A_1)\} > 0.$$

This can always be achieved by taking X_1 small enough. Hence there exists $\mu > 0$ such that

$$\|T_L(t)x_1\| \geq e^{\mu t}\|x_1\| \quad \text{for } t \geq 0 \text{ and } x_1 \in X_1.$$

Let $0 < \eta < \mu$ be such that

$$\|T_L(t)x_2\| \leq e^{\eta t}\|x_2\| \quad \text{for } t \geq 0 \text{ and } x_2 \in X_2.$$

Let $t_0 > 0$ be fixed and define

$$\sigma := -\frac{1}{t_0}\log\left(\frac{e^{\mu t_0} - e^{\eta t_0}}{2}\right). \tag{$*$}$$

By Theorem 2.5 there exists an $\varepsilon > 0$ such that

$$\|T_L(t_0)x - u(t_0; x)\| \leq \frac{1}{2}e^{-\sigma t_0}\|x\| \quad \text{if } \|x\| \leq \varepsilon.$$

Let x be such that $\|x\| \leq \varepsilon$ and $\|Qx\| \leq \|Px\|$. Assume that

$$\|u(mt_0; x)\| < \varepsilon \text{ for all } m \in \mathbb{N}. \tag{$**$}$$

(In particular $t_{\max}(x) = \infty$.)
Then we obtain

$$\begin{aligned}
\|Pu(t_0;x)\| &\geq \|PT_L(t_0)x\| - \|P(u(t_0;x) - T_L(t_0)x)\| \\
&\geq e^{\mu t_0}\|Px\| - \|u(t_0;x) - T_L(t_0)x\| \\
&\geq e^{\mu t_0}\|Px\| - \frac{1}{2}e^{-\sigma t_0}\|x\| \\
&\geq e^{\mu t_0}\|Px\| - \frac{1}{2}e^{-\sigma t_0}(\|Px\| + \|Qx\|) \\
&\geq e^{\mu t_0}\|Px\| - e^{-\sigma t_0}\|Px\|
\end{aligned}$$

and also

$$\begin{aligned}
\|Qu(t_0;x)\| &\leq \|QT_L(t_0)x\| + \|Q(u(t_0;x) - T_L(t_0)x)\| \\
&\leq \|QT_L(t_0)x\| + \|u(t_0;x) - T_L(t_0)x\| \\
&\leq e^{\eta t_0}\|Qx\| + \frac{1}{2}e^{-\sigma t_0}\|x\| \\
&\leq e^{\eta t_0}\|Qx\| + e^{-\sigma t_0}\|Px\| \\
&\leq (e^{\eta t_0} + e^{-\sigma t_0})\|Px\| \\
&= (e^{\mu t_0} - e^{-\sigma t_0})\|Px\| \quad \text{by } (*).
\end{aligned}$$

Consequently $\|Pu(t_0;x)\| \geq \|Qu(t_0;x)\|$ and by iteration we now find that

$$\|Pu(nt_0;x)\| \geq (e^{\eta t_0} + e^{-\sigma t_0})^n\|Px\|,$$

hence $\|Pu(nt_0;x)\| \to \infty$ since $\eta > 0$. Thus

$$\|u(nt_0;x)\| = \|Pu(nt_0;x)\| + \|Qu(nt_0;x)\| \to \infty \quad \text{for } n \to \infty.$$

This yields a contradiction to $(**)$ and the result follows. □

3. The quasilinear equation

In this paper we are primarily interested in the quasilinear Cauchy problem

$$\begin{aligned}&\frac{d}{dt}u(t) = \gamma(u(t))A_0u(t) + F(u(t))\\&u(0) = x,\end{aligned} \tag{P_t}$$

where A_0 is the infinitesimal generator of a C_0-semigroup $\{T_0(t), t \geq 0\}$ on a Banach space X (see Section 2), γ is a function from X to $\mathbb{R}_+$, and F is a nonlinear mapping from X into X. Throughout the remainder of this paper we make the following assumptions on F and γ.

Assumption 3.1.

(i) γ is a continuous, strictly positive, and locally bounded (i.e., γ is bounded on bounded subsets of X) function.

(ii) The operator B defined by

$$B(u) := F(u)/\gamma(u)$$

is locally Lipschitz continuous.

In particular, these assumptions imply that F is a continuous operator. We also consider the semilinear problem

$$\begin{aligned}&\frac{d}{d\tau}v(\tau) = A_0v(\tau) + B(v(\tau))\\&v(0) = x.\end{aligned} \tag{P_τ}$$

In Section 1 we already briefly indicated the relation between (P_t) and (P_τ). Below we shall give a very precise description of this relation. We need the following notation. For every continuous function $u : [0, t_0] \mapsto X$ we define

$$\tau_u(t) := \int_0^t \gamma(u(s))ds, \qquad t \in [0, t_0]. \tag{3.1}$$

We denote by $t_u(\cdot)$ the inverse of τ_u. Thus t_u is defined on $[0, \tau_0]$, with $\tau_0 = \tau_u(t_0)$. Analogously, if $v : [0, \tau_0] \mapsto X$ is a continuous function, we define

$$t_v(\tau) := \int_0^\tau \left[\gamma(v(\sigma))\right]^{-1} d\sigma, \qquad \tau \in [0, \tau_0], \tag{3.2}$$

and denote by $\tau_v(\cdot)$ the inverse of t_v on $[0, t_0]$ (with $t_0 = t_v(\tau_0)$).

Throughout the remainder of this paper we use the following notational convention. If we write u we always mean a function of t, whereas v will be used to denote a function of τ. Furthermore, t_u, t_v, τ_u, and τ_v are defined as above.

LEMMA 3.2. *To every $t_0 \geq 0$ and $u \in C([0,t_0];X)$ there corresponds a unique $\tau_0 \geq 0$ and a unique $v \in C([0,\tau_0];X)$ such that the following relations hold:*

$$\tau_0 = \tau_u(t_0) \tag{3.3}$$

$$t_0 = t_v(\tau_0) \tag{3.4}$$

$$t_u(\tau) = t_v(\tau), \qquad 0 \leq \tau \leq \tau_0 \tag{3.5}$$

$$\tau_u(t) = \tau_v(t), \qquad 0 \leq t \leq t_0 \tag{3.6}$$

$$v(\tau) = u(t_u(\tau)), \qquad 0 \leq \tau \leq \tau_0 \tag{3.7}$$

$$u(t) = v(\tau_v(t)), \qquad 0 \leq t \leq t_0. \tag{3.8}$$

Conversely, to every $\tau_0 \geq 0$ and $v \in C([0,\tau_0];X)$ there corresponds a unique $t_0 \geq 0$ and a unique $u \in C([0,t_0];X)$ such that (3.3)–(3.8) *hold.*

PROOF. Let $t_0 \geq 0$ and $u \in C([0,t_0];X)$ be given. Let τ_u be defined by (3.1) and let t_u be its inverse. Define τ_0, v, and t_v by (3.3), (3.7), and (3.2) respectively. By construction, both t_u and t_v are continuously differentiable, and we have

$$\frac{dt_v}{d\tau}(\tau) = \frac{1}{\gamma(v(\tau))} = \frac{1}{\gamma(u(t_u(\tau)))} = \frac{dt_u}{d\tau}(\tau).$$

Thus $t_u = t_v$ and hence also $\tau_u = \tau_v$. This proves (3.5) and (3.6). In particular,

$$t_0 = t_v(\tau_v(t_0)) = t_v(\tau_u(t_0)) = t_v(\tau_0),$$

which establishes (3.4). Finally, substituting $\tau = \tau_v(t)$ in (3.7), we find

$$v(\tau_v(t)) = u(t_u(\tau_v(t))) = u(t_u(\tau_u(t))) = u(t),$$

which proves (3.8). The converse result is proved in an analogous way. □

If $u \in C([0,t_0];X)$ then we denote by $v = v[u]$ the element of $C([0,\tau_0];X)$ defined by (3.7), where τ_0 is given by (3.3). Similarly, $u = u[v]$ is defined. It follows immediately that

$$u[v[u]] = u \text{ and } v[u[v]] = v.$$

DEFINITION 3.3. We call $u : [0,t_0] \mapsto X$ a *(local) classical solution* of (P_t) if u is continuously differentiable, $u(t) \in D(A_0)$ for all $t \in [0,t_0]$, and u satisfies equation (P_t). Analogously, (local) classical solutions of (P_τ) are defined.

THEOREM 3.4. *The function $u \in C([0,t_0];X)$ is a (local) classical solution of* (P_t) *if and only if $v = v[u]$ is a (local) classical solution of* (P_τ).

Proof. Let u be a classical solution of (P_t). Thus $u \in C^1([0,t_0];X)$, $u(t) \in D(A_0)$ for $0 \le t \le t_0$, and u satisfies (P_t). Let v be given by (3.7), then $v \in C^1([0,\tau_0];X)$ and $v(\tau) \in D(A_0)$ for $0 \le \tau \le \tau_0$. Furthermore

$$\begin{aligned}\frac{d}{d\tau}v(\tau) &= \frac{d}{d\tau}t_u(\tau)\cdot u'(t_u(\tau))\\ &= [\gamma(v(\tau))]^{-1}\Big[\gamma(u(t_u(\tau)))\cdot A_0u(t_u(\tau)) + F(u(t_u(\tau)))\Big]\\ &= A_0v(\tau) + B(v(\tau)).\end{aligned}$$

The other direction is proved analogously. □

We can write down the variation–of–constants formulas corresponding to (P_t) and (P_τ) respectively:

$$u(t) = T_0(\tau_u(t))x + \int_0^t T_0(\tau_u(t)-\tau_u(s))F(u(s))ds, \tag{VOC$_t$}$$

$$v(\tau) = T_0(\tau)x + \int_0^\tau T_0(\tau-\sigma)B(v(\sigma))d\sigma. \tag{VOC$_\tau$}$$

Definition 3.5. A continuous function $u : [0,t_0] \mapsto X$ is called a *(local) mild solution* of (P_t) if u satisfies (VOC_t). Similarly, (local) mild solutions of (P_τ) are defined.

In the following section we shall justify this definition. There we will study regularity properties of solutions of $(\mathrm{VOC}_t)$and specify the sense in which a solution of (VOC_t) is a solution of (P_t). Moreover we shall give a condition under which a mild solution is automatically a classical solution.

Analogously to the case of classical solutions the following correspondence between mild solutions of (P_t) and (P_τ) holds.

Theorem 3.6. *A function u is a (local) mild solution to (P_t) if and only if $v = v[u]$ is a local mild solution of (P_τ).*

Proof. This result follows immediately by substituting $t_u(\tau)$ for t in (VOC_t). □

Theorem 3.4 and Theorem 3.6 provide a rigorous justification of the intuitive idea that solving the quasilinear equation (P_t) amounts to solving the semilinear equation (P_τ), a problem which, as we showed in the previous section, is well understood. By Assumption 3.1, the operator B is locally Lipschitz continuous, and hence, by Theorem 2.1, there exists a local solution $v(\tau;x)$ of (VOC_τ) on $[0,\tau_{\max}(x))$, for every $x \in X$. As we have seen, $\tau_{\max}$ depends only on the norm $\|x\|$. Furthermore, this solution v has the semigroup property, that is,

$$v(\tau;v(\sigma;x)) = v(\tau+\sigma;x), \qquad \text{for } \tau,\ \sigma > 0 \text{ with } \tau+\sigma < \tau_{\max}(x). \tag{3.9}$$

Thus using the relation between solutions of (P_t) and (P_τ) given in Theorem 3.4 and 3.6 we can conclude that there exists a local mild solution of (P_t), for every $x \in X$. More precisely, for every $x \in X$ there exists a local solution $u(t;x)$ of (VOC_t) on some interval $[0,t_{\max}(x))$, where

$$t_{\max}(x) := \lim_{\tau\uparrow\tau_{\max}(x)} t_v(\tau) = \int_0^{\tau_{\max}(x)} \big[\gamma(v(\sigma;x))\big]^{-1}d\sigma. \tag{3.10}$$

We shall prove that this solution $u(\cdot;x)$ inherits the semigroup property from $v(\cdot;x)$. But first we introduce some further notation. For $x \in X$ we define

$$t(\tau;x) := t_{v(\cdot;x)}(\tau) = \int_0^\tau \big[\gamma(v(\sigma;x))\big]^{-1}d\sigma, \tag{3.11}$$

and let $\tau(\cdot;x)$ be its inverse. Thus the solution $u(\cdot;x)$ of (VOC_t) is given by

$$u(t;x) := v(\tau(t;x);x), \qquad \text{for } t \in [0,t_{\max}(x)). \tag{3.12}$$

Proposition 3.7. *The solution $u(\cdot;x)$ of (VOC_t) satisfies the semigroup property.*

PROOF. We have to show that

$$u(t+s;x) = u(t;u(s;x))$$

for $s,t \geq 0$ with $t+s < t_{\max}(x)$. From (3.11) we easily deduce that

$$t(\tau;v(\sigma;x)) + t(\sigma;x) = t(\tau+\sigma;x).$$

Substituting $\tau = \tau(t;v(\tau(s;x);x))$ and $\sigma = \tau(s;x)$ we obtain

$$t+s = t(\tau(t;v(\tau(s;x);x)) + \tau(s;x);x),$$

or equivalently,

$$\tau(t+s;x) = \tau(t;v(\tau(s;x);x)) + \tau(s;x).$$

Thus, using (3.12) and the fact that v satisfies the semigroup property, we find that

$$\begin{aligned} u(t+s;x) &= v(\tau(t+s;x);x) \\ &= v(\tau(t;v(\tau(s;x);x)) + \tau(s;x);x) \\ &= v(\tau(t;v(\tau(s;x);x));v(\tau(s;x);x)) \\ &= v(\tau(t;u(s;x));u(s;x)) \\ &= u(t;u(s;x)), \end{aligned}$$

which is true as long as $t+s < t_{\max}(x)$. □

We now address ourselves to the question of *global* existence of solutions. Since (P_τ) is a semilinear equation of the type studied in Section 2 we know that a local solution $v(\cdot;x)$ on $[0,\tau_0)$, where $\tau_0 < \infty$, which stays bounded can be extended beyond τ_0. Our next theorem shows that a similar result holds for solutions of the quasilinear problem (P_t).

THEOREM 3.8. *Let $x \in X$ and suppose that, for some finite constant C, $\|u(t;x)\| \leq C$ for $t \in [0,t_{\max}(x))$. Then $t_{\max}(x) = \infty$.*

PROOF. Suppose that $u(\cdot;x)$ is a local mild solution of (P_t) on $[0,t_0)$ (where $t_0 < \infty$), and suppose also that $\|u(t;x)\| \leq C$ for all $t \in [0,t_0)$. Define

$$\tau_0 := \lim_{t\uparrow t_0} \tau(t;x) = \int_0^{t_0} \gamma(u(s;x))ds.$$

Assume now that $\tau_0 = \infty$. Then $\gamma(u(t;x)) \to \infty$ for $t \uparrow t_0$. By Assumption 3.1, γ is locally bounded, thus $\|u(t;x)\| \to \infty$ as $t \uparrow t_0$. This contradicts the assumption that u is bounded. Therefore $\tau_0 < \infty$. Since $\|v(\tau;x)\| = \|u(t(\tau;x);x)\| \leq C$ for $\tau \in [0,\tau_0)$ we may conclude that $v(\cdot;x)$ can be extended beyond τ_0. But this also implies that $u(\cdot;x)$ can be extended beyond t_0. From this the result follows. □

If $t_{\max}(x) = \infty$ for every $x \in X$, then the quasilinear Cauchy problem (P_t) has a unique global mild solution $u(\cdot;x)$ which, by Proposition 3.7, satisfies the semigroup property. By setting

$$T(t)x := u(t;x), \qquad t \geq 0,\ x \in X, \tag{3.13}$$

we obtain a nonlinear strongly continuous semigroup $\{T(t), t \geq 0\}$ on X.

We conclude this section with an example which illustrates that it is not sufficient merely to assume that γ is nonnegative, but that strict positivity is a requirement which cannot be omitted.

EXAMPLE 3.9. Consider the quasilinear Cauchy problem

$$\frac{d}{dt}u(t) = \gamma(u(t))A_0u(t), \quad u(0) = x, \tag{3.14}$$

where A_0 is the generator of a C_0-semigroup $\{T_0(t), t \geq 0\}$ on X. The "integral equation" corresponding to this problem is:

$$u(t;x) = T_0\Big(\int_0^t \gamma(u(s;x))\,ds\Big)x. \tag{3.15}$$

Indeed, for $x \in D(A_0)$, every continuous solution of (3.15) is automatically continuously differentiable and satisfies (3.14). Nevertheless solutions to (3.15) are not always unique as we now show.

Let X be the Banach space

$$X := \{x \in C(\mathbb{R}_+) \mid \theta \mapsto e^{-\theta}x(\theta) \text{ is bounded and uniformly continuous}\}$$

with norm $\|x\| := \sup_{\theta\geq 0} e^{-\theta}|x(\theta)|$. Let $\{T_0(t), t \geq 0\}$ be a C_0-semigroup of translations on X, that is, we set

$$(T_0(t)x)(\theta) := x(t+\theta), \quad t,\theta \geq 0.$$

Let $\gamma(u) := |u(0)|$ and let $x \in X$ be given by $x(\theta) := 2\sqrt{\theta}$. It is easy to check that for every $c \in \mathbb{R}_+$

$$u_c(t;x)(\theta) := \begin{cases} 2\sqrt{\theta} & \theta \geq 0,\ t < c, \\ 2\sqrt{\theta + (t-c)^2}, & \theta \geq 0,\ t \geq c. \end{cases}$$

is a solution of integral equation (3.15). In particular, $u \equiv x$ is an equilibrium solution. Using the definition of $T_0(t)$ the integral equation (3.15) amounts to

$$u(t;x)(\theta) = x\Big(\theta + \int_0^t |u(s;x)(0)|ds\Big). \tag{3.16}$$

Note that the initial condition x is an equilibrium of the system (3.15) although $x \notin D(A_0)$. In fact, every initial condition x with $x(0) = 0$ is an equilibrium. By substitution of $\theta = 0$, equation (3.16) can be reduced to the scalar integral equation

$$u(t;x)(0) = x\Big(\int_0^t u(s;x)(0)\,ds\Big),$$

as long as we consider only nonnegative initial data. Moreover, for $\psi(t) := \int_0^t u(s;x)(0)\,ds$ this equation reduces to the ordinary differential equation:

$$\psi'(t) = x(\psi(t)), \quad \psi(0) = 0.$$

Clearly, the nonuniqueness of solutions of this differential equation is a consequence of the fact that x is not Lipschitz continuous.

4. Positivity, boundedness and regularity

As motivated in the introduction we are interested in equations of form (P_t) which are derived from models for structured populations, and whose solutions describe the distribution of its individuals over some structuring variable(s) such as age and/or size. Thus the only biologically relevant solutions $u(\cdot;x)$ of (P_t) are those which are positive as long as we start with a positive initial distribution x. We thus are interested in conditions, preferably on A_0 and F, which assure that solutions $u(\cdot;x)$ of (P_t) are positive given that the initial data is positive. Whenever we speak of positivity, we assume that X is a Banach lattice and denote its positive cone by X_+ (see SCHAEFER [1974]). If x is an element of X_+ then we write $x \geq 0$.

We start our considerations concerning positivity for the semilinear equation (P_τ) and afterwards extend our results to the quasilinear equation (P_t). An easy consequence of the variation–of–constants formula (VOC_τ) is the following

PROPOSITION 4.1. *Let A_0 be the generator of a linear positive C_0-semigroup $\{T_0(t), t \geq 0\}$ on X and let B map X_+ into X_+. If $x \geq 0$ then $v(\tau;x) \geq 0$ for all $\tau \in [0, \tau_{\max}(x))$, where $v(\cdot;x)$ is the solution of (VOC_τ) given by Theorem 2.1.*

In applications it seems too strong to assume that B is positive in order to obtain positive solutions of (VOC_τ) for positive initial values. We want to allow some "local" nonpositivity. Thus we claim that the assertion of Proposition 4.1 remains valid under a weaker positivity assumption on B.

THEOREM 4.2. *Let A_0 be the generator of a linear positive C_0-semigroup $\{T_0(t), t \geq 0\}$ and let B satisfy the following "positive–off–diagonal" property*

$$\langle B(x), x^* \rangle \geq 0 \quad \text{for all } x \geq 0,\ x^* \geq 0 \text{ with } \langle x, x^* \rangle = 0. \tag{POD}$$

Then $x \geq 0$ implies $v(\tau;x) \geq 0$ for all $\tau \in [0, \tau_{\max}(x))$.

To prove this theorem we need some auxiliary results.

REMARK. The conclusion of Theorem 4.2 can easily be proved if B is a linear bounded operator on X. Indeed by Proposition 4.1 we know that $A_0 + B + \|B\|\mathrm{I}$ generates a positive semigroup, since $B + \|B\|\mathrm{I} \geq 0$: see NAGEL [1986, Section C–II, Theorem 1.11]. Hence $A_0 + B = A_0 + B + \|B\|\mathrm{I} - \|\mathrm{B}\|\mathrm{I}$ generates a positive semigroup as well.

In the nonlinear case the analysis is somewhat more tedious. We will need some results concerning the geometry of the positive cone X_+. For $x \in X$ we denote by $x_+ := \sup\{x, 0\}$, $x_- := \sup\{-x, 0\}$ and $|x| := \sup\{x, -x\}$ the *positive part*, the *negative part* and the *modulus* of x, respectively. Furthermore let $\mathrm{dist}(x, X_+) := \inf_{y \in X_+} \|x - y\|$ denote the *distance function*.

LEMMA 4.3. *Let X be a Banach lattice. Then*

(a) $x = x_+ - x_-$ *and* $|x| = x_+ + x_-$ *for all* $x \in X$;
(b) $x \mapsto x_+$, $x \mapsto x_-$ *and* $x \mapsto |x|$ *are continuous;*
(c) $\|x - y\| \geq \|x_-\| - \|y_-\|$ *for all* $x, y \in X$;
(d) $x \mapsto \|x_-\|$ *is convex and positive homogeneous, i.e.* $\|\lambda x_-\| = \lambda \|x_-\|$ *for all* $x \in X$ *and* $\lambda \geq 0$;
(e) $\|x_-\| = \mathrm{dist}(x, X_+)$ *for all* $x \in X$;
(f) $\mathrm{dist}(x + y, X_+) \leq \mathrm{dist}(x, X_+) + \mathrm{dist}(y, X_+)$ *for all* $x, y \in X$;
(g) $\|[x + y]_-\| \leq \|x_-\| + \|y_-\| \leq \|x_-\| + \|y\|$ *for all* $x, y \in X$;
(h) *Let T be a positive linear operator on X. Then $(Tx)_- \leq Tx_-$ for all $x \in X$.*

PROOF. (a),(b): easy.

(c): For $z_1, z_2 \in X$ we have $(z_1+z_2)_- \le (z_1)_- + (z_2)_-$. Indeed let $z_3 = z_1 + z_2$. Since $(z_i)_- = \frac{1}{2}(|z_i| - z_i)$ $(i = 1,2,3)$ we have $(z_3)_- \le \frac{1}{2}(|z_1| + |z_2| - z_1 - z_2) = (z_1)_- + (z_2)_-$. Let $x, y \in X$. Then $x_- = (x-y+y)_- \le (x-y)_- + y_-$. For the norm we obtain $\|x_-\| \le \|[x-y]_-\| + \|y_-\| \le \|x-y\| + \|y_-\|$. Thus $\|x-y\| \ge \|x_-\| - \|y_-\|$ for all $x, y \in X$.

(d): easy.

(e): Let $x, y \in X$. Then $\operatorname{dist}(x, X_+) = \inf_{y \in X_+} \|x-y\| \le \|x - x_+\| = \|x_-\|$. For the converse estimate we conclude from (c) that $\|x-y\| \ge \|x_-\| - \|y_-\| = \|x_-\|$ for all $y \in X_+$. Hence $\operatorname{dist}(x, X_+) = \inf_{y \in X_+} \|x-y\| \ge \|x_-\|$ which proves assertion (e).

(f): Let $x, y \in X$. Then $\operatorname{dist}(x+y, X_+) = \inf_{z \in X_+} \|x+y-z\| = \inf_{u \in X_+} \inf_{v \in X_+} \|x+y-u-v\| \le \inf_{u \in X_+} \|x-u\| + \inf_{v \in X_+} \|y-v\| = \operatorname{dist}(x, X_+) + \operatorname{dist}(y, X_+)$.

(g): easy.

(h): Let $0 \le T \in \mathcal{L}(X)$ and $x \in X$. Then $(Tx)_- = \frac{1}{2}(|Tx| - Tx) \le \frac{1}{2}(T|x| - Tx) = Tx_-$. □

LEMMA 4.4. *Let* $x \ge 0$. *Equivalent are:*

(i) *If* $x^* \ge 0$ *and* $\langle x, x^* \rangle = 0$, *then* $\langle B(x), x^* \rangle \ge 0$.

(ii) $\lim_{h \downarrow 0} \frac{1}{h} \operatorname{dist}(x + hB(x), X_+) = \lim_{h \downarrow 0} \frac{1}{h} \|[x + hB(x)]_-\| = 0$.

PROOF. We can assume without loss of generality that $x \in \partial X_+$. Indeed for $x \in \operatorname{int}(X_+)$ the conditions (i) and (ii) are trivially satisfied.

(i)⇔(ii): Consider $\Phi : X \to \mathbb{R}_+$ given by $\Phi(x) = \operatorname{dist}(x, X_+)$. One easily verifies that Φ is a sublinear, continuous function (Lemma 4.3). We thus can define the *subdifferential* $d\Phi(x)$ of Φ in x (see e.g. NAGEL [1986, A–II, Section 2] or CLÉMENT, HEIJMANS et al. [1987,Appendix A.1]).

$$\begin{aligned} d\Phi(x) :=& \{x^* \in X^* : \langle y, x^* \rangle \le \Phi(y) \text{ for all } y \in X, \text{ and } \langle x, x^* \rangle = \Phi(x)\} \\ =& \{x^* \in X^* : \langle y, x^* \rangle \le \operatorname{dist}(y, X_+) \text{ for all } y \in X, \text{ and } \langle x, x^* \rangle = \operatorname{dist}(x, X_+)\} \\ =& \{x^* \in X^* : \|x^*\| \le 1, \langle x, x^* \rangle = 0 \text{ and } -x^* \ge 0\}. \end{aligned}$$

Since $x \ge 0$ we have

$$\lim_{h \downarrow 0} \frac{1}{h} \operatorname{dist}(x + hB(x), X_+) = \lim_{h \downarrow 0} \frac{1}{h} (\Phi(x + hB(x)) - \Phi(x)) = D^+_{B(x)}(x)$$

where $D^+_{B(x)}(x)$ denotes the right sided Gateaux–derivative of Φ at x in the direction of $B(x)$. It is well known (see e.g. CLÉMENT, HEIJMANS et al. [1987, Proposition A.1.24]) that

$$D^+_{B(x)}\Phi(x) = \sup\{\langle B(x), x^* \rangle : x^* \in d\Phi(x)\}. \tag{4.1}$$

From the explicit form of $d\Phi(x)$ we now conclude that condition (i) is equivalent to $\langle B(x), x^* \rangle \le 0$ for all $x^* \in d\Phi(x)$. By formula (4.1) this is equivalent to $D^+_{B(x)}\Phi(x) \le 0$, and thus to $\lim_{h \downarrow 0} \frac{1}{h} \operatorname{dist}(x + hB(x), X_+) \le 0$, hence to condition (ii). □

REMARK. Obviously condition (i) of Lemma 4.4 is equivalent to:

if $x^* \in X^*$ and $\langle x, x^* \rangle = \sup_{y \in X_+} \langle y, x^* \rangle$, then $\langle B(x), x^* \rangle \le 0$.

Note that for $x \in \partial X_+$ the elements $x^* \in X^*$ with $\langle x, x^* \rangle = \sup_{y \in X_+} \langle y, x^* \rangle$ can be interpreted as the normal vectors to X_+ in x. Condition (i), or equivalently (ii), is also called the *subtangential condition* of B in $x \in \partial X_+$ (compare DEIMLING [1977]).

The following lemma can be found in MARTIN [1976, Lemma 1.3, p.326].

LEMMA 4.5. *Let $x \in X_+$. Assume that one of the equivalent conditions of Lemma 4.4 holds. Then*

$$\frac{1}{h}[T_0(h)x + \int_0^h T_0(s)B(x)\,ds]_- \to 0 \quad \text{as } h \downarrow 0. \tag{4.2}$$

We are now prepared to prove Theorem 4.2:

PROOF OF THEOREM 4.2. Without loss of generality we may assume that

$$B(x) = B(x_+) \qquad \text{for all } x \in X. \tag{4.3}$$

If this is not satisfied we define $B_0 : X \to X$ by $B_0(x) := B(x_+)$ $(x \in X)$. Then by construction $B_0(x) = B_0(x_+)$. Furthermore (POD) remains valid. If solutions of (VOC_τ) with B replaced by B_0 are positivity preserving, then they coincide with solutions of the original (VOC_τ) for positive initial data x.
We first consider the case where $\|T_0(t)\| \le Me^{\omega t}$ with $M = 1$ for all $t \ge 0$ (sometimes called the quasi-contractive case). In a second step we will reduce the general case (with $M > 1$) to this situation. Let $x \ge 0$ and let $v(\tau) = v(\tau; x)$ be the continuous solution of (VOC_τ) on $[0, \tau_{\max}(x))$. We show that $v(\tau) \ge 0$ or equivalently that $v_-(\tau) := [v(\tau)]_-$ is zero. For $\tau < \tau_{\max}$ we define

$$\phi(\tau) := e^{-\omega\tau}\,\|v_-(\tau)\|.$$

Now

$$v(\tau + h) = T_0(h)v(\tau) + \int_0^h T_0(h-s)B(v(\tau+s))\,ds.$$

Thus by Lemma 4.3(c), Lemma 4.5 and (4.3) we have

$$\begin{aligned}
\|v_-(\tau+h)\| &\le \left\|v(\tau+h) - T_0(h)v_+(\tau) - \int_0^h T_0(h-s)B(v_+(\tau+s))\,ds\right\| \\
&\qquad + \left\|[T_0(h)v_+(\tau) + \int_0^h T_0(h-s)B(v_+(\tau+s))\,ds]_-\right\| \\
&\le \|T_0(h)v_-(\tau)\| + \left\|v(\tau+h) - T_0(h)v(\tau) - \int_0^h T_0(h-s)B(v(\tau+s))\,ds\right\| \\
&\qquad + \left\|\left[T_0(h)v_+(\tau) + \int_0^h T_0(h-s)B(v_+(\tau))\,ds\right]_-\right\| + \mathrm{o(h)} \\
&\le e^{\omega h}\,\|v_-(\tau)\| + \mathrm{o(h)}.
\end{aligned}$$

Hence $\phi(\tau + h) \le \phi(\tau) + \mathrm{o(h)}$ for $h \downarrow 0$ and $\tau < \tau_{\max}$. In other words

$$D_+\phi(\tau) := \liminf_{h\downarrow 0} \frac{1}{h}(\phi(\tau+h) - \phi(\tau)) \le 0.$$

Since $\phi(0) = \|u_-(0)\| = \|x_-\| = 0$, a well known result from the theory of differential inequalities (see e.g. MARTIN [1976, Lemma 7.4, p.260]) implies $\phi = 0$.

It remains to consider the case where $\|T_0(t)\| \le Me^{\omega t}$ with $M > 1$. We use a renormalization of X which allows us to reduce this situation to the the case $M = 1$. For $x \in X$ let $\|x\|' := \sup_{t\ge 0} e^{-\omega t}\|T_0(t)|x|\,\|$. We have $\|x\| \le \|x\|' \le M\|x\|$, thus $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms and

$$\begin{aligned}\|T_0(t)x\|' &= \sup_{s\ge 0} e^{-\omega s}\|T_0(s)|T_0(t)x|\,\| \le \sup_{s\ge 0} e^{-\omega s}\|T_0(s+t)|x|\,\| \\ &\le e^{\omega t}\sup_{s\ge 0} e^{-\omega(s+t)}\|T_0(t+s)|x|\,\| \le e^{\omega t}\sup_{s\ge 0} e^{-\omega s}\|T_0(s)|x|\,\| = e^{\omega t}\|x\|'.\end{aligned}$$

Let $x, y \in X$ with $|x| \le |y|$, then $\|\,|x|\,\|' := \sup_{t\ge 0} e^{-\omega t}\|T_0(t)|x|\,\| \le \sup_{t\ge 0} e^{-\omega t}\|T_0(t)|y|\,\| = \|\,|y|\,\|'$. This shows that $\|\cdot\|'$ defines a lattice norm on X. Hence the properties listed in Lemma 4.3 hold for $\|\cdot\|'$ and dist$'$ (dist$'$ defined in the obvious way) as well. Thus $\{T_0(t)\,,\, t \ge 0\}$ is a positive quasi-contraction semigroup on $(X, \|\cdot\|')$ and we are in the before discussed situation and thus the assertion follows. □

REMARK. Note that our perturbation B needs not to be defined on the whole space X but only on X_+, which is of course the situation we usually meet in biological examples.

For the solutions $u(t;x)$ of the quasilinear Cauchy problem (P_t) the following positivity result follows:

THEOREM 4.6. *Assume that F satisfies* (POD). *Then, for any $x \ge 0$ we have $u(t;x) \ge 0$ for all $t \in [0, t_{\max}(x))$.*

PROOF. The assertion follows directly from Theorem 4.2 using relation (3.8) and Theorem 3.6. □

Another important property of solutions of (P_t) or (VOC_t) which has to be investigated is the boundedness. Recall from Theorem 3.8 that solutions of (VOC_t) which are bounded on a finite time interval can be extended. Thus in order to obtain global existence of a solution of (VOC_t), we have to show that it stays bounded on any finite time interval.

THEOREM 4.7. *Let A_0 be generator of a linear bounded positive C_0–semigroup $\{T_0(t)\,,\, t \ge 0\}$. Assume that $u(t;x) \ge 0$ if $x \ge 0$ and $t < t_{\max}(x)$. Furthermore let F satisfy the following "off–diagonal–boundedness" property: there exists an operator F_0 on X such that*

(i) *$F(x) \le F_0(x)$ for all $x \ge 0$;*

and (4.4)

(ii) $\|F_0(x)\| \le C\|x\|$ for all $x \ge 0$.

Then $t_{\max}(x) = \infty$ for all $x \ge 0$.

PROOF. In Section 3 we have seen that a mild solution $u(t) = u(t;x)$ of (P_t) is by definition a continuous solution of the variation–of–constants formula

$$u(t) = T_0(\tau_u(t))x + \int_0^t T_0(\tau_u(t) - \tau_u(s))F(u(s))\,ds, \qquad t \in [0, t_0), \qquad (\mathrm{VOC}_t)$$

where τ_u is given by formula (3.1). Let $x \in X, x \ge 0$ and $t < t_0$. Using assumption (i) for F and the positivity of $u(s)$ we have

$$u(t) \le T_0(\tau_u(t))x + \int_0^t T_0(\tau_u(t) - \tau_u(s))F_0(u(s))\,ds,$$

since every $T_0(t)$ is a positive operator. Thus by assumption (ii) and the boundedness of the semigroup $\{T_0(t), t \geq 0\}$ there exists a constant $M > 0$ such that

$$\|u(t)\| \leq M\|x\| + \int_0^t MC\|u(s)\|\, ds.$$

By the lemma of Gronwall we have $\|u(t)\| \leq M \cdot e^{MCt}\|x\|$ for all $t < t_0$. Hence $\|u(t)\| \leq M \cdot e^{MCt_0}\|x\|$. Thus the assumptions of Theorem 3.8 are satisfied and we can conclude that there is a globally defined continuous solution of (VOC_t), or equivalently a global mild solution of (P_t). □

REMARKS. 1) To assume that $\|F(x)\| \leq M\|x\|$ for all $x \geq 0$ is much stronger than the "off-diagonal-boundedness" property (4.4). Indeed in many examples from structured population dynamics such an assumption is quite unrealistic due to the presence of a death term, whereas the existence of an operator F_0 as in (4.4) can usually be established.
2) One can easily construct examples which show that the assumption "$\{T_0(t), t \geq 0\}$ bounded" is essential in order to have global solutions of (P_t).

COROLLARY 4.8. *Let A_0 be the generator of a linear, positive, bounded C_0-semigroup $\{T_0(t), t \geq 0\}$. Furthermore, assume that F satisfies the "positive-off-diagonal" property (POD) and the "off-diagonal-boundedness" property (4.4). Then $t_{\max}(x) = \infty$ for all $x \geq 0$.*

PROOF. The assertion follows directly from Theorem 4.6 and Theorem 4.7. □

In the remainder of this section we study the regularity properties of the solutions of (VOC_t) and justify the name "mild solution" of (P_t). We will use the relation between (P_t) and (P_τ) (recall Lemma 3.2, and Theorem 3.4 and 3.6) and the knowledge we have about regularity of solutions of the semilinear equation (P_τ) (see Theorem 2.2 and Theorem 2.3).

THEOREM 4.9. *Let $u(\cdot) := u(\cdot; x)$ be a local (continuous) solution of (P_t) on $[0, t_0)$ with initial value $x \in X$. Then $\langle u(\cdot), x^* \rangle$ is continuously differentiable for every $x^* \in D(A_0^*)$ and moreover*

$$\frac{d}{dt}\langle u(t), x^* \rangle = \gamma(u(t))\langle u(t), A_0^* x^* \rangle + \langle F(u(t)), x^* \rangle, \qquad \text{for all } t \in [0, t_0). \tag{4.5}$$

PROOF. Since u is related to v by formula (3.8) it is immediately clear that $\langle u(\cdot; x), x^* \rangle$ is continuously differentiable for all $x^* \in D(A_0^*)$ since the same is true for $v(\cdot; x)$. Moreover by formula (2.3):

$$\begin{aligned}
\frac{d}{dt}\langle u(t;x), x^* \rangle &= \frac{d}{dt}\langle v(\tau(t;x);x), x^* \rangle \\
&= \frac{d}{dt}\tau(t;x)\Big[\frac{d}{d\tau}\langle v(\tau;x), x^* \rangle\Big]_{\tau=\tau(t;x)} \\
&= \gamma(u(t;x))\big[\langle v(\tau(t;x);x), A_0^* x^* \rangle + \langle B(v(\tau(t;x);x)), x^* \rangle\big] \\
&= \gamma(u(t;x))\langle u(t;x), A_0^* x^* \rangle + \langle F(u(t;x)), x^* \rangle
\end{aligned}$$

for all $t \in [0, t_0)$. Thus (4.5) holds. □

Continuous solutions of (VOC_t) are thus "weak solutions" of (P_t). It is an important task to find conditions under which the continuous solutions of (VOC_t) are classical C^1-solutions of (P_t).
Once again we use the corresponding result from the semilinear theory. In Theorem 2.3 we proved that a continuous solution $v(\cdot; x)$ of (VOC_τ) is a classical solution of (P_τ) if B is continuously Fréchet-differentiable and $x \in D(A_0)$. This result immediately carries over to our quasilinear equation (P_t).

Corollary 4.10. *Assume that* $B = \gamma^{-1} \cdot F$ *is continuously Fréchet–differentiable. Then* $u(\cdot; x)$ *is a classical solution of* (P_t) *if* $x \in D(A_0)$.

Proof. The map $t \mapsto u(t;x)$ is C^1 if and only if $\tau \mapsto v(\tau, x)$ is C^1. By Theorem 2.3 the latter is true if $x \in D(A_0)$ and we have

$$\frac{d}{d\tau} v(\tau) = A_0 v(\tau) + B(v(\tau)).$$

Hence by Theorem 3.4 the result follows. □

5. Principle of linearized (in)stability

In this section we prove that the principle of linearized (in)stability holds for quasilinear equations of the form (P_t). Obviously an equilibrium $\overline{u}$ of (P_t) is an equilibrium of (P_τ), and vice versa. The main result of this section is the following.

Theorem 5.1. *An equilibrium* $\overline{u}$ *of* (P_t) *is stable if and only if it is stable for* (P_τ).

As in Section 2 we may assume without loss of generality that $\overline{u} = 0$ is an equilibrium of (P_t), i.e.,

$$F(0) = 0. \tag{5.1}$$

Then 0 is also an equilibrium of (P_τ).

Proof of Theorem 5.1. We only prove the *if* part. The *only if* part is proved in the same way. Assume that 0 is a stable equilibrium of (P_τ). Let $\epsilon > 0$. We must show that there is a $\delta > 0$ such that for $\|x\| \leq \delta$:

(i) $t_{\max}(x) = \infty$, and

(ii) $\|u(t;x)\| \leq \epsilon, \qquad t \geq 0$.

By hypothesis there exists $\delta > 0$ such that for $\|x\| \leq \delta$:

(i') $\tau_{\max}(x) = \infty$, and

(ii') $\|v(\tau;x)\| \leq \epsilon, \qquad \tau \geq 0$.

Recall from Section 3 that

$$t_{\max}(x) = \int_0^\infty \frac{d\sigma}{\gamma(v(\sigma;x))}.$$

Since γ is locally bounded there exists $\eta > 0$ such that for all v with $\|v\| \leq \epsilon$ we have $\gamma(v) \leq \eta$. Hence, if $\|x\| \leq \delta$ we have

$$t_{\max}(x) \geq \int_0^\infty \frac{d\sigma}{\eta} = \infty.$$

Furthermore,

$$\|u(t;x)\| = \|v(\tau(t;x);x)\| \leq \epsilon, \qquad t \geq 0.$$

Thus we have proved (i) and (ii). □

From Theorems 2.6 and 2.7 we know that (in)stability of the equilibrium 0 of the semilinear problem (P_τ) and therefore of the quasilinear problem (P_t), hinges upon the spectral properties of the generator $A_0 + B'(0)$. Here we have assumed that $B = F/\gamma$ is Fréchet differentiable at 0. If both F and γ are differentiable at 0, then the linearization of (P_t) at $u = 0$ is given by

$$\frac{dw}{dt} = \gamma(0)A_0 w + F'(0)w. \tag{5.2}$$

Since

$$\gamma(0)A_0 + F'(0) = \gamma(0)\big(A_0 + B'(0)\big)$$

and $\gamma(0) > 0$ we may equivalently state that the stability of the equilibrium is determined by (5.2), or more precisely, by the spectral properties of the linearized operator $\gamma(0)A_0 + F'(0)$.

Let us denote by $\{S(t), t \geq 0\}$ the semigroup corresponding to the linear Cauchy problem (5.2). One might wonder if the analogue of Theorem 2.5 holds: is $x \mapsto u(t;x)$ Fréchet differentiable at $x = 0$ with derivative $S(t)$? We now present an example which shows that this is not true in general.

EXAMPLE 5.2. Let S^1 be the unit circle in $\mathbb{R}^2$ and $X = C(S^1)$. We will identify S^1 with $\mathbb{R}/[0, 2\pi)$. Define the C_0-group $\{T_0(t), t \geq 0\}$ on X by:

$$\big(T_0(t)x\big)(\theta) := x(t - \theta), \qquad \theta \in S^1,\ t \in \mathbb{R}.$$

Let $\gamma(x) := 1/2 + |1/2 + x(0)|$ and $F \equiv 0$. Obviously, $x = 0$ is a stable equilibrium of

$$\frac{du}{dt} = \gamma(u)A_0 u, \qquad u(0) = x. \tag{5.3}$$

Note that $\gamma(u)$ represents the speed of rotation along the circle. Obviously, γ is differentiable at $u = 0$ and the linearization around 0 is given by

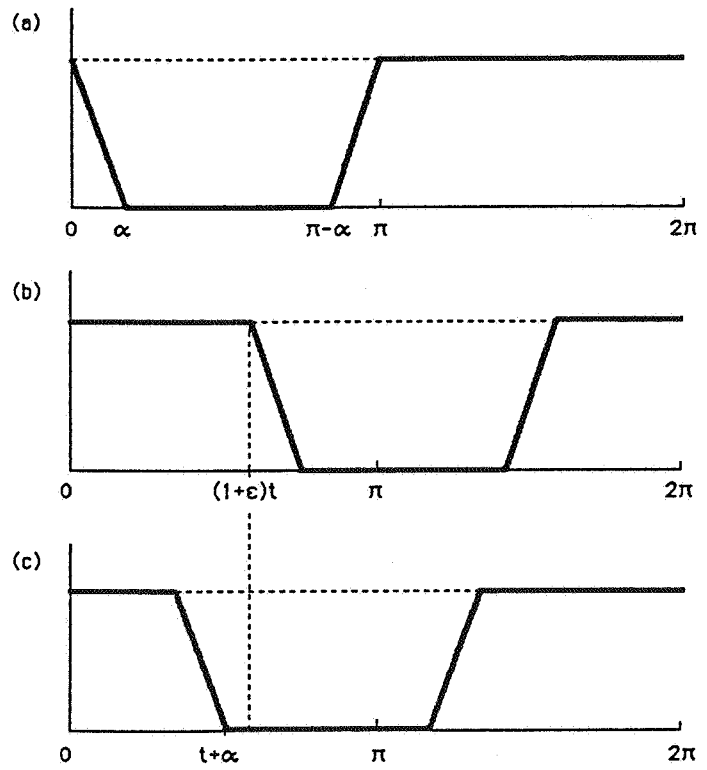
$$\frac{dw}{dt} = A_0 w. \tag{5.4}$$

Let $\{T(t), t \geq 0\}$ be the nonlinear semigroup generated by (5.3) (note that this problem has a global solution for every $x \in X$).

We show that $T_0(t)$ is *not* the Fréchet derivative of $x \mapsto T(t)x$ at $x = 0$. Actually we prove more, namely that for some $t > 0$ (but one can extend this for all $t > 0$) the following holds: for all ϵ with $0 < \epsilon < 1$ there exists $x \in X$ with $\|x\| = \epsilon$, such that $\|T(t)x - T_0(t)x\| = \|x\|$. Namely, fix $t \in (0, \pi/2]$ and $0 < \epsilon < 1$. Choose $\alpha > 0$ such that $\alpha \leq \epsilon t$.
Let x be as depicted in Figure (a). During the time interval $[0, t]$ the rotation velocity in (5.3) equals

$$\gamma(u(t;x)) = 1/2 + \big|1/2 + u(t;x)(0)\big| = 1 + \epsilon$$

since $u(t;x)(0) = (T(t)x)(0) = \epsilon$ for all $t \in (0, \pi/2]$, whereas this speed is constantly 1 in (5.4). Therefore $T(t)x$ and $T_0(t)x$ are as depicted in Figure (b) and (c) respectively. Since $\alpha \leq \epsilon t$ we have $(1 + \epsilon)t \geq t + \alpha$ and we find that $\|T(t)x - T_0(t)x\| = \epsilon = \|x\|$.



For the sake of completeness we want to conclude this section by mentioning the explicit formula for the linearization of (P_t) in a nontrivial equilibrium point $\overline{u}$. Again we first consider the linearized equation corresponding to (P_τ). We obtain:

$$\begin{aligned}\frac{dw}{dt} &= A_0 w + \frac{1}{\gamma(\overline{u})} F'(\overline{u})w - \frac{\langle \gamma'(\overline{u}), w\rangle}{\gamma(\overline{u})^2} F(\overline{u}) \\ &= A_0 w + \langle \gamma'(\overline{u}), w\rangle \cdot \frac{1}{\gamma(\overline{u})} A_0 \overline{u} + \frac{1}{\gamma(\overline{u})} F'(\overline{u}) w.\end{aligned}$$

This system "corresponds" to the linearization of (P_t) in $\overline{u}$ which is given by:

$$\frac{dw}{dt} = \gamma(\overline{u}) A_0 w + \langle \gamma'(\overline{u}), w\rangle \cdot A_0 \overline{u} + F'(\overline{u}) w.$$

6. Application to the cell division model

After having studied the general quasilinear equation in some detail we now return to our starting point: the model describing size-dependent cell growth and division described by equation (1.2). Before starting the analysis we have to specify the setting. Let us recall from Section 1 the initial value problem which we want to investigate:

$$\frac{\partial}{\partial t} p(t,s) + \hat{\gamma}(E(t)) \frac{\partial}{\partial s}(g(s)p(t,s)) = -\mu(s)p(t,s) - b(s)p(t,s) + 4b(2s)p(t,2s) \qquad (6.1\text{a})$$

$$g(\alpha/2)p(t,\alpha/2) = 0 \tag{6.1b}$$

$$\frac{d}{dt}E(t) = -\sigma E(t) + \int_{\alpha/2}^{1} h(s)p(t,s)\,ds \tag{6.1c}$$

$$p(0,s) = p_0(s) \geq 0 \tag{6.1d}$$

$$E(0) = E_0 \geq 0 \tag{6.1e}$$

The biological interpretation of this equation, outlined in Section 1, suggests to look for "densities" p satisfying (6.1). Mathematically it thus makes sense to look for solutions $p(t,\cdot) \in L^1([\alpha/2,1])$ which are positive.

We assume that the minimal possible cell division size of a cell is α. Thus the possible cell size s of an individual cell is restricted to values between a minimal size $\alpha/2$ and a maximal size 1. Furthermore we make the following assumptions on $\hat{\gamma}, g, \mu, b$ and h:

ASSUMPTION 6.1.

- ($A_{\hat{\gamma}}$) $\hat{\gamma} : X \to \mathbb{R}_+\backslash\{0\}$ is locally bounded and Fréchet differentiable.
- (A_g) $g : [\alpha/2, 1] \to \mathbb{R}_+\backslash\{0\}$ is continuous.
- (A_μ) $\mu : [\alpha/2, 1] \to \mathbb{R}_+$ is measurable and bounded.
- (A_b) $b : [\alpha/2, 1] \to \mathbb{R}_+$ is measurable and bounded, $b(x) = 0$ for a.e. $x \in [\alpha/2, \alpha]$ and $b(x) > 0$ for a.e. $x \in (\alpha, 1]$.
- (A_h) $h : [\alpha/2, 1] \to \mathbb{R}_+$ is measurable and bounded.

REMARK. In (A_b) we assume that the division rate b is bounded on $[\alpha/2, 1]$. The biological interpretation of this mathematical assumption has quite drastic consequences for our model. The boundedness of b implies that a cell which does not divide before reaching size 1 will never divide. It will just exist and grow until it eventually dies. Nevertheless these "large, quiescent" cells may have an effect on our system (6.1), since we assumed that the total number of cells (weighted by h) will contribute to the production of the enzyme E. Thus, also cells of size $s > 1$ would be of importance here, but they are not included in our model formulation. There are at least two ways out of the dilemma. We could make the assumption that cells with size $s > 1$ will automatically belong to another "type" of cells (for example by differentiation), which is out of our consideration. Thus they have no (direct) influence on the amount of the enzyme E present. Alternatively we could assume that cells which reach the threshold $s = 1$ die instantaneously. This may seem rather drastic but one should realize that the fraction of cells which indeed reaches this size can be negligible small depending on b and μ.

We plan to discuss a more elaborated model describing the blood cell production system of man in the next future. The model we have in mind is based on the idea that cells which eventually will become blood ingredients have to pass through several different cell type compartments during their development.

The system (6.1) is of the form (P_t). To show this we have to define A_0, F and γ.
Let $X := L^1[\alpha/2, 1] \times \mathbb{R}$. We define γ, A_0 and F by

$$\gamma\big((p,E)\big) := \hat{\gamma}(E),$$

$$A_0\big((p,E)\big) := (-(gp)', 0),$$

with domain

$$D(A_0) := \{(p,E) \in L^1[\alpha/2,1] \times \mathbb{R} \mid p \in AC[\alpha/2,1] \text{ and } p(\alpha/2) = 0\},$$

and

$$F\big((p,E)\big) := \Big(s \mapsto [-\mu(s)p(s) - b(s)p(s) + 4b(2s)p(2s)],\ -\sigma E + \int_{\alpha/2}^{1} h(s)p(s)\,ds\Big),$$

where we set $b(2s)p(2s) = 0$ if $s > 1/2$.

If we define

$$u(t) := (p(t,\cdot), E(t)) \quad \text{and} \quad x := (p_0(\cdot), E_0)$$

we see that (6.1) is of the form (P_t).

It is obvious that A_0 generates a bounded, positive C_0–semigroup $\{T_0(t), t \geq 0\}$ and that Assumption 3.1 is satisfied. Furthermore one easily verifies that F/γ satisfies the "positive–off–diagonal" property (POD) introduced in Section 4. Since F is linear it automatically satisfies the "off-diagonal–boundedness" property (4.4). Summarizing we thus have:

Theorem 6.2. *Let Assumption (6.1) hold and suppose that $E_0 \geq 0$ and p_0 is a positive, absolutely continuous function satisfying $p_0(\alpha/2) = 0$. Then there exists a unique positive, continuously differentiable solution $p(t,\cdot), E(t)$ of (6.1) defined for all $t \geq 0$.*

Remark. The abstract theory gives us "solutions" of (6.1) for all initial data p_0 in $L^1[\alpha/2, 1]$. In general these solutions are not continuously differentiable but satisfy a related integral equation which can be obtained by integration of (6.1) along characteristics.

Next we will investigate the existence and stability properties of positive equilibrium solutions of the nonlinear system (6.1). Of course $(0,0)$ is an equilibrium point, but in general there may be others. Such nonzero equilibria $(\overline{p}, \overline{E})$ can be obtained from the functional differential equation

$$\hat{\gamma}(\overline{E})\frac{d}{ds}(g\overline{p}) = -\mu(s)\overline{p}(s) - b(s)\overline{p}(s) + 4b(2s)\overline{p}(2s), \tag{6.3a}$$

$$g(\alpha/2)\overline{p}(\alpha/2) = 0, \tag{6.3b}$$

$$\sigma\overline{E} = \int_{\alpha/2}^{1} h(s)\overline{p}(s)\,ds. \tag{6.3c}$$

To make life easy we assume that $\alpha > 1/2$. Then we can solve (6.3a), (6.3b) successively for $s \geq 1/2$ and $s < 1/2$ and obtain

$$\overline{p}(s) = cq(s) \quad \text{for all } s \in [\alpha/2, 1], \tag{6.4}$$

for a constant c (which must still be determined) and

$$q(s) := \begin{cases} \frac{H_\theta(s)}{g(s)} & \text{if } s \geq 1/2, \\ \frac{4}{\theta}\frac{H_\theta(s)}{g(s)} \int_{\alpha/2}^{s} \frac{b(2\xi)}{g(2\xi)}\frac{H_\theta(2\xi)}{H_\theta(\xi)}\,d\xi & \text{if } s < 1/2. \end{cases}$$

Here $\theta := \hat{\gamma}(\overline{E})$ and

$$H_\theta(s) := \exp\Big(-\frac{1}{\theta}\int_{\alpha/2}^{s} \frac{b(\xi) + \mu(\xi)}{g(\xi)}\,d\xi\Big) \quad \text{for all } s \in [\alpha/2, 1].$$

Since $\overline{p}$ is an element of $D(A_0)$ it has to be a continuous function in particular it must be continuous at $s = 1/2$. From this we obtain a scalar equation which determines $\overline{E}$, or more precisely $\theta = \hat{\gamma}(\overline{E})$, namely by the fixed point equation:

$$\theta = Q(\theta) \tag{6.5}$$

where

$$Q(\theta) := 4\int_{\alpha/2}^{1/2} \frac{b(2\xi)}{g(2\xi)} \exp\Big(-\frac{1}{\theta}\int_{\xi}^{2\xi} \frac{\mu(\eta) + b(\eta)}{g(\eta)}\,d\eta\Big)\,d\xi \quad \text{for all } \theta > 0.$$

We thus have proved the following result:

THEOREM 6.3. *The pair $(\overline{p}, \overline{E}) \neq (0,0)$ is an equilibrium of (6.1) if and only if $\hat{\gamma}(\overline{E})$ is a solution of (6.5) and $\overline{p} = cq$, with $c := \sigma\overline{E}\left(\int_{\alpha/2}^{1} h(s)q(s)\,ds\right)^{-1}$.*

Note that Q is a continuous increasing function and that $Q(0) = 0$. The quantity $Q(\theta)$ can be interpreted as the *reproductive value*: see METZ and DIEKMANN [1986, Interlude 4.3.2, p.36].

From our previous results we know that the local stability properties of an equilibrium are determined by the linearized system:

$$\frac{\partial}{\partial t}w(t,s) + \hat{\gamma}(\overline{E})\frac{\partial}{\partial s}(g(s)w(t,s)) = -\hat{\gamma}'(\overline{E})\frac{\partial}{\partial s}(g(s)\overline{p}(s)) \cdot \mathcal{E}(t) - \mu(s)w(t,s) - b(s)w(t,s) + 4b(2s)w(t,2s), \tag{6.6a}$$

$$g(\alpha/2)w(t,\alpha/2) = 0, \tag{6.6b}$$

$$\frac{d}{dt}\mathcal{E}(t) = -\sigma\mathcal{E}(t) + \int_{\alpha/2}^{1} h(s)w(t,s)\,ds. \tag{6.6c}$$

The characteristic equation corresponding to this system can be obtained by a straightforward but lengthy computation.

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