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SOME INEQUALITIES INVOLVING RIEMANN'S ZETA-FUNCTION

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Some inequalities involving Riemann's zeta-function

by

J. van de Lune

ABSTRACT

Littlewood showed that for infinitely many $n \in \mathbb{N}$, $\psi(n)$ is considerably larger than n, ψ being Chebychef's function occurring in prime number theory.

Since

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_0^\infty e^{-st}\,\psi(e^t)\,dt,\quad (s > 1)$$

it seems reasonable to expect that

$$-\frac{\zeta'(s)}{\zeta(s)} > s\int_0^\infty e^{-st}\,e^t\,dt = \frac{s}{s-1}$$

for at least some values of $s > 1$.

However, in this note it will be shown that, for example,

$$-\frac{\zeta'(s)}{\zeta(s)} < \frac{1}{s-1},\ \forall\, s > 0.$$

KEY WORDS & PHRASES: *Riemann zeta function, inequalities.*

0. INTRODUCTION

The subject of this note was motivated by the following observation: If, as usual, we define the number theoretical functions Λ and ψ by (cf. [2], p.252 and p.344)

(1) $$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^m \\ 0 & \text{if} \quad n \neq p^m \end{cases}$$

and

(2) $$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

then (cf. [1], p.188)

(3) $$-\frac{\zeta'(s)}{\zeta(s)} = s \int_0^\infty e^{-st}\psi(e^t)dt, \qquad (s > 1).$$

It is well known that there exist arbitrarily large values of x for which

(4) $$\psi(x) > x$$

or, more precisely (cf. [4]), that there exists a positive constant c_0 such that

(5) $$\psi(n) > n + c_0\sqrt{n} \log \log \log n$$

for infinitely many positive integers n. Since, as will be shown later on, the (entire) function $(s-1)\zeta(s)$ is increasing on the positive real axis with a positive derivative

(6) $$\frac{d}{ds}(s-1)\zeta(s) = (s-1)\zeta'(s) + \zeta(s) > 0, \qquad (s > 0)$$

we certainly have

(7) $$-\frac{\zeta'(s)}{\zeta(s)} < \frac{1}{s-1} \left(= \int_0^\infty e^{-st}e^t dt \right), \qquad (s > 1).$$

From (3) and (7) it is easily seen that

(8) $$\int_0^\infty e^{-st}\psi(e^t)dt < \int_0^\infty e^{-st}e^t dt, \qquad (s > 1)$$

which is quite an intriguing inequality in view of the fact that for certain arbitrarily large values of t, the function $\psi(e^t)$ is considerably larger than e^t.

1. For later use we first prove the following

LEMMA.

(9) $$\zeta(s) - \frac{1}{s-1} > \frac{1}{2}, \qquad (s > 0).$$

PROOF. For Re $s = \sigma > 1$ we have

(10) $$\frac{1}{s-1} = \int_1^\infty x^{-s}dx$$

so that

(11) $$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \frac{1}{s-1} - \int_1^\infty x^{-s}dx + \sum_{n=1}^\infty n^{-s} =$$

$$= \frac{1}{s-1} + \sum_{n=1}^\infty \left\{n^{-s} - \int_n^{n+1} x^{-s}dx\right\}, \qquad (\sigma > 1).$$

Since $\zeta(s) - (s-1)^{-1}$ is an entire function and since the last series in (11) represents an analytic function for $\sigma > 0$ (the proof of which is easily supplied) we have by analytic continuation

(12) $$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^\infty \left\{n^{-s} - \int_n^{n+1} x^{-s}dx\right\}, \qquad (\sigma > 0).$$

Now observe that for any fixed $s > 0$ the function x^{-s} is convex on $\mathbb{R}^+$ so that for all $n \in \mathbb{N}$

(13) $$\int_n^{n+1} x^{-s}dx < \frac{1}{2}\left\{n^{-s} + (n+1)^{-s}\right\}.$$

From (12) and (13) it follows that

$$(14)\qquad \zeta(s) - \frac{1}{s-1} > \frac{1}{2}\sum_{n=1}^{\infty}\left\{n^{-s} - (n+1)^{-s}\right\} = \frac{1}{2}, \qquad (s > 0)$$

proving the lemma.

Next we prove

<u>THEOREM 1</u>.

$$(15)\qquad \frac{d}{ds}(s-1)\zeta(s) = (s-1)\zeta'(s) + \zeta(s) > 0, \qquad (s > 0).$$

<u>PROOF</u>. For $s > 0$ we have (cf. [5], p.14)

$$(16)\qquad \zeta(s) = \frac{1}{s-1} + 1 - s\int_1^{\infty}\frac{x-[x]}{x^{s+1}}\,dx.$$

Writing $p(x) = x - [x]$ we thus have

$$(17)\qquad \zeta(s) = \frac{1}{s-1} + 1 - s\int_1^{\infty}\frac{p(x)}{x^{s+1}}\,dx \qquad (s > 0)$$

and

$$(18)\qquad \int_1^{\infty}\frac{p(x)}{x^{s+1}}\,dx = \frac{1}{s} - \zeta(s) + \frac{1}{s-1} + 1\ , \qquad (s > 0),$$

so that (note that $p(x) > 0$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$)

$$(19)\qquad \zeta'(s) = -\frac{1}{(s-1)^2} + s\int_1^{\infty}\frac{p(x)\log x}{x^{s+1}}\,dx - \int_1^{\infty}\frac{p(x)}{x^{s+1}}\,dx >$$

$$> -\frac{1}{(s-1)^2} - \int_1^{\infty}\frac{p(x)}{x^{s+1}}\,dx = -\frac{1}{(s-1)^2} + \frac{1}{s}\left\{\zeta(s) - \frac{1}{s-1} - 1\right\} =$$

$$= -\frac{1}{(s-1)^2} + \frac{\zeta(s)}{s} - \frac{1}{s-1} \qquad (s > 0).$$

Taking $s > 1$ it follows that

$$(20)\qquad (s-1)\zeta'(s) > -\frac{1}{s-1} + \frac{s-1}{s}\zeta(s) - 1 = -\frac{s}{s-1} + \frac{s-1}{s}\zeta(s)$$

so that

$$(21) \qquad (s-1)\zeta'(s) + \zeta(s) > -\frac{s}{s-1} + \left(\frac{s-1}{s} + 1\right)\zeta(s) =$$

$$= -\frac{s}{s-1} + \frac{2s-1}{s}\zeta(s).$$

Since

$$\frac{2s-1}{s} > 0, \qquad (s > \tfrac{1}{2})$$

and

$$\zeta(s) - \frac{1}{s-1} > \frac{1}{2}, \qquad (s > 0),$$

it follows from (21) that

$$(22) \qquad (s-1)\zeta'(s) + \zeta(s) > -\frac{s}{s-1} + \frac{2s-1}{s}\left(\frac{1}{s-1} + \frac{1}{2}\right) = \frac{1}{2s}, \qquad (s > 1),$$

from which it is clear that (15) holds true for $s > 1$ and by continuity also for $s = 1$. Actually for $s = 1$ the left hand side of (22) takes the value γ = Euler's constant as may be seen from the Laurent expansion of $\zeta(s)$ about the point $s = 1$ (cf. [5], p. 16)

$$(23) \qquad \zeta(s) = \frac{1}{s-1} + \gamma + a_1(s-1) + \dots .$$

In order to show that (15) also holds for $0 < s < 1$ we observe that (cf. [5], p. 14) for $s > 0$

$$(24) \qquad \zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{x - [x] - \frac{1}{2}}{x^{s+1}}$$

$$= \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{P_1(x)}{x^{s+1}}\,dx =$$

$$= \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{1}{x^{s+1}}\,d\left(P_2(x) - \frac{1}{12}\right) =$$

$$= \frac{1}{s-1} + \frac{1}{2} - s\left\{\frac{P_2(x) - \frac{1}{12}}{x^{s+1}}\Bigg|_1^\infty - \int_1^\infty \left\{P_2(x) - \frac{1}{12}\right\} dx^{-s-1}\right\} =$$

$$= \frac{1}{s-1} + \frac{1}{2} + s(s+1) \int_1^\infty \frac{\frac{1}{12} - P_2(x)}{x^{s+2}} \, dx$$

where (cf. [3], pp.523-525)

$$(25) \qquad P_1(x) = x - [x] - \frac{1}{2}$$

and $P_2(x)$ is the continuous periodic function defined by

$$(26) \qquad P_2'(x) = P_1(x), \qquad (x \in \mathbb{R} \setminus \mathbb{Z})$$

and

$$(27) \qquad \int_0^1 P_2(x)dx = 0.$$

Since (cf. [3], pp.536-537)

$$(28) \qquad q(x) \stackrel{\text{def}}{=} \frac{1}{12} - P_2(x) > 0 \qquad \text{for all} \qquad x \in \mathbb{R} \setminus \mathbb{Z}$$

it follows from (24) that

$$(29) \qquad \zeta'(s) = -\frac{1}{(s-1)^2} + (2s+1) \int_1^\infty \frac{q(x)}{x^{s+2}} \, dx - s(s+1) \int_1^\infty \frac{q(x) \log x}{x^{s+2}} <$$

$$< -\frac{1}{(s-1)^2} + (2s+1) \int_1^\infty \frac{q(x)}{x^{s+2}} \, dx <$$

$$< -\frac{1}{(s-1)^2} + (2s+1) \int_1^\infty \frac{q(x)}{x^2} \, dx, \qquad (s > 0).$$

From (24) we also obtain

$$(30) \qquad \int_1^\infty \frac{q(x)}{x^2} \, dx = \lim_{s \to 0} \int_1^\infty \frac{q(x)}{x^{s+2}} \, dx = \lim_{s \to 0} \frac{1}{s(s+1)} \left\{ \zeta(s) - \frac{1}{s-1} - \frac{1}{2} \right\}.$$

Since

$$(31) \qquad \zeta(0) = -\frac{1}{2}$$

by (24), and (cf. [5], p.20)

$$(32) \qquad \zeta'(0) = -\frac{1}{2} \log 2\pi$$

it follows from (30) that

$$(33) \qquad \int_1^\infty \frac{q(x)}{x^2}\,dx = \left\{\zeta'(x) + \frac{1}{(s-1)^2}\right\}_{s=0} =$$

$$= \zeta'(0) + 1 = 1 - \frac{1}{2}\log 2\pi = 0.08106\ldots .$$

In combination with (29) it follows that

$$(34) \qquad \zeta'(s) < -\frac{1}{(s-1)^2} + (2s+1) * 0.0811,$$

so that for $0 < s < 1$

$$(35) \qquad (s-1)\zeta'(s) > -\frac{1}{s-1} + (s-1)(2s+1) * 0.0811.$$

Hence, for $0 < s < 1$ we have

$$(36) \qquad (s-1)\zeta'(s) + \zeta(s) > \zeta(s) - \frac{1}{s-1} + (s-1)(2s+1) * 0.0811 >$$

$$> \frac{1}{2} - (1+s-2s^2) * 0.0811 \geqq \frac{1}{2} - \frac{9}{8} * 0.0811 > 0.4$$

so that (15) also holds true for $0 < s < 1$, completing the proof. □

COROLLARY 1.1. *The function* $(s-1)\zeta(s)$ *is increasing on the positive real axis.*

COROLLARY 1.2. *Using the same notation as in the introduction we have*

$$(37) \qquad \int_0^\infty e^{-st}\psi(e^t)dt < \frac{1}{s(s-1)} \left(< \frac{1}{s-1}\right), \qquad (s > 1).$$

2. THEOREM 2.

$$(38) \qquad \zeta'(s) + \frac{1}{(s-1)^2} > 0, \qquad (s > 0).$$

PROOF. From (see (24))

$$(39) \qquad \zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{P_1(x)}{x^{s+1}}\,dx, \qquad (s > 0)$$

we obtain by partial integration that

$$(40)\qquad \zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - s(s+1)(s+2)\int_1^\infty \frac{P_3(x)}{x^{s+3}}\,dx$$

where $P_3(x)$ is as in [3], p.524-525.

Since (cf. [3], p.527)

$$(41)\qquad |P_3(x)| \leqq \frac{4}{(2\pi)^3} < \frac{4}{6^3} < \frac{1}{50}$$

we have

$$(42)\qquad P_3(x) + \frac{1}{50} > 0 \qquad \text{for all } x \in \mathbb{R}$$

so that by (40) for $s > 0$

$$(43)\qquad \int_1^\infty \frac{P_3(x) + \frac{1}{50}}{x^{s+3}}\,dx = \frac{\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} + \frac{1}{50}s(s+1) - \zeta(s)}{s(s+1)(s+2)}.$$

Now observe that by (42) the left-hand side of (43) is decreasing in s for $s > 0$ so that for $s > 0$

$$(44)\qquad \frac{\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} + \frac{1}{50}s(s+1) - \zeta(s)}{s(s+1)(s+2)} < \lim_{s\downarrow 0}\{\text{RHS of (43)}\} =$$

$$= \frac{1}{2}\left\{-\frac{1}{(s-1)^2} + \frac{1}{12} + \frac{2s+1}{50} - \zeta'(s)\right\}_{s=0} =$$

$$= \frac{1}{2}\left\{-1 + \frac{1}{12} + \frac{1}{50} + \frac{1}{2}\log 2\pi\right\} < 0.01114.$$

By the same argument the derivative of the RHS of (43) is negative for $s > 0$ which is equivalent to

$$(45)\qquad \zeta'(s) + \frac{1}{(s-1)^2} - \frac{1}{12} - \frac{2s+1}{50} >$$

$$> -(3s^2+6s+2)\,\frac{\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} + \frac{s(s+1)}{50} - \zeta(s)}{s(s+1)(s+2}$$

In view of (44) it follows that

(46) $$\zeta'(s) + \frac{1}{(s-1)^2} > \frac{1}{12} + \frac{2s+1}{50} - (3s^2+6s-2) * 0.01114 =$$

$$= -0.03342s^2 - 0.02684s + 0.08105.$$

Since this polynomial is decreasing on the positive real axis and at $s = 1$ it takes the value

(47) $$-0.03342 - 0.02684 + 0.08105 > 0$$

it follows that

(48) $$\zeta'(s) + \frac{1}{(s-1)^2} > 0, \qquad (0 < s \leqq 1).$$

It is easily verified that for $s > 0$

(49) $$\int_1^\infty \frac{P_4(x) + \frac{1}{720}}{x^{s+4}}\,dx = \frac{\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \zeta(s)}{s(s+1)(s+2)(s+3)} .$$

Since

(50) $$P_4(x) + \frac{1}{720} > 0 \qquad \text{for all} \qquad x \in \mathbb{R} \setminus \mathbb{Z}$$

the left-hand side of (49) is decreasing in s so that

(51) $$\frac{\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \zeta(s)}{s(s+1)(s+2)(s+3)} < \lim_{s\to 1} \{\text{RHS of } (49)\} =$$

$$= \frac{\frac{1}{2} + \frac{1}{12} - \gamma}{4!} < 0.00026, \quad (s > 1).$$

Since the derivative of the RHS of (49) is negative we also have after some calculations

(52) $$\zeta'(s) + \frac{1}{(s-1)^2} > \frac{1}{12} - \left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}\right\} \left\{\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \zeta(s)\right\} >$$

$$> \frac{1}{12} - s(s+1)(s+2)(s+3) \left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}\right\} *$$

$$* 0.00026.$$

This last polynomial is decreasing on the positive real axis and at $s = 2$ it takes the value

$$(53)\qquad \frac{1}{12} - 120\left\{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right\} * 0.00026 = \frac{1}{12} - 154 * 0.00026 > 0.04$$

so that also

$$(54)\qquad \zeta'(s) + \frac{1}{(s-1)^2} > 0, \qquad (1 < s \leqq 2).$$

Similarly as before we have for $s > 2$

$$(55)\qquad \frac{\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \zeta(s)}{s(s+1)(s+2)(s+3)} < \frac{1 + \frac{1}{2} + \frac{1}{6} - \zeta(2)}{120} < 0.000187$$

and (since the derivative of the RHS of (49) is negative)

$$(56)\qquad \zeta'(s) + \frac{1}{(s-1)^2} > \frac{1}{12} - s(s+1)(s+2)(s+3)\left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}\right\} * \\ * 0.000187.$$

This polynomial is decreasing on $\mathbb{R}^+$ and at $s = 3$ it takes the value

$$(57)\qquad 0.0833\ldots - 0.0639\ldots > 0$$

from which it follows that (15) also holds true for $2 < s \leqq 3$.

The following finishing touch of the proof is due to E. WATTEL. It is easily verified that

$$(58)\qquad \frac{\log 2}{2^s} < \frac{1}{2(s-1)^2}, \qquad (s > 3)$$

and

$$(59)\qquad \frac{\log 3}{3^s} < \frac{1}{6(s-1)^2}, \qquad (s > 3).$$

Since for any fixed $s > 3$ the function $\frac{\log x}{x^s}$ is convex on the inteval $[3,\infty)$ we have

$$(60)\qquad \sum_{n=4}^{\infty} \frac{\log n}{n^s} < \int_{3\frac{1}{2}}^{\infty} \frac{\log x}{x^s}\,dx = \frac{1 + (s-1)\log(3.5)}{(s-1)^2(3.5)^{s-1}}$$

and from this it is easily seen that

(61) $$\sum_{n=4}^{\infty} \frac{\log n}{n^s} < \frac{1}{3(s-1)^2}, \qquad (s > 3).$$

From (58), (59) and (61) it is clear that

(62) $$-\zeta'(s) = \frac{\log 2}{2^s} + \frac{\log 3}{3^s} + \sum_{n=4}^{\infty} \frac{\log n}{n^s} < \frac{1}{(s-1)^2}, \quad (s > 3)$$

so that (15) also holds for $s > 3$, completing the proof of theorem 2. □

COROLLARY 2.1. The entire function $\zeta(s) - \frac{1}{s-1}$ is increasing on $\mathbb{R}^+$.

COROLLARY 2.2.

(63) $$\zeta(s) - \frac{1}{s-1} > \gamma, \qquad (s > 1)$$

This follows from corollary 2.1 and (23).

COROLLARY 2.3.

(64) $$(s-1)\zeta'(s) + \zeta(s) > \gamma, \qquad (s > 1)$$

Indeed, it follows from (15) that

(65) $$(s-1)\zeta'(s) > -\frac{1}{s-1}, \qquad (s > 1)$$

so that

(66) $$(s-1)\zeta'(s) + \zeta(s) > \zeta(s) - \frac{1}{s-1}, \qquad (s > 1)$$

and (64) follows from (63).

3. In order to show that the technique illustrated above also applies to alternating series we prove in this section

THEOREM 3.

(63) $$\eta'(s) > 0, \qquad (s > 0)$$

where

(64) $$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \qquad (\mathrm{Re}\ s = \sigma > 0).$$

<u>PROOF</u>. Define the function $\lambda_1 : \mathbb{R} \to \mathbb{R}$ by

(65) $$\lambda_1(x) = \begin{cases} -\frac{1}{2} & \text{if} \quad 2m < x < 2m+1, \qquad m \in \mathbb{Z} \\ 0 & \text{if} \quad x \in \mathbb{Z} \\ \frac{1}{2} & \text{if} \quad 2m-1 < x < 2m, \qquad m \in \mathbb{Z} \end{cases}$$

Then for $s > 0$ we have

(66) $$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \int_{1-}^{\infty} x^{-s} d\lambda_1(x) = \frac{\lambda_1(x)}{x^s}\Big|_{1-}^{\infty} - \int_1^{\infty} \lambda_1(x) dx^{-s} =$$

$$= \frac{1}{2} + s \int_1^{\infty} \frac{_1(x)}{x^{s+1}} dx.$$

Define

(67) $$\lambda_2(x) = \int_1^x \lambda_1(t) dt, \qquad (x \in \mathbb{R}).$$

Then

(68) $$\eta(s) = \frac{1}{2} + s \int_1^{\infty} \frac{1}{x^{s+1}} d\lambda_2(x) = \frac{1}{2} + s \left\{ \frac{\lambda_2(x)}{x^{s+1}}\Big|_1^{\infty} - \int_1^{\infty} \lambda_2(x) dx^{-s-1} \right\} =$$

$$= \frac{1}{2} + s(s+1) \int_1^{\infty} \frac{\lambda_2(x)}{x^{s+2}} dx = \frac{1}{2} + \frac{s}{2} - s(s+1) \int_1^{\infty} \frac{\frac{1}{2} - \lambda_2(x)}{x^{s+1}} dx,$$

$$(s > 0)$$

so that

(69) $$\eta(s) = \frac{1}{2} + \frac{s}{2} - s(s+1) \int_1^{\infty} \frac{p(x)}{x^{s+1}} dx, \qquad (s > 0)$$

where

(70) $$p(x) = \frac{1}{2} - \lambda_2(x).$$

It is easily seen that $p(x)$ is continuous and that

(71) $$p(x) > 0, \qquad (x \neq \tfrac{1}{2} + m;\ m \notin \mathbb{Z}),$$

from which it follows that

$$(72)\qquad \eta'(s) = \frac{1}{2} + s(s+1)\int_1^\infty \frac{p(x)\log x}{x^{s+2}}\,dx - (2s+1)\int_1^\infty \frac{p(x)}{x^{s+2}}\,dx >$$

$$> \frac{1}{2} - (2s+1)\int_1^\infty \frac{p(x)}{x^2}\,dx, \qquad (s > 0).$$

Since

$$(73)\qquad \eta(s) = (1-2^{1-s})\zeta(s), \qquad (\forall x \in \mathbb{C})$$

it follows by logarithmic differentiation that

$$(74)\qquad \frac{\eta'(s)}{\eta(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{2^{1-s}\log 2}{1-2^{1-s}}\ .$$

From (73) and (31) it follows that

$$(75)\qquad \eta(0) = \frac{1}{2}$$

so that, using (32), we obtain

$$(76)\qquad \eta'(0) = \frac{1}{2}\left\{\log 2\pi + \frac{2\log 2}{1-2}\right\} = \frac{1}{2}\log\frac{\pi}{2}\ .$$

Hence

$$(77)\qquad \int_1^\infty \frac{p(x)}{x^2}\,dx = \lim_{s\to 0}\int_1^\infty \frac{p(x)}{x^{s+2}}\,dx = \lim_{s\to 0}\frac{-\eta(s)+\frac{1}{2}+\frac{s}{2}}{s(s+1)} =$$

$$= \left\{-\eta'(s) + \frac{1}{2}\right\}_{s=0} = -\eta'(0) + \frac{1}{2} = \frac{1}{2} - \frac{1}{2}\log\frac{\pi}{2} < 0.275.$$

In view of (72) it follows that

$$(78)\qquad \eta'(s) > \frac{1}{2} - (2s+1) * 0.275, \qquad (s > 0).$$

The RHS of (78) is decreasing on $\mathbb{R}^+$ and is still positive at $s = 0.41$ so that

$$(79)\qquad \eta'(s) > 0, \qquad (0 < s \leqq 0.41).$$

We may proceed somewhat more accurately as follows. From (68) we obtain

(80) $$\eta(s) = \frac{1}{2} + s(s+1) \int_1^\infty \frac{\lambda_2(x)}{x^{s+2}}\, dx, \qquad (s > 0)$$

so that

(81) $$\eta(s) = \frac{1}{2} + \frac{s}{4} + s(s+1) \int_1^\infty \frac{\lambda_2(x) - \frac{1}{2}}{x^{s+2}}\, dx, \qquad (s > 0).$$

Defining

(82) $$\lambda_3(x) = \int_1^x \left(\lambda_2(t) - \frac{1}{4}\right) dt$$

we obtain from (81) that

(83) $$\eta(s) = \frac{1}{2} + \frac{s}{4} + s(s+1)(s+2) \int_1^\infty \frac{\lambda_3(x)}{x^{s+3}}\, dx, \qquad (s > 0).$$

Observing that

$$\lambda_3(x) \leqq 0, \qquad (\forall x \in \mathbb{R})$$

it follows from (83) that

(85) $$\eta'(s) \geqq \frac{1}{4} + x(s+1)(s+2) \left\{\frac{1}{2} + \frac{1}{s+1} + \frac{1}{s+2}\right\} \int_1^\infty \frac{\lambda_3(x)}{x^{s+3}}\, dx \geqq$$
$$\geqq \frac{1}{4} + s(s+1)(s+2) \left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}\right\} \int_1^\infty \frac{\lambda_3(x)}{x^3}\, dx.$$

Similarly as before one easily finds that

(86) $$\int_1^\infty \frac{\lambda_3(x)}{x^3} dx = \lim_{s\to 0} \int_1^\infty \frac{\lambda_3(x)}{x^{s+3}}\, dx = \lim_{s\to 0} \frac{\eta(s) - \frac{1}{2} - \frac{s}{4}}{s(s+1)(s+2)} = \frac{1}{2}\left\{\eta'(0) - \frac{1}{4}\right\} =$$
$$= \frac{1}{2}\left\{\frac{1}{2}\log\frac{\pi}{2} - \frac{1}{4}\right\} > -\, 0.01211$$

so that for $s > 0$

(87) $$\eta'(s) > \frac{1}{4} - s(s+1)(s+2)\left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}\right\} * 0.01211.$$

The RHS of (87) is decreasing on $\mathbb{R}^+$ and assumes the value $0.25 - 11 * 0.01211 >$ > 0.116 at $s = 1$ so that

(88) $$\eta'(s) > 0, \qquad (0 < s \leqq 1).$$

If $s > 1$ then it follows from (85) that

$$(89)\qquad \eta'(s) \geqq \frac{1}{4} + s(s+1)(s+2)\left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}\right\}\int_1^\infty \frac{\lambda_3(x)}{x^{s+3}}\,dx \geqq$$

$$\geqq \frac{1}{4} + s(s+1)(s+2)\left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}\right\}\int_1^\infty \frac{\lambda_3(x)}{x^{4}}\,dx$$

so that in view of

$$(90)\qquad \int_1^\infty \frac{\lambda_3(x)}{x^4}\,dx = \frac{\eta(1) - \frac{1}{2} - \frac{1}{4}}{6} = \frac{\log 2 - 0.5 - 0.25}{6} > -\,0.0095$$

we find that

$$(91)\qquad \eta'(s) > \frac{1}{4} - s(s+1)(s+2)\left\{\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}\right\} * 0.0095.$$

The RHS of (91) is decreasing on $\mathbb{R}^+$ and assumes the value $0.25 - 26 * 0.0095 = 0.003$ at $s = 2$ so that

$$(92)\qquad \eta'(s) > 0, \qquad\qquad (1 < s \leqq 2).$$

In order to complete the proof we observe that for any fixed $s > 2$ the function $\frac{\log x}{x^s}$ is decreasing on $[2,\infty)$ so that

$$(93)\qquad \eta'(s) = \left(\frac{\log 2}{2^s} - \frac{\log 3}{3^s}\right) + \left(\frac{\log 4}{4^s} - \frac{\log 5}{5^s}\right) + \ldots \quad 0,$$

proving the theorem. □

COROLLARY 3.1. $\eta(s)$ *is increasing on* $\mathbb{R}^+$.

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