Asymptotic Expansions Connected With Truncated Series Of Exponential And Bessel Type.

by

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Asymptotic expansions connected with truncated series of exponential and Bessel type.

1. Introduction.

A conjecture of Ramanujan [1] was the starting-point of some papers: Watson[2] and Szegö[3] treated the function y(n), defined by

$$\frac{e^n}{2} = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \dots + \frac{n^{n-1}}{(n-1)!} + y(n)\,\frac{n^n}{n!} \quad , \qquad (1.1)$$

and they found the asymptotic expansion for y (n):

$$y(n) \sim \frac{1}{3} + \frac{4}{135\,n} + \dots \qquad (1.2)$$

A similar result, connected with e^{-n}, was discovered by Aitken and proved by Copson, that, connected with cos n and sin n is given also.

Theorems of the same kind are given by Furch[6], Mirakyan[7] and Liouville[8].

The intention of this report is: To give in the first place an expansion for the function $\varphi(n,w)$ defined by

$$e^{nw} = 1 + \frac{nw}{1!} + \frac{(nw)^2}{2!} + \frac{(nw)^3}{3!} + \dots + \frac{(nw)^{n-1}}{(n-1)!} + \frac{(nw)^n}{n!}\,\phi(n,w), \qquad (1.3)$$

where w is a complex number. From this expansion, all results, mentioned above, can be derived.

To give in the second place, an expansion for $\varphi_k(n,x)$ defined by

$$\sum_{h=0}^{\infty} \frac{x^h}{h!(h+k)} = \sum_{h=0}^{n} \frac{x^h}{h!(h+k)!} + \frac{x^n}{n!(n+k)!}\,\varphi_k(n,x), \qquad (1.4)$$

where $k \geqslant 0$ and x is a negative number. One may be acquainted with the fact, that the functions in the left-hand side of (1.4) are closely connected with Bessel-functions. Finally an application of this last expansion is given.

2. An integral representation.

The first object is to find an integral representation for $\phi(n,w)$ defined by:

$$1 + \frac{nw}{1!} + \frac{(nw)^2}{2!} \dots + \frac{(nw)^{k-1}}{(n-1)!} + \frac{(nw)^n}{n!}\,\phi(n,w) = e^{nw} \qquad (2.1)$$

It is easily seen that $\sum_{0}^{n} k \frac{(nw)^k}{k!} = \frac{(nw)^n}{n!} \int_0^\infty e^{-u} (1+\frac{u}{nw})^n du.$

Suppose w real:

$$\int_0^\infty e^{-u} (1 + \frac{u}{nw})^n du = \int_0^{-nw} e^{-u}(1+ \frac{u}{nw})^n du + \int_{-nw}^\infty e^{-u}(1+\frac{u}{nw})^n du =$$

$$= - nw \int_0^1 e^{nwu} (1 - u)^n du + \frac{e^{nw} n!}{(nw)^n} .$$

$$\sum_0^n k \frac{(nw)^k}{k!} = e^{nw} - \frac{(nw)^{n+1}}{n!} \int_0^1 e^{nwu} (1-u)^n du$$

$$\sum_0^{n-1} k \frac{(nw)^k}{k!} + \frac{(nw)^n}{n!} \Phi (n,w) = e^{nw}$$

$$\Phi (n,w) = 1 + nw \cdot \int_0^1 e^{nwu} (1 - u)^n du. \qquad (2.2)$$

This integral representation holds by means of analytic continuation for all complex w.

3. Transformation of u.

Next the complex variable

$$t = -wu - \ln (1-u) \qquad (3.1)$$

is substituted in (2.2), yielding:

$$\Phi (n,w) = 1 + nw \int_C e^{-nt} \frac{du}{dt} dt. \qquad (3.2)$$

where C denotes an integration path in the t-plane given by (3.1) when u varies from 0 to 1 along the real axis.

To get an asymptotic expansion for Φ (n,w) one has to expand the integral $I = \int_C e^{-ut} \frac{du}{dt} dt$. (3.3)

This can be done by replacing C by the real positive axis and by using a lemma of Watson[9]: Let F(t) be analytic when $|t| \leqslant a + \delta$, $a > 0, \delta > 0$ save for a branch-point at the origin and let

$$F(t) = \sum_1^\infty m \; a_m t^{m/r^{-1}} \quad \text{when } |t| < a, \; r > 0;$$

also let $|F(t)| < Ke^{br}$, K, b positive numbers independent of t when t positive and $t \geqslant a$.

Then the asymptotic expansion

$$\int_0^\infty e^{-\nu t} F(t)dt \sim \sum_1^\infty a_m \Gamma(\tfrac{m}{r}) \nu^{-m/r}$$

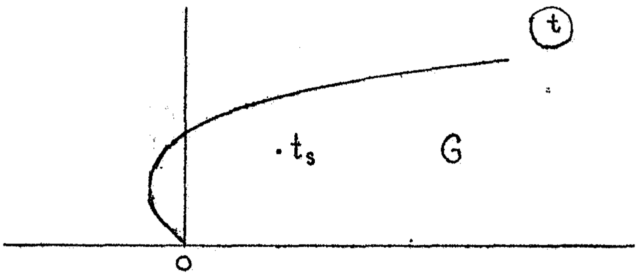
is valid in the sense of Poincare when $|\nu|$ is sufficiently large and $|\arg \nu| \leqslant \pi/2 - \Delta$, arbitrary > 0.

But to replace C by the positive real axis one has to investigate the singularities of $\frac{du}{dt}$. Writing $w = re^{i\varphi}$, one has:

$$\frac{dt}{du} = -w + \frac{1}{1-u} \tag{3.4}$$

The critical points are $u = 1$ and $u_s = 1 - \frac{1}{w}$, resp. $t = \infty$ and $t_s = 1 - w + \ln w$, where log w be defined real for positive w; by cutting the w-plane along the negative axis, so that $-\pi < \arg w \leqslant \pi$, log w is made single valued.

The t-plane is divided by the positive real axis and C in two parts. Now the condition will be derived that the point t_s does not lie between C and positive axis, i.e. in G



$t = -wu - \ln(1-u)$ $\quad t_s = 1 - w + \ln w$

$\text{Im } t = -ur \sin\varphi$ $\quad \text{Im } t_s = -r \sin\varphi + \varphi$

$\text{Re } t = -ur \cos\varphi - \ln(1-u)$ $\quad \text{Re } t_s = 1 - r\cos\varphi + \ln r$.

If there is a point t_s, inside G, it must be possible to find such u, that $\text{Im } t = \text{Im } t_s$, thus

$$u = 1 - \frac{\varphi}{r \sin\varphi}$$

Since $0 \leqslant u \leqslant 1$, must $r \geqslant \frac{\varphi}{\sin\varphi}$.

It is thus proved that for $r < \frac{\varphi}{\sin\varphi}$ there cannot be a singularity inside G. In that case C may be replaced by the positive real axis.

If $\varphi = \pi$, one has $\text{Im } t = 0$ and $\text{Im } t_s = \pi$. In that case C is real and t_s is a complex point.

If $r \geqslant \frac{\varphi}{\sin\varphi}$ there exists $u = 1 - \frac{\varphi}{r \sin\varphi}$, $0 \leqslant u \leqslant 1$,

$$\text{Re } t_s - \text{Re } t = 1 - \frac{\varphi}{\text{tg}\varphi} + \ln \frac{\varphi}{\sin\varphi} \geqslant 0 \text{ for } -\pi < \varphi < \pi.$$

As Im t is a monotone function of u, it is proved that for $r \geqslant \frac{\varphi}{\sin\varphi}$ one singularity lies inside G.

4. Determination of the character of the singularity.

One has from (3.4)

$$\frac{du}{dt} = \frac{(1-u)(1-u_s)}{u-u_s} = \frac{1}{w^2(u-u_s)} - \frac{1}{w} \tag{4.2}$$

Now $\frac{du}{dt}$ has to be expanded in a power series of $t-t_s$:

$$t-t_s = w(1-u)-1 - \ln(1-u)w = \frac{1-u}{1-u_s} - 1 - \ln\frac{1-u}{1-u_s} =$$

$$= - w(u-u_s) - \ln\left[1-w(u-u_s)\right] = -w(u-u_s) + \sum_{1}^{\infty} k \frac{1}{k}(w[u-u_s])^k =$$

$$= \sum_{2}^{\infty} k \frac{1}{k}(w[u-u_s])^k \tag{4.3}$$

The t-transformation has therefore a branchpoint at $t = t_s$.

For the convergence of (4.3) in u = 0 must $|u_s| < |1-u_s|$, or Re u_s = Re$(1 - \frac{1}{w}) = 1 - \frac{1}{r}\cos\varphi < \frac{1}{2}$, which gives the condition $r < 2\cos\varphi$, that is a inner region of a circle with radius one and with origin in point 1.

5. Power-series expansions for $\frac{du}{dt}$.

First the case $r < \frac{\varphi}{\sin\varphi}$.

One has: $\frac{du}{dt} = \frac{1-u}{1-w(1-u)}$.

By means of (3.1) and reversing of series one finds:

$$\frac{du}{dt} = \frac{1}{1-w} - \frac{1}{(1-w)^3}\frac{t}{1!} + \frac{1+2w}{(1-w)^5}\frac{t^2}{2!} - \frac{1+8w+6w^2}{(1-w)^7}\frac{t^3}{3!} +$$

$$+ \frac{1+22w+58w^2+24w^3}{(1-w)^9}\frac{t^4}{4!} - \frac{1+52w+328w^2+444w^3+120w^4}{(1-w)^{11}}\frac{t^5}{5!} \ \cdots \tag{5.2}$$

Now the case $r \geqslant \frac{\varphi}{\sin\varphi}$, $r < 2\cos\varphi$.

In replacing C by the positive real axis one has to make a loop around the branchpoint t_s.

(4.2):

$$\frac{du}{dt} = \frac{1}{w^2(u-u_s)} - \frac{1}{w}$$

From (4.3) $\quad t-t_s = \sum_{\frac{k}{2}}^{\infty} \frac{w^k(u-u_s)^k}{k}$ one has :

$$\frac{1}{u-u_s} = \frac{w}{\sqrt{2}}\,(t-t_s)^{-\frac{1}{2}} + \frac{w}{3} + \frac{w}{6\sqrt{2}}\,(t-t_s)^{\frac{1}{2}} + \frac{4w}{135}\,(t-t_s) + \ldots$$

and

$$\frac{du}{dt} = \frac{1}{w\sqrt{2}}\,(t-t_s)^{-\frac{1}{2}} - \frac{2}{3w} + \frac{1}{6w\sqrt{2}}\,(t-t_s)^{\frac{1}{2}} + \frac{4}{135w}\,(t-t_s) + \ldots =$$

$$= \frac{1}{w}\sum_{-1}^{\infty} c_k (t-t_s)^{k/2} \qquad (5.3)$$

where $(t-t_s)^{\frac{1}{2}}$ is defined as $-\sqrt{t-t_s}$ as long as one does not pass the branchpoint and as $+\sqrt{t-t_s}$ as one has passed the branchpoint.

6. Asymptotic expansion of $\phi(n,w)$.

As the conditions of the Watson-lemma are satisfied, one finds, using (5.2) for the asymptotic expansion of ϕ (n,w):

$$\phi(n,w) \sim \frac{1}{1-w} - \frac{w}{(1-w)^3}\frac{1}{n} + \frac{(1+2w)w}{(1-w)^5}\frac{1}{n^2} - \frac{(1+8w+6w^2)w}{(1-w)^7}\frac{1}{n^3} +$$

$$+ \frac{(1+22w+58w^2+24w^3)w}{(1-w)^9}\frac{1}{n^4} - \frac{(1+52w+328w^2+444w^3+120w^4)w}{(1-w)^{11}}\frac{1}{n^5} + \cdots$$

$$+ \ldots\ldots \qquad (6.1)$$

which holds for $r < \frac{\varphi}{\sin\varphi}$, if $w = r\,e^{i\varphi}$, $-\pi < \varphi \leqslant \pi$.

Now the case $r \geqslant \frac{\varphi}{\sin\varphi}$, $r < 2\cos\varphi$.

One then makes use of expansion (5.3):

$$\phi(n,w) \sim 1 + n\int_0^{-t_s} \sum_{-1}^{\infty} c_k\, e^{-nt}(t-t_s)^{k/2}\,(-)^k\,dt + n\int_{-t_s}^{\infty} \sum_{-1}^{\infty} c_k e^{-nt}(t-t_s)^{k/2}dt \qquad (6.2)$$

$$= 1 + 2\,e^{-nt_s}\sum_{k=-1}^{\infty} \frac{c_{2k+1}\,(k+\frac{1}{2})!}{n^{k+\frac{1}{2}}} + e^{-nt_s}\,n^{-k/2}\sum_{-1}^{\infty}(-)^k\int_{-nt_s}^{\infty} e^{-v}v^{k/2}dv \qquad (6.3)$$

where $t_s = 1 - w + \ln w$.

As the expansion (6.1) becomes bad when w is nearly equal 1, one can better use (5.3). One then has

$$\phi(n,w) \sim 1 + n \sum_{-1}^{\infty} k \int_{0}^{\infty} c_k \, e^{-nt} (t-t_s)^{k/2} \, dt =$$

$$1 + e^{-nt_s} \sum_{-1}^{\infty} \frac{(k/2)!}{n^{k/2}} - e^{-nt_s} \sum_{-1}^{\infty} k \int_{0}^{-nt_s} \frac{c_k}{n^{k/2}} e^{-v} v^{k/2} \, dv. \qquad (6.4)$$

For $w = 1 - \frac{1}{3n}$ an expansion of this kind was obtained by Furch (6).

7. Some special cases.

Taking w = -1 in (6.1) one has:

$$\phi(n,-1) \sim \frac{1}{2} + \frac{1}{8n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{13}{512n^4} - \frac{47}{2048n^5} - \dots \qquad (7.1)$$

which corresponds with Copson's result (4), except for the coefficient of n^{-4} given there as $-\frac{13}{256}$.

One has: $$\cos nw = \sum_{0}^{n-1} \frac{(-)^k (nw)^{2k}}{2k!} + \frac{(-)^n (nw)^{2n}}{2n!} \operatorname{Re} \phi(2n, iw)$$

and $$\sin nw = \sum_{0}^{n-1} \frac{(-)^k (nw)^{2k+1}}{(2k+1)!} + \frac{(-)^n (nw)^{2n+1}}{(2n+1)!} \operatorname{Re} \phi(2n, iw)$$

and for $\operatorname{Re}\phi(2n, iw)$ one finds:

$$\operatorname{Re}\phi(2n,iw) \sim \frac{1}{1+w^2} + \frac{w^2(3-w^2)}{2(1+w^2)^3}\frac{1}{n} - \frac{w^2(7-30w^2+11w^4)}{4(1+w^2)^5}\frac{1}{n^2} +$$

$$+ \frac{w^2(15-245w^2+511w^4-183w^6+6w^8)}{8(1+w^2)^7}\frac{1}{n^3} +$$

$$- \frac{w^2(31-1422w^2+8634w^4-12216w^6+4304w^8-274w^{10})}{16(1+w^2)^9}\frac{1}{n^4} \dots \qquad (7.2)$$

Taking w = 1 one has:

$$\operatorname{Re}\phi(2n,i) \sim \frac{1}{2} + \frac{1}{8n} + \frac{3}{32n^2} + \frac{13}{128n^3} + \frac{59}{512n^4} \dots \qquad (7.3)$$

which corresponds with the result given in (5) except that the coefficient of n^{-4} given there is $-\frac{59}{512}$.

Finally the case $w = 1$; one has $t_s = 0$.

Using (6.2) one has, after inserting the coefficients c_k according to (5.3),

$$\phi(n,1) \sim \sqrt{\frac{\pi}{2}}\, n^{\frac{1}{2}} + \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2}}\, n^{-1/2} + \frac{4}{135} n^{-1} + \frac{1}{288}\sqrt{\frac{\pi}{2}}\, n^{-3/2} \ldots \tag{7.4}$$

Watson [2] and Szego [3] gave an expansion for y defined by

$$\tfrac{1}{2} e^{n} = \sum_{0}^{n-1} \frac{n^k}{k!} + y \frac{n^n}{n!} .$$

Comparing this with $e^n = \sum_{0}^{n-1} \frac{n^k}{k!} + \phi(n,1) \cdot \frac{n^n}{n!}$ one has

$y = \phi(n,1) - \frac{n! e^n}{2n^n}$.

Stirling's expansion gives:

$$\frac{n! e^n}{2n^n} \sim \sqrt{\frac{\pi n}{2}} \left[1 + \frac{1}{12n} + \frac{1}{288 n^2} \ldots \right]$$

Thus:

$$y \sim \frac{1}{3} + \frac{4}{135} n^{-1} \ldots \tag{7.5}$$

and these are the first two terms Watson gave.

8. The approximation connected with $J_k(x)$.

As in the preceding sections one can try to give an analogue expansion for $J_k(x)$ and $I_k(x)$. It appears, however, that it is easier to treat the function

$$I_{\infty,k}(x) = \sum_{h=0}^{\infty} \frac{x^h}{h!(h+k)!} . \tag{8.1}$$

It is shown already in [10] that

$$I_{\infty,k}(-x) = J_k(2\sqrt{x})/(\sqrt{x})^k, \tag{8.2}$$

and

$$I_{\infty,k}(x) = I_k(2\sqrt{x})/(\sqrt{x})^k, \tag{8.2}$$

where $x > 0$.

Putting again

$$I_{n,k}(x) = \sum_{h=0}^{n} \frac{x^h}{h!(h+k)!} , \tag{8.3}$$

one defines $\varphi_k(n,-x)$ by the equation

$$I_{\infty,k}(-x) - I_{nk}(-x) = (-1)^{n+1}\, \varphi_k(n,-x)\, \frac{x^n}{n!(k+n)!} \tag{8.4}$$

and one can derive the following representation(10)

$$\varphi_k(n,-x) = 2^{k+n}(k+n)!\, .\int_0^{2\sqrt{x}} dt . J_{k+n+1}(t) . \left[1 - \frac{t^2}{4x}\right]^n t^{-k-n} . \tag{8.5}$$

Substituting now into (5) the well-known integral-representation for the Besselfunction of the first kind (11)

$$J_{k+n+1}(t) = \frac{2^{-k-n} t^{k+n+1}}{(k+n+\frac{1}{2})!\sqrt{\pi}} \int_0^1 dy . (1-y^2)^{k+n+\frac{1}{2}} . \cos t\, y, \tag{8.6}$$

it results after application of the transformation

$$u = 2\sqrt{x}\; y,$$

that

$$\varphi_k(n,-x) = \frac{(k+n)!}{2(k+n+\frac{1}{2})!\sqrt{\pi x}} \int_0^{2\sqrt{x}} du \int_0^{2\sqrt{x}} dt . t . \cos\left(\frac{tu}{2\sqrt{x}}\right) . \left[1 - \frac{t^2}{4x}\right]^n \left[1 - \frac{u^2}{4x}\right]^{k+n+\frac{1}{2}} \tag{8.7}$$

In order to derive the desired asymptotic formulae one has to expand first

$$\left(1 - \frac{t^2}{4x}\right)^n = \exp\left[n \log\left(1 - \frac{t^2}{4x}\right)\right] =$$

$$= \exp\left(- n \sum_{h=1}^{\infty} \frac{t^{2h}}{h . 4^h . x^h}\right) =$$

$$= \exp\left\{-\frac{t^2 n}{4x}\right\} . \left[1 - \frac{t^4 n}{32x^2} - \frac{t^6 n}{192x^3} + \frac{t^8 n^2}{2048x^4} - \frac{t^8 n}{1024x^4} \cdots \right. \tag{8.8}$$

The expansion for the logarithme converges uniformly in the region $|t| < 2\sqrt{x}$, so the expansion mentioned in the last member of (8.8) will converge uniformly in the same set.

By introducing

$$\nu = k + n + \tfrac{1}{2} ,$$

one sees easily

$$\left[1 - \frac{u^2}{4x}\right]^{\nu} = \exp\left\{-\frac{u^2 \nu}{4x}\right\} . \left[1 - \frac{u^4 \nu}{32x^2} - \frac{u^6 \nu}{192x^3} + \frac{u^8 \nu^2}{2048x^4} - \frac{u^8 \nu}{1024x^4} \cdots \right] \tag{8.9}$$

the series in the right-hand side converges again in a region $|u| < 2\sqrt{x}$.

For convenience let be introduced now

$$f(t,u) = \left[1 - \frac{t^2}{4x}\right]^n \left[1 - \frac{u^2}{4x}\right]^{\nu}. \qquad (8.10)$$

Of course $f(t,u)$ is also dependent of n, ν and x. It is shown already that $f(t,u)$ can be expanded in a series

$$f(t,u) = \exp\left\{-\frac{t^2 n + u^2 \nu}{4x}\right\} \cdot \left[\sum_{\substack{l=0\\h=0}}^{\infty} f_{l,h}\, t^l\, u^h\right] \qquad (8.11)$$

$f_{l,h}$ being suitable coefficients and the series converges in a neighbourhood $|u| < 2\sqrt{x}$, $|t| < 2\sqrt{x}$.

From this it can be proved that

$$\varphi_k(n,-x) = \frac{(k+n)!}{2\sqrt{\pi x}\,(k+n+\frac{1}{2})!} \int_0^{2\sqrt{x}} du \int_0^{2\sqrt{x}} dt.t.\cos\left(\frac{tu}{2\sqrt{x}}\right). f(t,u) =$$

$$= \frac{(k+n)!}{2\sqrt{\pi x}\,(k+n+\frac{1}{2})!} \sum_{\substack{l=0\\h=0}}^{h+l=M} f_{l,h} \int_0^{2\sqrt{x}} du \int_0^{2\sqrt{x}} dt.t^{l+1}\, u^h \cos\left(\frac{tu}{2\sqrt{x}}\right).$$

$$.\exp\left\{-\frac{t^2 n + u^2 \nu}{4x}\right\} + R_{n,k}(-x,M) \qquad (8.12)$$

where $R_{n,k}(-x,M)$ stands for

$$\frac{(k+n)!}{2\sqrt{\pi x}\,(k+n+\frac{1}{2})!} \sum_{\substack{h>0\\l>0\\h+l>M}}^{\infty} f_{l,h} \int_0^{2\sqrt{x}} du \int_0^{2\sqrt{x}} dt.t^{l+1}.u^h.\cos\left(\frac{tu}{2\sqrt{x}}\right).\exp\left\{-\frac{t^2 n + u^2 \nu}{4x}\right\} \qquad (8.13)$$

Let be put the following condition

$$\lim_{n\to\infty} \frac{\sqrt{x}}{n} = r > 0. \qquad (8.14)$$

It may be possible that this condition is not necessary, but it is a sufficient condition. One can put now the upper limits of the double integral equal to infinity. The error, made by doing so, is asymptotically of such an order, that it can be neglected. For

$$\int_0^{\infty} du \int_{2rn}^{\infty} dt.t^{l+1}.u^h.\cos\left(\frac{tu}{2rn}\right).\exp\left\{-\frac{t^2+u^2}{4r^2 n}\right\} +$$

$$+ \int_0^{\infty} dt. \int_{2rn}^{\infty} du.t^{l+1}.u^h.\cos\left(\frac{tu}{2rn}\right).\exp\left\{-\frac{t^2+u^2}{4r^2 n}\right\} = O(e^{-n}).$$

So one has finally

$$\varphi_n(n,-x)=\frac{(k+n)!}{2\sqrt{\pi x}(k+n+\frac{1}{2})!}\sum_{\substack{h=0\\l=0}}^{h+l=M} f_{l,h}\int_0^{\infty}du\int_0^{\infty}dt.t^{l+1}.u^h.\cos(\frac{tu}{2\sqrt{x}}).\exp\left\{-\frac{t^2n+u^2\nu}{4x}\right\}+R_{n,k}(-x,M) \tag{8.15}$$

9. Some calculations.

Now one needs the following integrals:

$$G_{2a}=\int_0^{\infty}du.u^{2a}.\cos(\frac{tu}{2\sqrt{x}}).\exp\ (-\frac{u^2\nu}{4x}), \tag{9.1}$$

and

$$H_{2a+1}=\int_0^{\infty}dt.e^{-pt^2}.t^{2a+1}\ ,\ \text{where } p=(\frac{n}{4x}+\frac{1}{4\nu}). \tag{9.2}$$

These integrals may be calculated from the expressions found in (12).

So one finds:

$$G_0=\sqrt{\frac{\pi x}{\nu}}\ e^{-\frac{t^2}{4\nu}},$$

$$G_4=G_0\frac{x^2}{\nu^4}(12\nu^2-12t^2\nu+t^4),$$

$$G_6=G_0.\frac{x^3}{\nu^6}(120\nu^3-180t^2\nu^2+30t^4\nu-t^6),$$

$$G_8=G_0.\frac{x^4}{\nu^8}(1680\nu^4-3360t^2\nu^3+840t^4\nu^2-56t^6\nu+t^8)$$

$$G_{10}=G_0.\frac{x^5}{\nu^{10}}(30240\nu^5-75600t^2\nu^4+25200t^4\nu^3-2520t^6\nu^2+90t^8\nu-t^{10}),$$

and

$$H_{2a+1}=\frac{a!}{2p^{a+1}}.$$

One also needs an asymptotic expression for $\frac{(k+n)!}{(k+n+\frac{1}{2})!}$, which may be found in Nörlund (13)

$$\frac{(z+\frac{1}{2})!}{z!\sqrt{z}}=\sum_{s=0}^{\infty}\frac{(-1)^s\binom{s-\frac{1}{2}}{s}B_s^{(s+\frac{1}{2})}}{(z+\frac{3}{2})(z+\frac{5}{2})\dots(z+s+\frac{1}{2})},\qquad (z>0). \tag{9.3}$$

From this one derives

$$\frac{z!\sqrt{z+\frac{1}{2}}}{(z+\frac{1}{2})!}=1-\frac{1}{8(z+\frac{1}{2})}+\frac{1}{128(z+\frac{1}{2})^2}+\frac{5}{1024(z+\frac{1}{2})^3}+\dots\ . \tag{9.4}$$

Combining the results mentioned in the formulae (8.8), (8.9), (8.15), (9.1), (9.2) and (9.4) one finds after some more integrate calculations the series:

$$\varphi_k(n,-x) \sim \frac{1}{2\nu p} - \frac{1}{8\nu} - \frac{n}{32\nu x^2 p^3} - \frac{6\nu^2p^2-6\nu p+1}{32\nu^4 p^3}$$

$$+ \frac{1}{128\nu^2} - \frac{n}{64\nu x^3 p^4} - \frac{20\nu^3p^3-30\nu^2p^2+10\nu p-1}{64\nu^6 p^4}$$

$$+ \frac{n(3\nu^2p^2-9\nu p+3)}{256\, x^2\nu^4p^5} + \frac{3n^2}{512x^4p^5\nu} + \frac{210\nu^4p^4-420\nu^3p^3+210\nu^2p^2-42p+3}{512\nu^7 p^5} \tag{9.5}$$

The special case $x = -n^2$ gives the theorem: The function $I_{\infty,k}(x)$ can be written for negative x in the form

$$I_{\infty,k}(x) = \sum_{h=0}^{n} \frac{x^h}{h!(h+k)!} + \frac{x^n}{n!(n+k)!}\,\varphi_k(n,-n^2)$$

where $\varphi_k(-n^2)$ possesses the asymptotic expansion:

$$\varphi_k(n,-n^2) \sim \frac{1}{2} - \frac{k+1}{4n} + \frac{k^2-1}{8n^2} + \dots .$$

10. <u>The approximation connected with $I_k(x)$</u>.

The way given by section 8 can also be used to obtain the integral-expression for $\varphi_k(n,+x)$, $x > 0$

$$\varphi_k(n,x) = \frac{(k+n)!}{2(k+n+\frac{1}{2})!\sqrt{\pi x}} \int_0^{2\sqrt{x}} du \int_0^{2\sqrt{x}} dt.t.\cosh\left(\frac{tu}{2\sqrt{x}}\right).\left(1-\frac{t^2}{4x}\right)^n\left(1-\frac{u^2}{4x}\right)^{k+n+\frac{1}{2}} \tag{10.1}$$

But now the trouble begins, especially in the case $x = n^2$ one has

$$\varphi_k(n,n^2) = \frac{(k+n)!}{2(k+n+\frac{1}{2})!\sqrt{\pi}\,n} \int_0^{2n} du \int_0^{2n} dt.t.\cosh\left(\frac{tu}{2n}\right).\left(1-\frac{t^2}{4n^2}\right)^n.\left(1-\frac{u^2}{4n^2}\right)^{k+n+\frac{1}{2}}, \tag{10.2}$$

and one sees that replacing $\left(1-\frac{t^2}{4n^2}\right)^n$ and $\left(1-\frac{u^2}{4n^2}\right)^{k+n+\frac{1}{2}}$ by the expansions (8.8) and (8.9) yields a false result. Replacing the upper limits of the integrals by infinity is not possible.

It must be possible, however, to write

$$\frac{1}{2} I_{\infty,k}(n^2) - I_{n,k}(n^2) = \frac{n^{2n}}{n!(n+k)!}\cdot\varphi_k(n,n^2),$$

with

$$\varphi_k(n,n^2) = \sum_{h=0}^{\infty} a_h\, n^{-h}.$$

The coefficients a_h are functions of k, and for k = 0 or 1 the values of a_0 are numerically found resp. $+\frac{5}{6}$ and $-\frac{2}{3}$.

11. An application.

The expansion (9.5) can be used to estimate the number of real zeros of the function $I_{n,k}(x)$. As already mentioned in R 173, these zeros are all negative.

It holds now for sufficiently large values of n and x

$$I_{n,k}(-x) = I_{\infty,k}(-x) + (-1)^n \varphi_k(n,-x)\, \frac{x^n}{n!\,(k+n)!} .$$

So, if x is a zero of $I_{n,k}(x)$, one has

$$(-1)^{n+1} \varphi_k(n,-x) . \frac{x^n}{n!\,(k+n)!} = I_{\infty,k}(-x) = J_k(2\sqrt{x})/(\sqrt{x})^k ,$$

or, by using only the first terms of the asymptotic representation for $J_k(2\sqrt{x})$ and of the expansion (9.5) one has

$$\frac{x^{n+1}}{n!\,(k+n)!\left\{(k+n+\frac{1}{2})n+x\right\}} \sim \frac{1}{2\sqrt{\pi}\, x^{k/2+1/4}} .$$

Introducing a number r by $x = -r\,n^2$, one gets in a manner similar to that used in R 173 an expansion for r. So it appears that $r \sim \frac{1}{e^2}$ is that value of r above which there are no zeros of $I_{n,k}(x)$ possible.

To determine the number of zeros lying in the interval $-r\,n^2 < x < 0$, one can use Schafheitlin's result (14); the number of zeros is equal to $\frac{n^2}{\pi e^2} + O(n)$.

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