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Homogeneous Behaviors*

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Abstract. Recently a smooth compactification of the space of linear systems with n states, m inputs, and p outputs has been discovered. In this paper we obtain a concrete interpretation of this compactification as a space of discrete-time behaviors. We use both homogeneous polynomial representations and generalized first-order representations, and provide a realization theory to link these to each other.

Key words. Linear behavior, Compactification of systems, Shift spaces, Grothendieck quotient scheme.

1. Introduction

An analogy that has been instrumental in the development of the "behavioral" theory of linear systems [W1]–[W3] is that between linear behaviors and linear subspaces of $\mathbb{R}^n$. In fact a subspace of $\mathbb{R}^n$ can be looked at as a static behavior, and the theory of linear time-invariant behaviors may be viewed as the most direct dynamic generalization of it. The analogy makes it natural for instance to look for "kernel" and "image" representations (corresponding to "AR" and "MA" representations in the terminology of [W3]). It is also suggestive in defining a notion of convergence. Starting from the representation of a subspace of given dimension m as the column space of a matrix of size $n \times m$, which is determined up to right multiplication by nonsingular matrices, a natural notion of convergence for subspaces is obtained by the construction of the quotient topology. The kernel representation may also be used in the same way; fortunately, this leads to the same notion of convergence. In the same manner, a topology can be defined for the the set of linear time-invariant systems of a given state space dimension n and a given number of inputs m. Starting for instance from a minimal first-order representa-

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tion, which is unique up to similarity transformations, a notion of convergence is again provided by the quotient topology. The set of m-dimensional subspaces of $\mathbb{R}^n$, with the indicated topology, is usually denoted by Grass(m, n); we use the symbol Sys(m, p, n) to indicate the set of time-invariant linear systems with m inputs, p outputs, and an n-dimensional state space.

In spite of the analogy between "subspaces" and "systems," there is an important topological difference between Grass(m, n) and Sys(m, p, n). Whereas the set of subspaces is compact, so that every sequence must have a limit point, the set of systems is not compact. To give an example in the very simple case $n = 1$, $m = 0$, $p = 1$, the discrete-time behavior spanned by the sequence $w_a = (1, a^{-1}, a^{-2}, \ldots)$ does not converge to a limit if a tends to zero. (Compare this with the sequence of one-dimensional subspaces spanned by the vector $(1, a^{-1}, \ldots, a^{-n})$; as a tends to zero, this sequence does converge to a limit, namely the subspace spanned by $(0, 0, \ldots, 1)$.) A limit behavior can only be obtained if the space of possible behaviors is enlarged. In our example, a suggestion of how to do this is arrived at as follows. Identify a sequence $w = (w_1, w_2, \ldots)$ with the "one-sided" power series $w(s) := \sum_{i=1}^{\infty} w_i s^{-i}$. In this way w_a is identified with the power series

$$w_a(s) := \sum_{i=1}^{\infty} a^{-i+1} s^{-i} = \frac{a}{as - 1}. \tag{1.1}$$

Given that the meromorphic function $(as - 1)^{-1}$ converges to the constant function -1 uniformly on compact subsets of $\mathbb{C}$ as a tends to zero, a natural candidate for a limit point of the one-dimensional subspaces generated by $w_a(s)$ would be the subspace generated by $w_0(s) = 1$. Note that this subspace is not generated by a power series of the form $w(s) = \sum_{i=1}^{\infty} w_i s^{-i}$. The example suggests that somehow some "generalized systems" should be added which could serve as additional limit points.

It was recently discovered that it is possible to compactify the set Sys(m, p, n) in a "smooth manner" using either kernel representations by homogeneous polynomial matrices [RR1] or first-order representations under weakened minimality conditions [GS1]. A continuous-time interpretation of generalized systems in terms of impulsive-smooth behaviors was given in [GS2], and a discrete-time interpretation on a rather abstract level has been provided in [L]. We still believe that it is of interest to add an interpretation in terms of concrete discrete-time behaviors, since the availability of an interpretation that is as simple and as concrete as possible may contribute to the understanding of the compactified space of systems.

The purpose of this paper is therefore to obtain a behavioral interpretation in discrete time of the systems given by homogeneous higher-order equations as in [RR1] or by generalized first-order equations as in [GS2]. The resulting set of behaviors, which is slightly bigger than the set considered by Willems for instance in [W1], is referred to as the set of "homogeneous behaviors."

The organization of this paper is as follows. The framework that we use is introduced in Section 2. Here we also introduce the various representations of homogeneous behaviors. The relations between these representations is explored in some detail in Section 3. In this section we also describe a very simple realization algorithm for homogeneous autoregressive systems first announced in [RRS]. In Section 4 we obtain the main result of the paper, establishing a one-to-one connection between (i) homogeneous polynomial matrices modulo homogeneous unimodular row transformations, as defined in Definition 3.9 of [RR1], (ii) generalized first-order representations modulo similarity transformations, and (iii) homogeneous behaviors. Conclusions follow in Section 5.

2. Definitions and Preliminaries

Denote by $\mathscr{L}$ the set of all functions from the integers $\mathbb{Z}$ to the reals $\mathbb{R}$ with support bounded on the left; that is, $\mathscr{L}$ is the set of two-sided infinite sequences of real numbers $(\ldots, a_{-1}, a_0, a_1, \ldots)$ for which there exists an N such that $a_i = 0$ for all $i < -N$. Of course, $\mathbb{Z}$ can be thought of as a time axis and $\mathscr{L}$ as a signal space. We write $\mathscr{L}^q$ for the analogous space of sequences with values in $\mathbb{R}^q$; in the terminology of [W1], this is our *universum.*

We use two notations for the elements of $\mathscr{L}^q$. The first one is the sequence notation

$$w = (w_{-N}, \ldots, w_{-1}, w_0 \mid w_1, w_2, \ldots), \tag{2.1}$$

where it is understood that $w_i = 0$ for $i < -N$, and in which we use a vertical bar to indicate the position between the signal values at time points 0 and 1. The second notation is the representation as a formal Laurent series:

$$w(s) = \sum_{i=-N}^{\infty} w_i s^{-i}. \tag{2.2}$$

Such a series may be thought of as consisting of a "polynomial part" $w_+(s) := \sum_{i=-N}^{0} w_i s^{-i}$ and a "strictly proper part" $w_-(s) := \sum_{i=1}^{\infty} w_i s^{-i}$, corresponding to the parts before and after the bar in the sequence notation. The Laurent series is written in powers of s^{-1} rather than in powers of s so that the polynomial part is a polynomial in s rather than in s^{-1}; of course this is just a matter of notation. We write $\mathscr{L}^q_+$ for the set of elements of $\mathscr{L}^q$ that appear as polynomials in the Laurent series notation, and the set of elements whose polynomial part is zero is denoted by $\mathscr{L}^q_-$. We remark that the spaces $\mathscr{L}^q$, $\mathscr{L}^q_+$, $\mathscr{L}^q_-$ have been well studied in the systems literature and we refer, e.g., to [HH].

Elements of $\mathscr{L}$ can be multiplied in the way suggested by the Laurent series notation (2.2). With respect to this multiplication and the usual addition, the set $\mathscr{L}$ is a *field* (compare with [HH]) and we make use of this fact in this paper. Note that the field of rational functions $\mathbb{R}(s)$ can be viewed as a subfield of $\mathscr{L}$ by identifying $f(s) \in \mathbb{R}(s)$ with its Laurent expansion around infinity.

On $\mathscr{L}^q$ we consider four shift operators, namely left and right shifts with and without cancellation. In sequence notation, the four operators are defined by

$$\begin{aligned}
\sigma: &\quad (w_{-N}, \ldots, w_{-1}, w_0 \,|\, w_1, w_2, \ldots) \mapsto (w_{-N}, \ldots, w_0, w_1 \,|\, w_2, w_3, \ldots), \\
\sigma_0: &\quad (w_{-N}, \ldots, w_{-1}, w_0 \,|\, w_1, w_2, \ldots) \mapsto (w_{-N}, \ldots, w_0, 0 \,|\, w_2, w_3, \ldots), \\
\tau: &\quad (w_{-N}, \ldots, w_{-1}, w_0 \,|\, w_1, w_2, \ldots) \mapsto (w_{-N}, \ldots, w_{-2}, w_{-1} \,|\, w_0, w_1, w_2, \ldots), \\
\tau_0: &\quad (w_{-N}, \ldots, w_{-1}, w_0 \,|\, w_1, w_2, \ldots) \mapsto (w_{-N}, \ldots, w_{-2}, w_{-1} \,|\, 0, w_1, w_2, \ldots).
\end{aligned} \tag{2.3}$$

The same operators can be given in Laurent series notation by

$$\begin{aligned}
(\sigma w)(s) &= s w(s), \\
(\sigma_0 w)(s) &= s w(s) - s w_-(s)|_{s=\infty}, \\
(\tau w)(s) &= s^{-1} w(s), \\
(\tau_0 w)(s) &= s^{-1}(w(s) - w_+(0)).
\end{aligned} \tag{2.4}$$

The following relations are easily seen to hold:

$$\sigma\tau = \tau\sigma = \text{identity} \tag{2.5}$$

and, for all $k \geq 0$,

$$\sigma_0^k \tau^k = \sigma^k \tau_0^k = \sigma_0^k \tau_0^k. \tag{2.6}$$

Consider now a homogeneous polynomial row vector of degree v, with q entries. Such a row vector may be written in the form

$$p(s,t) = \sum_{k=0}^{v} p_k s^k t^{v-k},$$

where the p_k are constant row vectors of length q. We want to associate a linear operator on $\mathscr{L}^q$ to this homogeneous vector, which in turn will determine a "behavior" as the set of all elements in $\mathscr{L}^q$ that are mapped to zero by this operator. Two linear operators that may be associated to $p(s,t)$ are the following:

$$p(\sigma_0, \tau) \stackrel{\text{def}}{=} \sigma_0^v \sum_{k=0}^{v} p_k \tau^{v-k} \tag{2.7}$$

and

$$p(\sigma, \tau_0) \stackrel{\text{def}}{=} \tau_0^v \sum_{k=0}^{v} p_k \sigma^k. \tag{2.8}$$

Both operators have the desirable property that, in the scalar case $q = 1$, the dimension of the associated behavior on $\mathscr{L}$ is equal to the degree of the polynomial from which the operator is derived. More is true: the following proposition shows that the behaviors determined by the two operators are in fact equal.

Proposition 2.1. *For any homogeneous polynomial $p(s,t)$, one has*

$$\{w \in \mathscr{L}^q \mid p(\sigma_0,\tau)w = 0\} = \{w \in \mathscr{L}^q \mid p(\sigma,\tau_0)w = 0\}. \tag{2.9}$$

Proof. The statement follows from the identities

$$\sigma_0^\nu \sum_{k=0}^{\nu} p_k \tau^{\nu-k} = \sigma_0^\nu \tau^\nu \sum_{k=0}^{\nu} p_k \sigma^k = \sigma^\nu \tau_0^\nu \sum_{k=0}^{\nu} p_k \sigma^k$$

and from the fact that σ^ν is invertible. ■

Remark 2.2. The proposition shows that from the point of view of behaviors it is immaterial whether $p(\sigma_0,\tau)$ or $p(\sigma,\tau_0)$ is taken as the operator associated to a homogeneous vector polynomial $p(s,t)$. It might be said, though, that the choice $p(\sigma_0,\tau)$ is closer to tradition in the sense that it acts on one-sided sequences (interpreted as sequences in $\mathscr{L}^q$) in the same way as the standard (left shift) operator associated to the dehomogenization $p(s) = p(s,1)$ of $p(s,t)$. In this context the effect of a factor t in $p(s,t)$ may be described as "cancel one more element to the left."

Remark 2.3. The association $p(s,t) \mapsto p(\sigma_0,\tau)$ is linear, but does not respect the multiplicative structure of homogeneous polynomials. For instance, the operator associated to the homogeneous polynomial t^2 is not the square of the operator associated to t.

Remark 2.4. Sequence spaces other than $\mathscr{L}^q$, which can be looked at as the space of bi-infinite sequences with support bounded to the left, might be considered. In particular, the space of *all* bi-infinite sequences may be taken. It seems to be hard, though, to associate to homogeneous polynomials a linear operator on this space in such a way that the dimension of the kernel of this operator is equal to the degree of the homogeneous polynomial that one started with. If the operator $p(\sigma_0,\tau)$ as defined above is associated to $p(s,t)$, then for instance the operator associated to $p(s,t) = s - t$, which is $\sigma_0(1-\tau)$, has a two-dimensional solution space associated to it (spanned by the sequences $(\ldots,1,1 \mid 0,0,\ldots)$ and $(\ldots,0,0 \mid 1,1,\ldots)$). On the other hand, if only sequences with finite support were considered, then the same operator would have a zero-dimensional solution space. Finally, the use of sequences with support bounded to the right would give rise to a theory that is essentially the same as the one developed in this paper.

We say that a polynomial matrix $P(s,t)$ is homogeneous if its rows are homogeneous. To a homogeneous polynomial matrix $P(s,t)$, we associate an operator $P(\sigma_0,\tau)$ by replacing each row $P_i(s,t)$ by the corresponding operator $P_i(\sigma_0,\tau)$. We now define:

Definition 2.5. The *homogeneous behavior* $\mathscr{B}(P)$ associated with a homogeneous polynomial matrix $P(s,t)$ is the set

$$\mathscr{B}(P) = \{w \in \mathscr{L}^q \mid P(\sigma_0, \tau)w = 0\}. \tag{2.10}$$

Example 2.6. In continuation of the example discussed in the Introduction, consider the set of scalar homogeneous polynomials of degree 1, i.e., the set of polynomials $p_{a,b}(s,t) = as + bt$ with $(a,b) \neq (0,0)$. It is easily verified that for $a \neq 0$ the homogeneous behavior $\mathscr{B}(p_{a,b})$ is the one-dimensional space generated by the sequence $(0 \mid 1, -b/a, b^2/a^2, -b^3/a^3, \dots)$, whereas for $a = 0$ the homogeneous behavior associated to $p_{a,b}$ is the space spanned by $(1 \mid 0, 0, \dots)$. Note in particular that $\mathscr{B}(p_{a,b})$ is one-dimensional for *all* values of $(a,b) \neq (0,0)$, since the singularity that occurs at $a = 0$ for nonhomogeneous behaviors parametrized by degree-1 nonhomogeneous polynomials of the form $p(s) = as + 1$ is exactly "filled up."

The above example is a special case of the main result of this paper, which states that there is a one-to-one connection between (i) homogeneous behaviors of degree n with $m+p$ external variables (inputs and outputs), (ii) triples of matrices (F, G, H), where F and G are of size $n \times (n+m)$ and H is of size $(p+m) \times (n+m)$, subject to certain minimality conditions and modulo similarity, as defined in [RR2], and (iii) homogeneous polynomial matrices of size $p \times (p+m)$ with row degrees summing up to n, modulo left multiplication by homogeneous unimodular matrices, a space studied in [RR1].

The dehomogenization $P(s,1)$ of a homogeneous polynomial matrix $P(s,t)$ is written simply as $P(s)$. By identifying polynomials with elements of $\mathscr{L}_+$, we can look at the matrix of polynomials $P(s)$ as a matrix with entries in $\mathscr{L}$, and so (using the multiplicative structure of $\mathscr{L}$) as a mapping from $\mathscr{L}^q$ to $\mathscr{L}^p$ where p is the number of rows of $P(s,t)$. Immediately from the definitions we now have the following characterization of $\mathscr{B}(P)$.

Lemma 2.7. *Let the row degrees of the homogeneous polynomial $P(s,t)$ be $\nu_1, \dots, \nu_p$. An element $w(s) \in \mathscr{L}^q$ belongs to $\mathscr{B}(P)$ if and only if the entries of the p-vector $P(s)w(s)$ are polynomials of degrees at most $\nu_1 - 1, \dots, \nu_p - 1$, respectively.*

In what follows we associate to a homogeneous polynomial matrix $P(s,t)$ of size $p \times q$ and of row degrees $\nu_1, \dots, \nu_p$ in a natural way a vector space of dimension $n := \sum_{i=1}^p \nu_i$. For this, note that the set of p-vectors having the property that the ith component is a homogeneous polynomial of degree $\nu_i - 1$, $i = 1, \dots, p$, has in a natural way the structure of an $\mathbb{R}$-vector space. Obviously the dimension of this space is $\sum_{i=1}^p \nu_i = n$, the McMillan degree of the associated homogeneous polynomial $P(s,t)$. Since this vector space is closely related with the state space we abbreviate it to X_ν. The analogous space of p-vectors whose ith component is a homogeneous polynomial of degree ν_i is denoted by $X_{\nu+1}$. The dimension of this space is $n + p$.

Definition 2.8. A $p \times n$ matrix $X(s,t)$ whose columns form an $\mathbb{R}$-basis of the vector space X_ν is called a *basis matrix* (of size ν).

We define the *canonical basis matrix* $\hat{X}(s,t)$ as the matrix of size $p \times n$ given by

$$\hat{X}(s,t) = \begin{bmatrix} s^{\nu_1-1} & s^{\nu_1-2}t & \dots & t^{\nu_1-1} & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & & 0 & s^{\nu_2-1} & \dots & & & \vdots \\ \vdots & \vdots & & & & & \ddots & & 0 \\ 0 & 0 & \dots & & & & & \dots & t^{\nu_p-1} \end{bmatrix}. \tag{2.11}$$

Remark 2.9. $\hat{X}(s,t)$ is also defined if $\nu_i = 0$ for some index i. In this case $\hat{X}(s,t)$ has a zero row in the ith row. Note also that every basis matrix has a unique description of the form $\hat{X}(s,t)S^{-1}$, where $S \in Gl_n$ is an $n \times n$ invertible matrix. In particular, any two basis matrices are related to each other through a simple Gl_n transformation.

The result stated in the lemma above can now be reformulated as follows. Let $X(s,t)$ be any basis matrix and let $X(s) = X(s,1)$. Then one has that

$$\mathscr{B}(P) = \{w(s) \in \mathscr{L}^q \mid P(s)w(s) \in \operatorname{span}_{\mathbb{R}} X(s)\}. \tag{2.12}$$

It is convenient to use polynomial representations that are of the above form but are not necessarily derived explicitly from a homogeneous polynomial matrix. For any pair of polynomial matrices $(R(s), V(s))$, where $R(s)$ and $V(s)$ have the same number of rows, we can define

$$\mathscr{B}(R,V) = \{w(s) \in \mathscr{L}^q \mid R(s)w(s) \in \operatorname{span}_{\mathbb{R}} V(s)\}. \tag{2.13}$$

Finally, a third representation that we use is the first-order representation. Consider a triple of real matrices (F,G,H) where F and G both have size $n \times (n+m)$ and H has size $q \times (n+m)$. With this triple we associate a behavior as follows:

$$\mathscr{B}(F,G,H) = \{w \in \mathscr{L}^q \mid w = Hz \text{ for some } z \in \mathscr{B}(sG - tF)\}. \tag{2.14}$$

Note that the operator associated to the homogeneous polynomial matrix $sG - tF$ is $\sigma_0(G - \tau F) = \sigma\tau_0(\sigma G - F)$. We may therefore also write the above definition in the form

$$\mathscr{B}(F,G,H) = \{w(s) \in \mathscr{L}^q \mid \exists z \in \mathscr{L}^{n+m},\ x_0 \in \mathbb{R}^n \text{ s.t. } (sG - F)z(s) = x_0 \text{ and } w(s) = Hz(s)\}.$$

We now turn to the relation between homogeneous behaviors and behaviors defined on $\mathbb{Z}_+$ such as for instance in [W1]. The latter are given in "AR representation" as follows. Let $P(s)$ be a polynomial matrix and let S denote the standard left shift that takes $(w_1, w_2, \dots)$ to $(w_2, w_3, \dots)$; then define $\mathscr{B}_-(P) = \{w \mid P(s)w = 0\}$. Identifying the space of one-sided sequences with $\mathscr{L}^q_-$, we get the following simple embedding of the set of "standard" behaviors in the space of homogeneous behaviors.

Proposition 2.10. *Let $P(s,t)$ be a homogeneous polynomial matrix, and let $P(s)$ be its dehomogenization. We have*

$$\mathscr{B}_-(P) = \mathscr{B}(P) \cap \mathscr{L}^q_-. \tag{2.15}$$

Proof. Let w be a sequence in $\mathscr{B}_-(P)$. The fact that $P(S)w = 0$ means that $P(s)w(s)$ is polynomial. Moreover, it is immediate from the multiplication rule that the ith entry of $P(s)w(s)$ is a polynomial of degree at most $\mu_i - 1$, where μ_i denotes the maximum of the degrees of the entries in the ith row of $P(s)$. Since $\mu_i \leq \nu_i$, where ν_i is the degree of the ith row of $P(s,t)$, it follows that $w \in \mathscr{B}(P)$. Conversely, let w be a one-sided (strictly proper) sequence in $\mathscr{B}(P)$. Then we know that $P(s)w(s)$ is polynomial and so $P(S)w = 0$. ∎

To get a similar result for (R, V)-representations, we need to impose a condition involving the space X_R that is defined as follows [F], [KS], [GS2]:

$$X_R = \{g(s) \in \mathscr{L}^p_+ \mid g(s) = R(s)w(s) \text{ for some } w(s) \in \mathscr{L}^q_-\}. \tag{2.16}$$

Using the same reasoning as above, we then obtain the following.

Proposition 2.11. *Let $(R(s), V(s))$ be a pair of polynomial matrices having the same number of rows. If $X_R \subset \operatorname{span}_{\mathbb{R}} V(s)$, then*

$$\mathscr{B}_-(R) = \mathscr{B}(R, V) \cap \mathscr{L}^q_-. \tag{2.17}$$

A subspace of $\mathscr{L}^q$ may be given in image representation as $\operatorname{im} M(s)$ or in kernel representation as $\ker N(s)$, where $M(s)$ and $N(s)$ are matrices over $\mathscr{L}$. If $M(s)$ and $N(s)$ are rational matrices, we may also look at $\operatorname{im} M(s)$ and $\ker N(s)$ as curves, that is, as Grass(m, n)-valued mappings defined almost everywhere (i.e., everywhere except for a finite number of points) on $\mathbb{C}$. The following lemma, which will be needed below, essentially says that these two points of view are equivalent.

Lemma 2.12. *Let $M(s)$ and $N(s)$ be rational matrices. In this case we have $\operatorname{im}_{\mathbb{C}} M(s) = \ker_{\mathbb{C}} N(s)$ for almost all $s \in \mathbb{C}$ if and only if $\operatorname{im}_{\mathscr{L}} M(s) = \ker_{\mathscr{L}} N(s)$.*

Proof. We may assume without loss of generality that $M(s)$ and $N(s)$ have full column rank and full row rank, respectively, as matrices over $\mathbb{R}(s)$ (or, equivalently, as matrices over $\mathscr{L}$). Let $M(s)$ have size $q \times m$ and let $N(s)$ have size $p \times q$. If $\operatorname{im} M(s) = \ker N(s)$ for almost all $s \in \mathbb{C}$, then $N(s)M(s) = 0$ almost everywhere and hence everywhere on $\mathbb{C}$. This shows that $\operatorname{im}_{\mathscr{L}} M(s) \subset \ker_{\mathscr{L}} N(s)$. From $\operatorname{im} M(s) = \ker N(s)$ it also follows that $p + m = q$, and together with the full rank assumption this implies that $\operatorname{im}_{\mathscr{L}} M(s) = \ker_{\mathscr{L}} N(s)$. The converse is obtained by a similar argument. ∎

The following lemma is standard; we provide a proof for completeness.

Lemma 2.13. *Let A, B, X, and Y be matrices such that $AX + BY = 0$. If $[A \mid B]$ and X have full row rank, then B also has full row rank.*

Proof. Let η be a row vector such that $\eta B = 0$. It follows from $\eta(AX + BY) = 0$ that $\eta AX = 0$ and consequently $\eta A = 0$ because X has full row rank. Then we have $\eta[A \mid B] = 0$, which implies that $\eta = 0$. ∎

The result in the lemma below is standard as well; it is related for instance to the fact that the observability indices and the controllability indices of a linear system both sum up to the same value, which coincides with the state space dimension n. We shall need the form below. By the *degree* of a polynomial matrix of full row or column rank, we mean the maximum of the degrees of its full-size minors. For typographical reasons we use transposition of matrices, indicated by a prime.

Lemma 2.14. *If $P(s)$ and $Q(s)$ are polynomial matrices of full column rank and full row rank, respectively, and we have*

$$\ker Q(s) = \operatorname{im} P(s) \tag{2.18}$$

for all $s \in \mathbb{C}$, then the degree of $P(s)$ is equal to the degree of $Q(s)$.

Proof. From the fact that the equality (2.18) holds for all s it follows that the Smith forms of $Q(s)$ and $P(s)$ must be $[I \mid 0]$ and $[I \mid 0]'$, respectively. Therefore we can find (see for instance p.382 of [K]) a polynomial matrix $T(s)$ such that $[T'(s) \mid Q'(s)]'$ is unimodular. Define $Z(s)$ by

$$\begin{bmatrix} T(s) \\ Q(s) \end{bmatrix} P(s) = \begin{bmatrix} Z(s) \\ 0 \end{bmatrix};$$

it follows that $Z(s)$ is unimodular. Without loss of generality, we may assume that $Q(s) = [Q_1(s) \mid Q_2(s)]$ where $Q_2(s)$ gives a minor with the maximal degree. Partitioning $T(s)$ and $P(s)$ accordingly, we get

$$\begin{bmatrix} T_1(s) & T_2(s) \\ Q_1(s) & Q_2(s) \end{bmatrix} \begin{bmatrix} P_1(s) & 0 \\ P_2(s) & I \end{bmatrix} = \begin{bmatrix} Z(s) & T_2(s) \\ 0 & Q_2(s) \end{bmatrix},$$

which shows that $\det P_1(s) = c \cdot \det Q_2(s)$ for some nonzero constant c. It follows that the degree of $P(s)$ must be at least as large as the degree of $Q(s)$. Since a similar reasoning provides the reverse inequality, the lemma is proved. ∎

3. Realization of Homogeneous Systems

Since we have associated homogeneous behaviors both to triples (F, G, H) and to homogeneous polynomial matrices $P(s,t)$, we can now state the following definition.

Definition 3.1. A triple of constant matrices (F, G, H) is said to be a *realization* of the polynomial matrix $P(s,t)$ if $\mathscr{B}(F, G, H) = \mathscr{B}(P)$.

The following lemma gives a sufficient condition for a triple (F, G, H) to be a realization of a homogeneous polynomial matrix $P(s, t)$.

Lemma 3.2. *Let $P(s,t)$ be a $p \times q$ homogeneous polynomial matrix with row degrees $\nu = (\nu_1, \ldots, \nu_p)$. Let $X(s,t)$ be a basis matrix of size ν. If a triple (F, G, H) is such that the equality*

$$\operatorname{im}\begin{bmatrix} sG - tF \\ H \end{bmatrix} = \ker[-X(s,t) \mid P(s,t)] \tag{3.1}$$

holds for almost all $(s,t) \in \mathbb{C}^2 \backslash \{(0,0)\}$, then $\mathscr{B}(F, G, H)$ is equal to $\mathscr{B}(P)$.

Proof. First take $w \in \mathscr{B}(F, G, H)$, and let $z(s)$ and the constant vector x_0 be such that $w(s) = Hz(s)$ and $(sG - F)z(s) = x_0$. We then have, by (3.1) and Lemma 2.12,

$$\begin{bmatrix} x_0 \\ w(s) \end{bmatrix} \in \operatorname{im}_{\mathscr{L}} \begin{bmatrix} sG - F \\ H \end{bmatrix} = \ker_{\mathscr{L}}[-X(s) \mid P(s)].$$

It follows that $P(s)w = X(s)x_0 \in \operatorname{span}_{\mathbb{R}} X(s)$ and so $w \in \mathscr{B}(P)$. For the reverse inclusion, take $w \in \mathscr{B}(P)$. Then we must have $P(s)w(s) = X(s)x_0$ for some constant x_0 and by the formula above we get $w \in \mathscr{B}(F, G, H)$. ∎

The following theorem states that every polynomial matrix has a realization. Moreover, it is shown that the matrices (F, G, H) in the realization can be chosen to satisfy certain requirements that will later be seen to be minimality conditions. Our proof is based on an elementary realization algorithm which applies *mutatis mutandis* (compare with [RS]) to standard (nonhomogeneous) systems as well. This algorithm was first announced in [RRS]. Though we formulate the proof for real homogeneous systems $P(s,t)$ we would like to remark that the same proof is also valid for homogeneous matrices $P(s,t)$ which are defined over an arbitrary base field.

Theorem 3.3. *Let $P(s,t)$ be a homogeneous polynomial matrix of size $p \times (p + m)$ and with row degrees $\nu_1, \ldots, \nu_p$. Assume $P(s,t)$ has generically rank p and let $n = \sum \nu_i$. Then $P(s,t)$ has a realization (F, G, H) satisfying the following properties:*

1. $\operatorname{rank}\begin{bmatrix} sG - tF \\ H \end{bmatrix} = m + n$ *for all* $(s,t) \in \mathbb{C}^2 \backslash \{(0,0)\}$,
2. $\operatorname{rank}[sG - tF] = n$ *for some (and hence almost all)* $(s,t) \in \mathbb{C}^2 \backslash \{(0,0)\}$.

Finally, $P(s_0, t_0)$ has rank p if and only if $s_0 G - t_0 F$ has rank n; in particular, $P(s,t)$ is controllable, that is, has full rank for all $(s,t) \in \mathbb{C}^2 \backslash \{(0,0)\}$, if and only if $sG - tF$ is controllable.

Proof. The existence of a realization is shown via a simple realization algorithm. Let $v = (v_1, \dots, v_p)$ be the set of row degrees of $P(s,t)$ and let $X(s,t)$ be a basis matrix for the vector space X_v.

Algorithm: Consider the $p \times (2n+m+p)$ matrix

$$A(s,t) := [tX(s,t) \mid -sX(s,t) \mid P(s,t)].$$

By definition the ith row of $A(s,t)$ contains homogeneous polynomials of degree v_i and so $A(s,t)$ defines a linear map from $\mathbb{R}^{2n+m+p}$ to the vector space X_{v+1}. Note that $\dim X_{v+1} = n+p$.

We claim that the mapping defined by $A(s,t)$ is surjective. To see this, assume without loss of generality that the row degrees $v_1, \dots, v_p$ satisfy $v_1 \geq \cdots \geq v_k \geq 1, v_{k+1} = \cdots = v_p = 0$. Then $P(s,t) = [P_1'(s,t) \mid P_2']'$ where $P_1(s,t)$ has positive row degrees and P_2 is constant. With the same partitioning, we have $X(s,t) = [X_1'(s,t) \mid 0]'$. Write $\tilde{v} = (v_1, \dots, v_k)$; then X_{v+1} is naturally isomorphic to $X_{\tilde{v}+1} \times \mathbb{R}^{p-k}$. Note that $[tX_1(s,t) \mid -sX_1(s,t)]$ is surjective as a mapping from $\mathbb{R}^{2n}$ to $X_{\tilde{v}+1}$ since every vector $x(s,t) \in X_{\tilde{v}+1}$ can be written as $tx_1(s,t) - sx_2(s,t)$ for some $x_1(s,t)$ and $x_2(s,t)$ in $X_{\tilde{v}}$. Also, the matrix P_2 is surjective from $\mathbb{R}^{m+p}$ to $\mathbb{R}^{p-k}$ by the full row rank assumption on $P(s,t)$. It follows that $A(s,t)$ is indeed surjective.

Consequently, the kernel of $A(s,t)$ is an $(n+m)$-dimensional subspace of $\mathbb{R}^{2n+m+p}$. Let an image representation for this subspace be given by the matrix $[F' \mid G' \mid H']'$ where the partitioning corresponds to the partitioning of $A(s,t)$. The triple (F, G, H) is our candidate realization.

We now have to show that the triple (F, G, H) is indeed a realization and satisfies properties 1 and 2 as claimed in the theorem. To show 1, we have to prove that if

$$\begin{bmatrix} s_0 G - t_0 F \\ H \end{bmatrix} z_0 = 0 \tag{3.2}$$

for some $(s_0, t_0) \in \mathbb{C}^2 \backslash \{(0,0)\}$ and $z_0 \in \mathbb{R}^{n+m}$, then $z_0 = 0$. So suppose that (3.2) holds. Then we have in particular that $s_0 G z_0 = t_0 F z_0$, so there must exist an $x_0 \in \mathbb{R}^n$ and constants α and β such that $Gz_0 = \alpha x_0$ and $Fz_0 = \beta x_0$. We then have $(sG - tF)z_0 = (\alpha s - \beta t)x_0$. Since $X(s,t)(sG - tF)z_0 = P(s,t)Hz_0 = 0$ and the columns of $X(s,t)$ are linearly independent, it follows that $x_0 = 0$ so that $Fz_0 = Gz_0 = 0$. Since also $Hz_0 = 0$ from (3.2), the full column rank property of $[F' \mid G' \mid H']'$ implies that $z_0 = 0$. It now follows from a dimension count that the equality (3.1) is satisfied, so that the triple (F, G, H) is indeed a realization of $P(s,t)$.

It remains to be shown that the generic rank of $sG - tF$ is n. This is a trivial consequence of the controllability part which we now prove.

Controllability: Take $(s_0, t_0) \neq (0,0)$. It follows from the identity $P(s,t)H = X(s,t)(sG - tF)$ that $Hz \in \ker P(s_0, t_0)$ for all $z \in \ker(s_0 G - t_0 F)$. Moreover, it follows from property 1 that $\ker H \cap \ker(s_0 G - t_0 F) = \{0\}$, so that

$$\dim \ker P(s_0, t_0) \geq \dim \ker(s_0 G - t_0 F) \geq m, \tag{3.3}$$

where the final inequality is obvious from the fact that $sG - tF$ has size $n \times (n+m)$. Since $P(s,t)$ has size $p \times (p+m)$, it follows that $s_0 G - t_0 F$ must have full row rank when $P(s_0, t_0)$ has full row rank. The converse is immediate from Lemma 2.13. ∎

Remark 3.4. Once we fix the row degrees, $(\nu_1, \ldots, \nu_p)$, we can fix the basis matrix $X(s,t)$. Then the coefficients of the realization (F, G, H) can be chosen to depend continuously (even analytically) on the coefficients of the homogeneous system $P(s,t)$, in a neighborhood of any given system. This follows from the presented realization algorithm, in particular from the way the matrices F, G, H have been computed. A similar result using the so-called "Fuhrmann realization" [F, Chapter I.10] has been established in [G].

In addition to this realization theorem, which describes the transformation from polynomial to first-order form, we also need a result that produces a polynomial representation starting from a first-order description. Such a result may be called an "elimination theorem," since essentially what is involved is the elimination of the internal variables.

Theorem 3.5. *Let F and G be $n \times (n+m)$ matrices and let H be an $(m+p)\times$ $(n+m)$ matrix, such that the triple (F, G, H) satisfies conditions 1 and 2 of Theorem 3.3. Then there exists a pair of homogeneous polynomial matrices $(X(s,t), P(s,t))$ such that the following holds:*

(i) $\ker[-X(s,t) \mid P(s,t)] = \operatorname{im}\begin{bmatrix} sG - tF \\ H \end{bmatrix}$ *for all* $(s,t) \neq (0,0)$,
(ii) *$P(s,t)$ has full row rank,*
(iii) *the columns of $X(s,t)$ form a basis for the vector space X_ν, where $\nu = (\nu_1, \ldots, \nu_p)$ is the set of row degrees of $P(s,t)$.*

In particular, we have $\mathscr{B}(F, G, H) = \mathscr{B}(P)$.

Proof. We dehomogenize with respect to a point (s_0, t_0) such that $s_0 G - t_0 F$ has full row rank. For ease of notation and without loss of generality we assume that we can take $(s_0, t_0) = (1, 0)$, so that the matrix G has full row rank. By any one of a variety of methods (see for instance p. 488 of [K] or p. 61 of [CD]), polynomial matrices $X(s)$ and $P(s)$ can be found such that

$$\ker[-X(s) \mid P(s)] = \operatorname{im}\begin{bmatrix} sG - F \\ H \end{bmatrix} \tag{3.4}$$

for all complex s. Since we can premultiply $X(s)$ and $P(s)$ by a unimodular matrix without affecting the above property, we may assume that the matrix $[-X(s) \mid P(s)]$ is row reduced. Let n be the number of rows of G. Because $sG - tF$ has full row rank n, it follows from the relation (3.4) and Lemma 2.14 that the sum of the row degrees of $[-X(s) \mid P(s)]$ must be equal to n. Let the row degrees of $P(s)$ be denoted by $\nu_1, \ldots, \nu_p$. Because G has full row rank and $X(s)(sG - F) = P(s)H$, the row degrees of $X(s)$ must be strictly less than those of $P(s)$. It follows that the ν_i are also the row degrees of $[-X(s) \mid P(s)]$ so that they must sum up to n. It also follows that $X(s)$ can be homogenized to a homogeneous polynomial matrix $X(s,t)$ of row degree $\nu - 1$. (Here and below we employ the standard homogenization in which $X(s) = X(s,1)$, in conformity with the notation that we have already been using.) Since $X(s,t)$ has n columns which all belong to the vector space X_ν and which are linearly independent by Lemma 3.1 in [GS1], it follows that actually $X(s,t)$ must be a basis matrix.

We can also homogenize $P(s)$ to a homogeneous matrix $P(s,t)$ of row degree ν. Note that $P(s)$ and hence $P(s,t)$ must have full row rank by Lemma 2.13. Property (i) in the statement of the theorem follows from (3.4) together with our assumptions that $[-X(s) \mid P(s)]$ is row reduced and that $[G' \mid H']'$ has full column rank. The final claim is immediate from Lemma 3.2.

4. Homogeneous Systems and Their Homogeneous Behaviors

We need the following uniqueness theorem for polynomial representations (compare Theorem 3.10 of [GS1]).

Theorem 4.1. *Let $(R_1(s), V_1(s))$ and $(R_2(s), V_2(s))$ be two pairs of polynomial matrices, and assume that for $i = 1, 2$ the following holds:*

(i) *$R_i(s)$ has full row rank,*
(ii) $\operatorname{span}_{\mathbb{R}} V_i(s) \supset X_{R_i}$,
(iii) *$V_i(s)$ has full column rank over $\mathbb{R}$.*

Under these conditions, we have $\mathscr{B}(R_1, V_1) = \mathscr{B}(R_2, V_2)$ if and only if there exists a unimodular polynomial matrix $U(s)$ and a nonsingular constant matrix S such that $R_2(s) = U(s)R_1(s)$ and $V_2(s) = U(s)V_1(s)S$.

Proof. The "if" part is immediate from the definition. For the converse, first note that we must have $\mathscr{B}_-(R_1) = \mathscr{B}_-(R_2)$ by Proposition 2.11. It then follows from the uniqueness theorem for polynomial representations of behaviors on $\mathbb{Z}_+$ (see Section 14 of [NW], Section 4 of [W1], and Corollary 2.5 of [S]) that there must exist a unimodular matrix $U(s)$ such that $R_2(s) = U(s)R_1(s)$. So we may assume that this unimodular transformation has already been carried out, and we write $R_1(s) = R_2(s) = R(s)$. Because $R(s)$ has full row rank, it follows from $\mathscr{B}(R, V_1) = \mathscr{B}(R, V_2)$ that $\operatorname{span}_{\mathbb{R}} V_1(s) = \operatorname{span}_{\mathbb{R}} V_2(s)$. Since both $V_1(s)$ and $V_2(s)$ have linearly independent columns, this proves that there exists a nonsingular constant matrix S such that $V_2(s) = V_1(s)S$. ∎

The crucial property that we have been aiming for is stated in the theorem below.

Theorem 4.2. *Let (F_1, G_1, H_1) and (F_2, G_2, H_2) be two triples satisfying the hypotheses of* (3.5). *Then $\mathscr{B}(F_1, G_1, H_1) = \mathscr{B}(F_2, G_2, H_2)$ if and only if there exist* *constant nonsingular matrices S and T such that $F_2 = SF_1T^{-1}$, $G_2 = SG_1T^{-1}$, and $H_2 = H_1T^{-1}$.*

Proof. By the elimination theorem of the previous section (Theorem 3.5), we can find for $i = 1, 2$ homogeneous polynomial matrices $X_i(s,t)$ and $P_i(s,t)$ satisfying conditions (i)–(iii) of that theorem. In particular it follows (see (2.12)) that $\mathcal{B}(P_1, X_1) = \mathcal{B}(F_1, G_1, H_1) = \mathcal{B}(F_2, G_2, H_2) = \mathcal{B}(P_2, X_2)$. The assumption of the uniqueness theorem, Theorem 4.1, are satisfied and so we can conclude that there exists a unimodular matrix $U(s)$ and a constant nonsingular matrix S such that $P_2(s) = U(s)P_1(s)$ and $X_2(s) = U(s)X_1(s)S^{-1}$. It follows that

$$\operatorname{im}\begin{bmatrix} sG_2 - F_2 \\ H_2 \end{bmatrix} = \operatorname{im}\begin{bmatrix} sSG_1 - SF_1 \\ H_1 \end{bmatrix} \tag{4.1}$$

and by our assumptions on both triples this implies the results claimed in the theorem (see the proof of Theorem 4.1 in [GS1] for details). ∎

Remark 4.3. It follows as in Theorem 4.2 of [GS1] that conditions 1 and 2 of Theorem 3.3 are actually minimality conditions. To be precise, we have the following: if $\mathcal{B}(F_1, G_1, H_1) = \mathcal{B}(F_2, G_2, H_2)$ where $sG_1 - tF_1$ has size $n_1 \times (n_1 + m_1)$ and $sG_2 - tF_2$ has size $n_2 \times (n_2 + m_2)$, and the triple (F_1, G_1, H_1) satisfies conditions 1 and 2, then $n_2 \geq n_1$ and $m_2 \geq m_1$. So it is justified to refer to triples (F, G, H) satisfying conditions 1 and 2 as *minimal* triples.

Remark 4.4. The main result in [RR2] states that there is one-to-one relation between minimal triples modulo similarity equivalence and full row rank homogeneous polynomial matrices modulo left multiplication by homogeneous unimodular matrices. So from the above theorem we also obtain a one-to-one connection between the latter quotient space and homogeneous behaviors. An alternative description of the same space (in terms of nonhomogeneous polynomial matrices) is provided by Theorem 4.1.

5. Conclusions

Using the set of formal Laurent series as a universum we introduced a new class of behaviors which we called homogeneous behaviors. Every homogeneous behavior can be described either through a homogeneous polynomial matrix or through a triple of matrices (F, G, H) inducing a generalized first-order representation of the homogeneous behavior. The relation between higher-order and first-order representation was explained. Both the equivalence classes of first-order representations and the equivalence classes of higher-order representations can be seen as points in a certain quotient scheme due to Grothendieck (see [RR2]), and therefore from a mathematical point of view our results come down to giving these points a concrete interpretation as elements in the Grassmannian of the space of vector-valued formal Laurent series.

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