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This paper presents a sampling strategy for grey-level functions based on mathematical morphology. Flexible sampling strategies are required for the construction of multiresolution image representations like pyramids and quad-trees. The sampling operator introduced here is a dilation. A number of different algorithms to reconstruct an image from its sampled version are presented. It is shown that a reconstruction which dominates the original image is obtained by application of the adjoint erosion.

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1. Introduction

Images may contain details of various sizes. To obtain a complete description an image must therefore be analyzed over a whole range of spatial scales. This is usually done by generating a hierarchy of images that consists of successively reduced resolution versions of the original image (hence the term "multiresolution representation"). Well known examples of these multiresolution image representations are the pyramid and quad-tree data structures: see [8,13].

Typically, a multiresolution representation is a sequence of images in which each image is a filtered and subsampled copy of its predecessor. The conventional multiresolution representations apply linear filters to reduce resolution. Linear filter techniques alter the object intensities and therefore the estimated location of their contours. Hierarchical image representation schemes based on these techniques are therefore of limited applicability to the

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description of size and shape. The underlying techniques of mathematical morphology are geometric in nature. As a result, multiresolution image representations constructed with morphological techniques are inherently more suited for the analysis of shape and size specific features in the image: see [11,12].

The construction of multiresolution image representations requires a simple but flexible sampling scheme. One particular example of such a sampling scheme was introduced and thoroughly studied by Haralick and his co-workers [5,6]. Some of their results which are relevant for the present discussion will be recapitulated below. But first we recall some basic concepts from mathematical morphology which will be used in the sequel. For a more complete exposition we refer to [4,9,10].

Let $\mathrm{Fun}(\mathbb{Z}^2;\mathcal{G})$, or $\mathrm{Fun}(\mathbb{Z}^2)$ for short, be the space of all grey-level functions mapping $\mathbb{Z}^2$ into the set of grey-levels $\mathcal{G}$ which we take here to be the finite set $\{0,1,2,\ldots,N\}$. For $h \in \mathbb{Z}^2$ we denote by F_h the translated function

$$F_h(x) = F(x-h) \quad \text{for } x \in \mathbb{Z}^2.$$

We denote by $\mathbf{id}$ the identity operator on $\mathrm{Fun}(\mathbb{Z}^2)$, that is $\mathbf{id}(F) = F$ for every $F \in \mathrm{Fun}(\mathbb{Z}^2)$. An operator $\psi : \mathrm{Fun}(\mathbb{Z}^2) \to \mathrm{Fun}(\mathbb{Z}^2)$ is called translation-invariant if

$$\psi(F_h) = (\psi(F))_h, \quad \text{for all } F \in \mathrm{Fun}(\mathbb{Z}^2) \text{ and } h \in \mathbb{Z}^2.$$

Let A be an arbitrary subset of $\mathbb{Z}^2$, called the *structuring element*. The *dilation* $F \oplus A$ and the *erosion* $F \ominus A$ of the function F are defined as

$$\begin{aligned}(F \oplus A)(x) &= \sup\{F(y) \mid y \in \check{A}_x\} = \sup_{h \in A} F_h(x)\\ (F \ominus A)(x) &= \inf\{F(y) \mid y \in A_x\} = \inf_{h \in A} F_{-h}(x).\end{aligned}$$

Here $\check{A}$ is the reflected set of A, that is, $\check{A} = \{-h \mid h \in A\}$.

One can extend the given formulas for dilation and erosion to the case where A is a grey-level function. In the special case that A only attains the value zero on its support we can replace A by a set (as we actually did) and in this case we call A a *flat structuring element*.

Dilation and erosion by A form an adjunction in the sense that for $F_1, F_2 \in \mathrm{Fun}(\mathbb{Z}^2)$,

$$F_1 \oplus A \le F_2 \quad \text{iff} \quad F_1 \le F_2 \ominus A.$$

The opening F_A and the closing F^A of F by A are given by

$$\begin{aligned}F_A &= (F \ominus A) \oplus A\\ F^A &= (F \oplus A) \ominus A.\end{aligned}$$

All these operators are translation-invariant. We recall that an operator ψ on $\mathrm{Fun}(\mathbb{Z}^2)$ is called an *opening* if

- ψ is monotone, that is $F \le G$ implies that $\psi(F) \le \psi(G)$

- ψ is idempotent, that is $\psi^2 = \psi$
- ψ is anti-extensive, that is $\psi \leq \mathbf{id}$.

To define closings one has to replace the adverb "anti-extensive" by "extensive".

First we shall briefly discuss some related ideas underlying the work by Haralick et al. in [6]. In particular, we will limit ourselves to *flat structuring elements*.

Let S be a subgroup of $\mathbb{Z}^2$; in the sequel S will be referred to as the *sampling set*. Let the structuring element $K \subset \mathbb{Z}^2$ be large enough so that the union of all K's positioned at $s \in S$ covers $\mathbb{Z}^2$, i.e.,

$$\mathbb{Z}^2 = S \oplus K := \bigcup_{s \in S} K_s. \tag{1.1}$$

Here $K_s = \{k + s \mid k \in K\}$. Furthermore assume that K is symmetric around $(0,0)$ (i.e., $K = \check{K}$), that $K \cap S = \{(0,0)\}$, and that

$$x \in K_y \implies K_x \cap K_y \cap S \neq \emptyset. \tag{1.2}$$

This latter condition is called the *sampling condition*; one easily shows that this condition combined with the fact that $(0,0) \in K$ yields (1.1). In accordance to the sampling theorem in linear signal analysis, this condition implies that the sampling distance must be less than half the diameter of the structuring element K. For an arbitrary element F of $\mathrm{Fun}(\mathbb{Z}^2)$, the *sampled function* $[F]$ (which is again an element of $\mathrm{Fun}(\mathbb{Z}^2)$) is defined as

$$[F](x) = \begin{cases} F(x), & x \in S \\ 0, & \text{elsewhere.} \end{cases}$$

Haralick et al. consider two reconstruction algorithms, namely the dilation reconstruction defined by $[F] \oplus K$, and the closing reconstruction defined by $[F]^K$. The restrictions on K and S result in the following bounding relationships:

$$\begin{aligned} F_K &\subseteq [F] \oplus K \subseteq F \oplus K \\ F \ominus K &\subseteq [F]^K \subseteq F^K. \end{aligned}$$

If F is both open and closed with respect to K, that is, $F = F_K = F^K$, then these estimates yield

$$[F]^K \subseteq F \subseteq [F] \oplus K.$$

Our approach differs from the approach by Haralick and co-workers in the sense that the sampled image is a restriction of $F \oplus \check{K}$ to S and not merely of F. Thereto we define a sampling operator $\sigma : \mathrm{Fun}(\mathbb{Z}^2) \to \mathrm{Fun}(S)$ by

$$\sigma(F)(s) = (F \oplus \check{K})(s).$$

Thus σ is a dilation as it distributes over arbitrary suprema (see [7,10]). The theory of adjunctions [7] states that this dilation σ is uniquely related to a corresponding erosion $\dot{\sigma}$ mapping $\mathrm{Fun}(S)$ to $\mathrm{Fun}(\mathbb{Z}^2)$ such that for every $F \in \mathrm{Fun}(\mathbb{Z}^2)$ and $G \in \mathrm{Fun}(S)$ one has

$$\sigma(F) \leq G \iff F \leq \dot{\sigma}(G). \tag{1.3}$$

It is easy to show that

$$\dot\sigma(G)(x) = \inf\{G(s) \mid s \in \check{K}_x \cap S\}.$$

Application of $\dot\sigma$ to a sampled image $\sigma(F)$ gives a reconstruction of the original image. Since $\dot\sigma\sigma$ is a closing we find immediately that $\dot\sigma\sigma(F) \geq F$.

It is clear that the sampling operator σ commutes with translations over vectors in S, that is, $\sigma(F_s) = \sigma(F)_s$, for $F \in \mathrm{Fun}(\mathbb{Z}^2)$ and $s \in S$. In Section 2 we discuss a more general sampling strategy which applies to arbitrary spaces and does not satisfy any translation invariance principle. From Section 3 onwards we restrict the discussion to regular sampling strategies and we define two different reconstruction operators. If the sampling element K can be decomposed as $A \oplus \check{A}$, then there exists a third useful reconstruction algorithm. This algorithm is discussed in Section 4. In Section 5 we examine the situation where the sets K_s ($s \in S$) are disjoint. In the construction of a morphological pyramid one has to repeat the sampling procedure several times. Problems related to iterative application of the sampling procedure are discussed in Section 6. Finally, in Section 7, we briefly discuss operations on sampled images.

The sampling strategy which is the topic of this paper can also be used to digitize continuous images, that is elements of $\mathrm{Fun}(\mathbb{R}^2)$. The conditions under which the sampled image converges to the original image will be the subject of a forthcoming paper by one of the authors.

2. A general sampling strategy

Let X, S be arbitrary sets and let $K : S \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the space of all subsets of X. The dual mapping $K^* : X \to \mathcal{P}(S)$ is defined as:

$$K^*(x) = \{s \in S \mid x \in K(s)\}. \tag{2.1}$$

It is easily seen that the second dual $K^{**} : S \to \mathcal{P}(X)$ equals K, i.e., $K = K^{**}$. One can also show that

$$\bigcup_{s \in S} K(s) = X \text{ if and only if } \forall x \in X : \ K^*(x) \neq \emptyset, \tag{2.2}$$

and in this case we say that K covers X. One easily sees that,

$$\forall x \in X : \ \text{card } K^*(x) = 1 \text{ if and only if the } K(s) \text{ are mutually disjoint},$$

if K covers X. Here "card" denotes the cardinality; for a finite set this amounts to the number of its elements. As before $\mathrm{Fun}(X)$ denotes the complete lattice of all functions $F : X \to \mathcal{G}$, where $\mathcal{G} = \{0, 1, 2, \ldots, N\}$ or any other complete lattice of grey-levels. The support of F, $\mathrm{supp}(F)$ is defined as the set of all $x \in X$ for which $F(x) \neq 0$. We define the dilation $\sigma : \mathrm{Fun}(X) \to \mathrm{Fun}(S)$ by

$$\sigma(F)(s) = \sup\{F(x) \mid x \in K(s)\}. \tag{2.3}$$

The adjoint erosion $\grave{\sigma}$: Fun$(S) \to$ Fun(X) which corresponds with σ by adjunction can be computed explicitly from (1.3). We have

$$\begin{aligned}\sigma(F) \leq G &\Leftrightarrow \forall s \in S: \ \sup\{F(x) \mid x \in K(s)\} \leq G(s)\\ &\Leftrightarrow \forall s \in S\ \forall x \in K(s): \ F(x) \leq G(s)\\ &\Leftrightarrow \forall x \in X\ \forall s \in K^*(x): \ F(x) \leq G(s)\\ &\Leftrightarrow \forall x \in X: \ F(x) \leq \inf\{G(s) \mid s \in K^*(x)\},\end{aligned}$$

which yields that

$$\grave{\sigma}(G)(x) = \inf\{G(s) \mid s \in K^*(x)\}. \tag{2.4}$$

The composition

$$\rho = \grave{\sigma}\sigma \tag{2.5}$$

is a closing on Fun(X), and

$$\sigma\rho = \sigma. \tag{2.6}$$

We call σ the *sampling operator*, $\grave{\sigma}$ the *reconstructing operator* and ρ the *reconstruction operator*. We tacitly assume that a reconstruction operator is of the form $\psi\tau$, where τ is a sampling operator mapping Fun$(\mathbb{Z}^2)$ into Fun(S) and ψ a reconstructing operator mapping Fun(S) back to Fun$(\mathbb{Z}^2)$. From the fact that ρ is a closing we have the lower estimate $\rho \geq \mathbf{id}$.

Proposition 2.1. *The closing $\rho(F)$ is the largest function G for which $\sigma(G) = \sigma(F)$.*

PROOF. If $\sigma(G) = \sigma(F)$ then $G \leq \grave{\sigma}\sigma(G) = \grave{\sigma}\sigma(F) = \rho(F)$.

∎

If K does not cover X and x is one of the elements which is not contained in any of the $K(s)$'s, then we have $\rho(F)(x) = N$. If K does cover X then we can derive an upper bound for the reconstruction $\rho(F)$ of F. Let $x, x' \in X$. We say that x' is a K-neighbour of x if $x, x' \in K(s)$ for some $s \in S$. The K-neighbours of a point x are denoted by $N(x)$. One derives immediately that

$$N(x) = \bigcup_{s \in K^*(x)} K(s). \tag{2.7}$$

So $y \in N(x)$ if and only if there is an $s \in S$ such that $x, y \in K(s)$. In particular, $x \in N(x)$ if and only if $K^*(x) \neq \emptyset$. So the K-neighbourship relation defines a symmetric relation on X which is reflexive if K covers X. We define the dilation δ : Fun$(X) \to$ Fun(X) by

$$\delta(F)(x) = \sup\{F(x') \mid x' \in N(x)\}. \tag{2.8}$$

One finds immediately that

$$\delta(F)(x) = \sup\{\sigma(F)(s) \mid s \in K^*(x)\}.$$

Since $\rho(F)(x) = \inf\{\sigma(F)(s) \mid s \in K^*(x)\}$, we find that $\rho(F)(x) \leq \delta(F)(x)$ if $K^*(x) \neq \emptyset$, and equality holds if $K^*(x)$ contains exactly one element. Thus we have proved the following result.

Proposition 2.2. *If K covers X then*

$$\rho \leq \delta, \tag{2.9}$$

where equality holds if the $K(s)$ are mutually disjoint.

Remark 2.3. Here we have applied a dilation to sample the image and obtained a reconstruction by application of the adjoint erosion. Instead, one could reverse the order and first erode to sample the image, and then dilate to get a reconstruction. The latter reconstruction operator, which we will denote by ρ^*, is an opening. The two approaches are closely related. This becomes apparent by defining the *negative* F^* of the function F as $F^*(x) = N - F(x)$. Then $\rho^*(F) = (\rho(F^*))^*$. Note that the negative of a function which takes values in $\mathbb{Z}_+ \cup \{\infty\}$ cannot be defined properly.

3. Regular sampling

In the sequel we will only consider the case $X = \mathbb{Z}^2$, and assume that S is a subset of $\mathbb{R}^2$. Unless otherwise stated, x will always denote an element of X and s will denote an element of S. Let $K(s)$ be the translate along s of some fixed set $K \subset \mathbb{R}^2$, called the *sampling element.* From now on we shall use the notation K_s rather than $K(s)$. Again we assume that $K_s \subset \mathbb{Z}^2$. Then

$$K^*(x) = \check{K}_x \cap S. \tag{3.1}$$

Here $\check{K}$ is the reflection of K, i.e., $\check{K} = \{-k \mid k \in K\}$, and $\check{K}_x$ is the translate of $\check{K}$ along the vector x. We say that K is *symmetric* if $\check{K} = K$, and that K is *shape-symmetric* if $\check{K} = K_h$ for some $h \in \mathbb{R}^2$. Obviously, every symmetric set is also shape-symmetric. The situation that K covers $\mathbb{Z}^2$ can be denoted mathematically as

$$S \oplus K = \mathbb{Z}^2. \tag{3.2}$$

In Figure 1 we have depicted some regular sampling strategies. Note that in Figure 1(f) the covering assumption (3.2) is not satisfied. The K's in Figure 1(a)-(f) are symmetric whereas the K in Figure 1(g) is only shape-symmetric.

For practical reasons we make the following assumption:

Assumption 3.1. S is a subgroup of $\mathbb{Z}^2$.

This assumption implies in particular that $(0,0) \in S$, and excludes the sampling strategies depicted in Figures 1(d),(e).

Remark 3.2. Note that Assumption 3.1 does not imply a severe restriction of generality. For example, the sampling scheme depicted in Figure 1(d) is, modulo a translation, equivalent to the one in Figure 1(g).

We say that the mapping ψ on $\mathrm{Fun}(\mathbb{Z}^2)$ is an S-operator if it commutes with translations along vectors in S, i.e.,

$$\psi(F_s) = \psi(F)_s, \quad F \in \mathrm{Fun}(\mathbb{Z}^2),\ s \in S.$$

The dilation sampling operator of (2.3) can now be written as

$$\sigma(F)(s) = (F \oplus \check{K})(s). \tag{3.3}$$

The reconstruction operator $\rho = \dot{\sigma}\sigma$ is given by

$$\rho(F)(x) = \inf\{(F \oplus \check{K})(s) \mid s \in \check{K}_x \cap S\}. \tag{3.4}$$

Obviously, ρ is an S-operator.

We give an upper- and lower-bound for ρ. Recall that the K-neighbourhood $N(x)$ of an element x is given by $N(x) = \{y \subset \mathbb{Z}^2 \mid x, y \in K_s \text{ for some } s \in S\}$. Let

$$L = K \oplus \check{K}. \tag{3.5}$$

Lemma 3.3. *For every $x \in \mathbb{Z}^2$,*

$$N(x) \subset L_x. \tag{3.6}$$

PROOF. From (2.7) we get that

$$N(x) = \bigcup_{s \in K^*(x)} K_s \subset \bigcup_{s \in \check{K}_x} K_s = (K \oplus \check{K})_x = L_x.$$

∎

Figure 2 illustrates that the size and shape of $N(x)$ generally depend on x, and that the inclusion in (3.6) may be strict.

Proposition 3.4. *For every $F \in \mathrm{Fun}(\mathbb{Z}^2)$ we have the following lower-bound for $\rho(F)$:*

$$F^{\check{K}} \leq \rho(F). \tag{3.7}$$

If $K \oplus S = \mathbb{Z}^2$ then we have the upper bound

$$\rho(F) \leq F \oplus L. \tag{3.8}$$

PROOF. From (3.4) we get that

$$\rho(F)(x) \geq \inf\{(F \oplus \check{K})(y) \mid y \in \check{K}_x\} = F^{\check{K}}(x).$$

The estimate (3.8) follows from Proposition 2.2 and Lemma 3.3.

∎

Corollary 3.5. *For every* $F \in \text{Fun}(\mathbb{Z}^2)$,

$$\sigma(F^{\check{K}}) = \sigma(F). \tag{3.9}$$

Proof. From (3.7) and the fact that $\sigma = \sigma\rho$ and σ are increasing, we get

$$\sigma(F^{\check{K}}) \geq \sigma(F) = \sigma\rho(F) \geq \sigma(F^{\check{K}}),$$

whence equality follows.

∎

We will now present a second reconstruction operator, called the *closing reconstruction* and denoted by ρ_k. Thereto we first define the operator σ_0 on $\text{Fun}(\mathbb{Z}^2)$ as follows:

$$\sigma_0(F)(x) = \begin{cases} \sigma(F)(x), & \text{if } x \in S \\ 0, & \text{elsewhere.} \end{cases} \tag{3.10}$$

From (3.3) it follows trivially that

$$\sigma_0(F) \leq F \oplus \check{K}. \tag{3.11}$$

Lemma 3.6. *If* $K \oplus S = \mathbb{Z}^2$ *then*

$$\rho(F) \leq \sigma_0(F) \oplus K, \tag{3.12}$$

for every $F \in \text{Fun}(\mathbb{Z}^2)$.

Proof. From $K \oplus S = \mathbb{Z}^2$ it follows that $K^*(x) = \check{K}_x \cap S \neq \emptyset$, for $x \in \mathbb{Z}^2$, and we get that

$$\begin{aligned} \rho(F)(x) &= \inf\{\sigma(F)(s) \mid s \in \check{K}_x \cap S\} \\ &\leq \sup\{\sigma(F)(s) \mid s \in \check{K}_x \cap S\} \\ &= \sup\{\sigma_0(F)(y) \mid y \in \check{K}_x\} \\ &= (\sigma_0(F) \oplus K)(x). \end{aligned}$$

This proves the result.

∎

We define the closing reconstruction ρ_k by

$$\rho_k(F) = \sigma_0(F)^{\check{K}} \tag{3.13}$$

Note that $\rho_k(F) = \sigma_0(F)^K$ if K is shape-symmetric, since closing with the translate K_h gives the same result as closing with K.

Proposition 3.7. *Let K be shape-symmetric and suppose that $K \oplus S = \mathbb{Z}^2$. For every $F \in \mathrm{Fun}(\mathbb{Z}^2)$,*

$$F \ominus K \leq \rho(F) \ominus K \leq \rho_k(F) \leq (F \oplus \check{K})^K = (F \oplus L) \ominus K. \tag{3.14}$$

If, in addition, $K \cap S = \{(0,0)\}$, then

$$\sigma\rho_k = \sigma, \quad \rho_k^2 = \rho_k, \quad \text{and} \quad \rho_k \leq \rho. \tag{3.15}$$

Proof. The estimates follow immediately from (3.11) and Lemma 3.6.
If $K \cap S = \{(0,0)\}$, then, for $F \in \mathrm{Fun}(\mathbb{Z}^2)$ and $s \in S$,

$$\begin{aligned}
\sigma\rho_k(F)(s) &= \sigma(\sigma_0(F)^{\check{K}})(s) = \left[\sigma_0(F)^{\check{K}} \oplus \check{K}\right](s) \\
&= \left[\sigma_0(F) \oplus \check{K}\right](s) = \sup_{x \in K_s} \sigma_0(F)(x) \\
&= \sigma_0(F)(s),
\end{aligned}$$

since the supremum takes its value in $S \cap K_s = \{s\}$. Thus $\sigma\rho_k = \sigma$. Now the second relation follows immediately.
To prove that $\rho_k \leq \rho$, note that $\rho\rho_k = \dot{\sigma}\sigma\rho_k = \dot{\sigma}\sigma = \rho$. Since $\rho \geq \mathrm{id}$ we find that $\rho \geq \rho_k$. ∎

We point out that (3.15) also holds if K is not shape-symmetric. Note that one may not conclude that ρ_k is a closing.

If $K \oplus S = \mathbb{Z}^2$ does not hold, in particular, if K contains only one point and S is a nontrivial subgroup of $\mathbb{Z}^2$, then the closing reconstruction ρ_k is rather useless. The same is true if the sampling element K is large compared to the spacing of the sample points in S, so that there is much overlap. However, ρ_k may yield a reasonable reconstruction if $K \oplus S = \mathbb{Z}^2$, and if K_s are mutually disjoint: this is actually the situation discussed in Section 5.

4. The case $K = A \oplus \check{A}$

In this section we present an important class of sampling elements K for which there exists a reconstruction algorithm which sometimes performs "better" than ρ and ρ_k. Let $A \subset \mathbb{Z}^2$. Throughout this section we will assume the following.

Assumption 4.1. A satisfies the covering assumption $S \oplus A = \mathbb{Z}^2$.

We define the symmetric sampling element K as

$$K = A \oplus \check{A}. \tag{4.1}$$

In Example 4.6 we will consider the important special case that $S = 2\mathbb{Z}^2 = \{(2i, 2j) \mid i, j \in \mathbb{Z}\}$, and $A = \{(0,0), (1,0), (0,1), (1,1)\}$. We start with a lemma.

Lemma 4.2. *Let Assumption 4.1 hold and let $K = A \oplus \check{A}$. For every $x \in \mathbb{Z}^2$,*

$$A_x \subset (A_x \cap S) \oplus K \quad \text{and} \quad \check{A}_x \subset (\check{A}_x \cap S) \oplus K.$$

PROOF. Because of the symmetry, we only have to prove the first inclusion. Let $a \in A$. We must show that $a + x \in (A_x \cap S) \oplus K$. Since $A \oplus S = \mathbb{Z}^2$ there exist $a' \in A$ and $s' \in S$ such that $-x = a' + s'$. Then $-s' = a' + x \in S$, because S is a group, and $-s' = a' + x \in A_x$. Therefore $-s' \in A_x \cap S$. Now $a + x = -s' + a - a' \in (A_x \cap S) \oplus K$, since $a - a' \in K$.

∎

Let σ_0 be as in (3.10). We define the reconstruction operator

$$\rho_a(F) := \sigma_0(F)^{\check{A}}.$$

Proposition 4.3. *Let Assumption 4.1 hold and let $K = A \oplus \check{A}$. Then*

$$\mathrm{id} \le \rho_a \le \rho_k. \tag{4.2}$$

PROOF. To prove the first inequality it suffices to show that $F \oplus \check{A} \le \sigma_0(F) \oplus \check{A}$, or equivalently, that for $x \in \mathbb{Z}^2$,

$$\begin{aligned} \sup\{F(y) \mid y \in A_x\} &\le \sup\{\sigma_0(F)(y) \mid y \in A_x\} \\ &= \sup\{\sigma(F)(s) \mid s \in A_x \cap S\} \\ &= \sup\{\sup\{F(z) \mid z \in K_s\} \mid s \in A_x \cap S\}. \end{aligned}$$

It is sufficient to prove that for every $y \in A_x$ we have $y \in K_s$ for some $s \in A_x \cap S$. But this is the same as $A_x \subset (A_x \cap S) \oplus K$, which was proved in Lemma 4.1.
We are now going to show that $\rho_a \le \rho_k$. We observe that any function which is closed with respect to K is also closed with respect to $\check{A}$, that is, $(G^K)^{\check{A}} = G^K$, for any function G on $\mathbb{Z}^2$. Since $G \le G^K$ this implies that $G^{\check{A}} \le G^K$. From this observation the proof follows immediately.

∎

Note that, because of the symmetry, we also have

$$F \le \sigma_0(F)^A \le \rho_k(F). \tag{4.3}$$

Proposition 4.4. *Let $K = A \oplus \check{A}$, with A satisfying Assumption 4.1, and suppose that $K \cap S = \{(0,0)\}$. Then*

$$\rho = \rho_a = \rho_k. \tag{4.4}$$

PROOF. We first show that, for any $F \in \mathrm{Fun}(\mathbb{Z}^2)$,

$$\sigma_0(F)^{\check{A}} \le \dot{\sigma}\sigma(F),$$

which is equivalent to

$$\sigma\big(\sigma_0(F)^{\check{A}}\big) \le \sigma(F).$$

From $K = A \oplus \check{A}$ we find that

$$\begin{aligned}\sigma(\sigma_0(F)^{\check{A}})(s) &= \left[((\sigma_0(F) \oplus \check{A}) \ominus \check{A}) \oplus \check{A} \oplus A\right](s) \\ &= \left[\sigma_0(F) \oplus \check{A} \oplus A\right](s) = \left[\sigma_0(F) \oplus K\right](s) \\ &= \sup_{x \in K_s} \sigma_0(F)(x) = \sigma_0(F)(s) \\ &= \sigma(F)(s),\end{aligned}$$

because $K_s \cap S = \{s\}$. Since $\sigma\rho = \sigma$ we have $\sigma_0\rho = \sigma_0$. Application of the closing by A at both sides results in $\rho_a\rho = \rho_a$. But since $\rho_a \geq \mathrm{id}$ this yields that $\rho_a\rho \geq \rho$, which gives $\rho_a \geq \rho$, and hence $\rho_a = \rho$. The proof that $\rho_a = \rho_k$ proceeds along the same lines: recall from (4.2) that $\rho_a \leq \rho_k$.

∎

Remark 4.5. Let $K = A \oplus \check{A}$. Then

$$K \cap S = \{(0,0)\} \quad \text{if and only if the } A_s \text{ are mutually disjoint.}$$

To prove "if", assume that the A_s are mutually disjoint and that $s \in K \cap S$. So $s \in A \oplus \check{A}$, and therefore can be written as $s = a_1 - a_2$ with $a_1, a_2 \in A$. But then $a_1 = a_2 + s \in A_s$. This implies that $a_1 \in A \cap A_s$ which is possible only if $s = (0,0)$.
To prove "only if", assume that $K \cap S = \{(0,0)\}$ and that $x \in A_r \cap A_s$ for some $r, s \in S$. Then $x = a_1 + r = a_2 + s$ for some $a_1, a_2 \in A$. So $r - s = a_2 - a_1 \in A \oplus \check{A} = K$. But $r - s$ is also an element of S and therefore must be equal to zero. So $r = s$.

Example 4.6. We consider now the important special case where $S = 2\mathbb{Z}^2$. We define

$$\begin{aligned}K^{(2)} &= \{(i,j) \mid 0 \leq i,j \leq 1\} \\ K^{(3)} &= \{(i,j) \mid -1 \leq i,j \leq 1\} \\ K^{(5)} &= \{(i,j) \mid -2 \leq i,j \leq 2\}.\end{aligned}$$

Then $K^{(3)}, K^{(5)}$ are symmetric and $K^{(2)}$ is shape-symmetric. See Figure 3(a). We denote by $\sigma^{(2)}, \sigma^{(3)}$ and $\sigma^{(5)}$ the corresponding sampling strategies. For $i = 2,3,5$ we define $\rho^{(i)} = \dot{\sigma}^{(i)}\sigma^{(i)}$. If $A^{(3)} := K^{(2)}$ then $K^{(3)} = A^{(3)} \oplus \check{A}^{(3)}$, and $A^{(3)}$ satisfies the assumptions of Proposition 4.4 whence we obtain that

$$\rho^{(3)} = \rho_a^{(3)} = \rho_k^{(3)},$$

where $\rho_a^{(3)}$ and $\rho_k^{(3)}$ have the obvious meaning. Defining $A^{(5)} := K^{(3)}$ we have $K^{(5)} = A^{(5)} \oplus \check{A}^{(5)}$. Now $A^{(5)}$ satisfies the covering assumption (4.1), that is,

$$S \oplus A^{(5)} = \mathbb{Z}^2.$$

It follows from Proposition 4.3 that

$$\rho_a^{(5)} \leq \rho_k^{(5)}.$$

However, $K^{(5)} \cap S$ contains more than just the element $(0,0)$, so the conclusions of Proposition 4.4 do not follow. By means of some simple examples it can easily be understood that $\rho^{(3)}$ and $\rho^{(5)}$ have a rather different performance. Thereto we consider a binary image consisting of only one point x. If $x \in \mathbb{Z}^2$ is a point with odd coordinates, then $\rho^{(3)}(\{x\}) = K_x^{(3)}$ and $\rho^{(5)}(\{x\}) = \{x\}$: see Figure 3(b). However, if x is a point with even coordinates, and hence an element of S, then $\rho^{(3)}(\{x\}) = \{x\}$ and $\rho^{(5)}(\{x\}) = K_x^{(3)}$: see Figure 3(c). (In fact, we should not insert sets but their characteristic functions as arguments of ρ.) For completeness we mention that in the first case $\rho^{(2)}(\{x\}) = \check{K}_x^{(2)}$ and $\rho_a^{(5)}(\{x\}) = \rho_k^{(5)}(\{x\}) = K_x^{(3)}$, whereas in the latter case $\rho^{(2)}(\{x\}) = K_x^{(2)}$ and $\rho_a^{(5)}(\{x\}) = \rho_k^{(5)}(\{x\}) = K_x^{(5)}$.

5. Non-overlapping sampling elements

Next we consider the case that the K_s's form a partition of the space $\mathbb{Z}^2$. Throughout this section we make the following assumptions, which are in particular satisfied for the sampling strategies depicted in Figures 1(b),(g),(h).

Assumption 5.1.
(a) $K \oplus S = \mathbb{Z}^2$
(b) the K_s are mutually disjoint, i.e. $K_r \cap K_s = \emptyset$ if $r \neq s$, $r, s \in S$
(c) $(0,0) \in K$
(d) K is shape-symmetric.

Note that (b) and (c) imply that $K \cap S = \{(0,0)\}$. So we can apply Proposition 3.7 which tells us that $\sigma\rho_k = \sigma$, $\rho_k^2 = \rho_k$ and $\rho_k \leq \rho$.

Remark 5.2. Note that (c) is not really a restriction. Because of (a),(b) there is a unique $s_0 \in S$ so that $(0,0) \in K_{s_0}$. Let $K' = K_{s_0}$ and let ρ' be the reconstruction operator corresponding to K'. Then $\rho' = \rho$.

Assumption 5.1 guarantees that there exists for every $x \in \mathbb{Z}^2$ a unique $s_x \in S$ such that

$$K^*(x) = \check{K}_x \cap S = \{s_x\} \tag{5.1}$$

Then the K-neighbourhood of x is

$$N(x) = K_{s_x}. \tag{5.2}$$

The erosion $\dot{\sigma}$ of (2.4) reduces to

$$\dot{\sigma}(G)(x) = G(s_x), \tag{5.3}$$

and it follows immediately that $\sigma\dot{\sigma}$ is the identity operator on $\mathrm{Fun}(S)$. The reconstruction operator $\rho = \dot{\sigma}\sigma$ becomes

$$\rho(F)(x) = \sigma(F)(s_x) = \sup\{F(y) \mid y \in K_{s_x}\} \quad \text{for all } x \in X. \tag{5.4}$$

In particular this implies that ρ is both a closing and a dilation. By $F|_S$ we denote the restriction of F to S.

Lemma 5.3. *Let Assumption 5.1 hold and let $F \in \mathrm{Fun}(\mathbb{Z}^2)$ with $\mathrm{supp}(F) \subset S$. Then*

$$\sigma(F) = \sigma(F \oplus K) = F|_S. \tag{5.5}$$

PROOF. The first equality is easy, and we only prove the second. Suppose that $\mathrm{supp}(F) \subset S$. Then

$$(F \oplus K)(x) = \sup\{F(y) \mid y \in \check{K}_x\} = \sup\{F(y) \mid y \in \check{K}_x \cap S\} = F(s_x),$$

for all $x \in X$. Therefore

$$\sigma(F \oplus K)(s) = \sup\{(F \oplus K)(x) \mid x \in K_s\} = \sup\{F(s_x) \mid x \in K_s\} = F(s),$$

for all $s \in S$.

∎

Proposition 5.4. *Let Assumption 5.1 hold, and let K be shape-symmetric. Then*

$$\rho(F) = \rho_k(F) \oplus K \quad \text{and} \quad \rho_k(F) = \rho(F) \ominus K, \tag{5.6}$$

and

$$\sigma\rho_k = \sigma \quad \text{and} \quad \rho_k^2 = \rho_k. \tag{5.7}$$

PROOF. We first show that $\rho(F) = \sigma_0(F) \oplus K$. For $x \in \mathbb{Z}^2$,

$$\begin{aligned}(\sigma_0(F) \oplus K)(x) &= \sup\{\sigma_0(F)(y) \mid y \in \check{K}_x\} \\ &= \sigma(F)(s_x) = \rho(F)(x),\end{aligned}$$

where we have used (5.4). This yields that

$$\rho_k(F) \oplus K = ((\sigma_0(F) \oplus K) \ominus K) \oplus K = \sigma_0(F) \oplus K = \rho(F),$$

and that

$$\rho(F) \ominus K = \sigma_0(F)^K = \rho_k,$$

which proves (5.6).

We now prove (5.7). From the first equality in (5.5) it follows that $\sigma = \sigma\sigma_0$. This implies that

$$\sigma = \sigma\sigma_0 \leq \sigma\rho_k \leq \sigma\rho = \sigma.$$

Thus $\sigma\rho_k = \sigma$ and hence $\rho_k^2 = \rho_k$.

∎

6. Repeated sampling

In case we want to repeat the sampling procedure of Section 3, we have to define a sampling element $K' \subset S$ and a sampling set S' in S such that S' is a subgroup of S. In this section we will restrict to the case where the same sampling scheme is used in every step. We make the following assumption:

Assumption 6.1. S is a subgroup of $\mathbb{Z}^2$ which is isomorphic to $\mathbb{Z}^2$.

We point out that every nontrivial subgroup S of $\mathbb{Z}^2$ which is not isomorphic to $\mathbb{Z}$ is isomorphic to $\mathbb{Z}^2$, and in that case the quotient group $\mathbb{Z}^2/S$ is finite. Let $i : \mathbb{Z}^2 \to S$ be a group isomorphism. Note that i is completely specified by its values at the points $(1,0)$ and $(0,1)$. Let $s_1, s_2 \in S$ be such that S is generated by s_1, s_2, i.e., every $s \in S$ can be written as $s = n_1 s_1 + n_2 s_2$ for some $n_1, n_2 \in \mathbb{Z}$. Then i is completely determined by $i(1,0) = s_1$, $i(0,1) = s_2$; namely $i(k,l) = ks_1 + ls_2$. In general, many choices for s_1, s_2 are possible; often they can be chosen orthogonal: see Figure 4.

We define $S^{(0)} = \mathbb{Z}^2$ and let $K^{(0)} = K \subset \mathbb{Z}^2$ be an arbitrary sampling element. We define $S^{(p)}, K^{(p)}$ recursively as

$$S^{(p)} = i(S^{(p-1)}), \quad K^{(p)} = i(K^{(p-1)}), \quad p \geq 1. \tag{6.1}$$

Note that $S^{(1)} = S$. By induction, it follows easily that every $S^{(p)}$ is a group which is isomorphic to $\mathbb{Z}^2$ and that

$$K^{(p)} \subset S^{(p)}. \tag{6.2}$$

Lemma 6.2. *For every $p \geq 0$,*

(a) if $(0,0) \in K$, then $(0,0) \in K^{(p)}$;

(b) if $K \oplus S = \mathbb{Z}^2$, then $K^{(p)} \oplus S^{(p+1)} = S^{(p)}$;

(c) if the sets K_s with $s \in S$ are mutually disjoint, then the sets $K_s^{(p)}$ with $s \in S^{(p+1)}$ are mutually disjoint as well;

(d) if K is (shape-) symmetric then $K^{(p)}$ is (shape-) symmetric as well.

Proof. (a),(c),(d) are trivial, and we shall only prove (b). The proof goes by induction. Let

$$K^{(p-1)} \oplus S^{(p)} = S^{(p-1)}$$

for some $p \geq 1$. Then

$$\begin{aligned} K^{(p)} \oplus S^{(p+1)} &= \bigcup \{k' + s' \mid k' \in K^{(p)},\ s' \in S^{(p+1)}\} \\ &= \bigcup \{i(k) + i(s) \mid k \in K^{(p-1)},\ s \in S^{(p)}\} \\ &= \bigcup \{i(k+s) \mid k \in K^{(p-1)},\ s \in S^{(p)}\} \\ &= i(\bigcup \{k+s \mid k \in K^{(p-1)},\ s \in S^{(p)}\}) \\ &= i(K^{(p-1)} \oplus S^{(p)}) \\ &= i(S^{(p-1)}) = S^{(p)}. \end{aligned}$$

This proves the result. ∎

Let, for $p \geq 0$, the sampling operator $\sigma^{(p,p-1)} : \mathrm{Fun}(S^{(p-1)}) \to \mathrm{Fun}(S^{(p)})$ be defined by (2.3), i.e.

$$\sigma^{(p,p-1)}(F)(s) = \sup\{F(x) \mid x \in K_s^{(p-1)}\},$$

where $F \in \mathrm{Fun}(S^{(p-1)})$ and $s \in S^{(p)}$. Thus $\sigma^{(p,p-1)}$ is the operator which brings us from the $(p-1)$'th to the p'th level. We can go immediately from the 0'th to the p'th level by means of the composed sampling operator

$$\sigma^{(p,0)} = \sigma^{(p,p-1)}\sigma^{(p-1,p-2)}\ldots\sigma^{(1,0)}.$$

This requires the sampling set $S^{(p)}$ and the sampling element $K^{(p-1)}\oplus K^{(p-2)}\oplus\ldots\oplus K^{(1)}\oplus K^{(0)}$. From Lemma 6.2(b) we deduce that, if $K \oplus S = \mathbb{Z}^2$, then $K^{(m-1)} \oplus S^{(m)} = S^{(m-1)}$ for $m \leq p$, and hence,

$$K^{(p-1)} \oplus K^{(p-2)} \oplus \ldots \oplus K^{(0)} \oplus S^{(p)} = \mathbb{Z}^2.$$

There is an alternative way to represent the sampled images $\sigma^{(p,0)}(F)$. Using the isomorphism $i : \mathbb{Z}^2 \to S$ one can construct a lattice isomorphism $\pi : \mathrm{Fun}(S) \to \mathrm{Fun}(\mathbb{Z}^2)$ as follows:

$$\pi(G)(x) = G(i(x)), \quad \text{for } G \in \mathrm{Fun}(S),\ x \in \mathbb{Z}^2.$$

Then $\pi\sigma$ maps $\mathrm{Fun}(\mathbb{Z}^2)$ into itself. Now repetition of the sampling procedure defined by K and S amounts to iteration of the mapping $\pi\sigma$. An example is presented in Figure 8.

7. Operations on sampled images

In this section we denote the closing and opening respectively with the symbols $\bullet$ and $\circ$. So instead of F^K (resp. F_K) we write $F \bullet K$ (resp. $F \circ K$).

We denote by $\oplus$ and $\ominus$ respectively the dilation and erosion on $\mathrm{Fun}(\mathbb{Z}^2)$, and by $\overline{\oplus}$ and $\overline{\ominus}$ the dilation and erosion on $\mathrm{Fun}(S)$. Analogously, $\overline{\bullet}$ and $\overline{\circ}$ will denote the closing and opening on $\mathrm{Fun}(S)$. If $C \subset S$ then both the expressions $F \oplus C$ for $F \in \mathrm{Fun}(\mathbb{Z}^2)$, and $G \,\overline{\oplus}\, C$ for $G \in \mathrm{Fun}(S)$ make sense.

Lemma 7.1. *Assume* $(0,0) \in K$, *and let* $F_i \in \mathrm{Fun}(\mathbb{Z}^2)$ *for* $i \in I$. *Then*

$$\bigwedge_{i\in I} \sigma(F_i) \leq \sigma(\bigwedge_{i\in I}(F_i \oplus \check{K})).$$

PROOF. Since $(0,0) \in K$, we have $s \in K_s$, for $s \in S$, hence

$$\begin{aligned}\sigma(\bigwedge_{i\in I}(F_i \oplus \check{K}))(s) &= \sup\{\bigwedge_{i\in I}(F_i \oplus \check{K})(x) \mid x \in K_s\}\\ &\geq \bigwedge_{i\in I}(F_i \oplus \check{K})(s)\\ &= \bigwedge_{i\in I} \sigma(F_i)(s).\end{aligned}$$

This proves the assertion. ∎

Proposition 7.2. *Assume that $(0,0) \in K$, let $F \in \mathrm{Fun}(\mathbb{Z}^2)$ and $C \subset S$. For every $F \in \mathrm{Fun}(\mathbb{Z}^2)$,*

$$\sigma(F \oplus C) = \sigma(F) \overline{\oplus} C \tag{7.1}$$

$$\sigma(F \ominus C) \leq \sigma(F) \overline{\ominus} C \leq \sigma((F \oplus \check{K}) \ominus C) \tag{7.2}$$

$$\sigma(F \circ C) \leq \sigma(F) \overline{\circ} C \leq \sigma((F \oplus \check{K}) \circ C) \tag{7.3}$$

$$\sigma(F \bullet C) \leq \sigma(F) \overline{\bullet} C \leq \sigma((F \oplus \check{K}) \bullet C) \tag{7.4}$$

$$\rho(F) \oplus C \leq \rho(F \oplus C) \tag{7.5}$$

$$\rho(F \ominus C) \leq \rho(F) \ominus C \leq \rho((F \oplus \check{K}) \ominus C) \tag{7.6}$$

$$\rho(F) \circ C \leq \rho((F \oplus \check{K}) \circ C) \tag{7.7}$$

$$\rho(F) \bullet C \leq \rho((F \oplus \check{K}) \bullet C). \tag{7.8}$$

Proof. Let $F \in \mathrm{Fun}(\mathbb{Z}^2)$ and $C \subset S$. First we note that for $s \in S$, $\sigma(F_s) = \sigma(F)_s$. The first identity expresses the fact that σ is dilation which commutes with S-translations.
We now prove (7.2):

$$\begin{aligned}\sigma(F \ominus C) &= \sigma\Big(\bigwedge_{c \in C} F_{-c}\Big) \leq \bigwedge_{c \in C} \sigma(F_{-c}) \\ &= \bigwedge_{c \in C} \sigma(F)_{-c} = \sigma(F) \overline{\ominus} C.\end{aligned}$$

To prove the second inequality in (7.2) we use Lemma 7.1:

$$\begin{aligned}\sigma(F) \overline{\ominus} C &= \bigwedge_{c \in C} \sigma(F)_{-c} = \bigwedge_{c \in C} \sigma(F_{-c}) \\ &\leq \sigma\Big(\bigwedge_{c \in C} (F_{-c} \oplus \check{K})\Big) = \sigma((F \oplus \check{K}) \ominus C).\end{aligned}$$

Now (7.3) and (7.4) follow immediately.
We prove (7.5):

$$\begin{aligned}\rho(F \oplus C) &= \dot{\sigma}\sigma(F \oplus C) = \dot{\sigma}(\sigma(F) \overline{\oplus} C) \\ &= \dot{\sigma}\Big(\bigvee_{c \in C} \sigma(F)_c\Big) \geq \bigvee_{c \in C} [\dot{\sigma}\sigma(F)]_c \\ &= \rho(F) \oplus C.\end{aligned}$$

(7.6) follows in a similar way.
To prove (7.7) and (7.8) one can use (7.5) and (7.6).

∎

If $\overline{\psi}$ is an operator on $\mathrm{Fun}(S)$ then

$$\psi = \dot{\sigma}\overline{\psi}\sigma \tag{7.9}$$

is an operator on $\mathrm{Fun}(\mathbb{Z}^2)$. Conversely, if ψ is an operator on $\mathrm{Fun}(\mathbb{Z}^2)$, then

$$\overline{\psi} = \sigma\psi\dot{\sigma} \tag{7.10}$$

is an operator on $\mathrm{Fun}(S)$. One can show that ψ is increasing if and only if $\overline{\psi}$ is increasing, and that ψ is an S-operator if and only if $\overline{\psi}$ is an S-operator.

Proposition 7.3.
(a) *If* $\overline{\psi}$ *is a closing on* $\mathrm{Fun}(S)$ *then* ψ *is a closing on* $\mathrm{Fun}(\mathbb{Z}^2)$.
(b) *If* ψ *is an opening on* $\mathrm{Fun}(\mathbb{Z}^2)$ *then* $\overline{\psi}$ *is an opening on* $\mathrm{Fun}(S)$.

PROOF. We only prove (a). Obviously

$$\psi = \dot{\sigma}\overline{\psi}\sigma \geq \dot{\sigma}\sigma \geq \mathbf{id},$$

and therefore $\psi^2 \geq \psi$. On the other hand,

$$\psi^2 = \dot{\sigma}\overline{\psi}\sigma\dot{\sigma}\overline{\psi}\sigma \leq \dot{\sigma}\overline{\psi}^2\sigma = \dot{\sigma}\overline{\psi}\sigma = \psi.$$

∎

8. Results and discussion

Each image in a multiresolution sequence is usually generated from its predecessor by reducing both the resolution and the sample density. In the morphological sampling scheme this is done by sampling the dilation. This generation process can be implemented efficiently as a single operation by assigning to each sample point at the coarser sample grid the maximum of the function values at neighbouring points in the finer sample grid.

A hierarchical decomposition of an image into a set of size limited components can be obtained by subtracting successive members of a multiresolution image sequence. This is the basic principle of the popular DOG, DOLP or Laplacian pyramids [1,2,3]. The subtraction of images with different resolution and sample densities requires an interpolation procedure that maps an image represented on a coarse sampling grid into one represented on a finer sampling grid. In the dilation sampling scheme the interpolated sample points can be obtained by the adjoint erosion. This interpolation process can be performed in a single operation by assigning to each sample point at the finer sample grid the minimum of the function values at neighbouring sample points in the coarser sample grid.

In the dilation sampling scheme there is a direct mapping between the original image and every other member of the hierarchical sequence. This is a result of the fact that the sequential application of dilations is equivalent to a single dilation.

These results are in contrast to Haralick's morphological samling scheme (which applies successive openings or closings to reduce the resolution and dilations or closings to interpolate sample points) where no simple mapping between neighbouring samples in successive image representations exists. Moreover, in Haralick's scheme there is no direct mapping between the original image and every other member of the hierarchical sequence. This is a result of the fact that the sequential application of openings or closings with structuring elements of increasing size is not equivalent to a single opening or closing (see [10, Chapter 10]).

The results of several different reconstruction operators are shown in Figures 5-7. Figure 5(b) illustrates that the use of a sampling element which is only shape-symmetric results in the asymmetric reconstruction of small circular objects in Figure 5(a). Figures 5(c) and 5(d) show that the accuracy of the reconstruction depends on the size of the sampling element and the position of the image details relative to the sampling grid. Figure 5(e) shows that the intersection of the reconstruction with sampling elements of different sizes gives a better approximation of the original image then either of the reconstructions by itself. This is also an immediate consequence of the theoretical result that either of the reconstruction operators involved is a closing. Figures 5(f) and 5(g) were obtained by sampling with $K^{(5)}$ followed by a closing with respectively $K^{(5)}$ and $A^{(5)} = K^{(3)}$. Although both reconstructions use the same dilation sampling element, the reconstruction in 5(f) is much coarser than the one in 5(g). This also follows from Proposition 4.3. These examples show that too much overlap of the sampling elements used by the reconstruct*ing* operator results in a reconstruction that contains no details with negative contrast smaller than the sampling element. As a comparison Figures 5(h) and 5(i) show respectively the results of Haralick's opening-sampling with dilation-reconstruction and closing-sampling with closing-reconstruction.

Figures 6 and 7 show the results of the different reconstruction operations on respectively the red and blue components of a colour picture of the Golden Gate Bridge in San Francisco. Note that the bridge has a positive contrast in the red component and a negative contrast in the blue component. These examples illustrate the effects of different dilation sampling reconstructions on image details with opposite contrast.

Figure 8 shows a morphological pyramid that was obtained by repeated application of the dilation sampling procedure. Note that (i) progressively larger details with negative contrast are filtered and (ii) a progressively coarser approximation of the remaining details is obtained with a progressive increase in the number of iteration steps.

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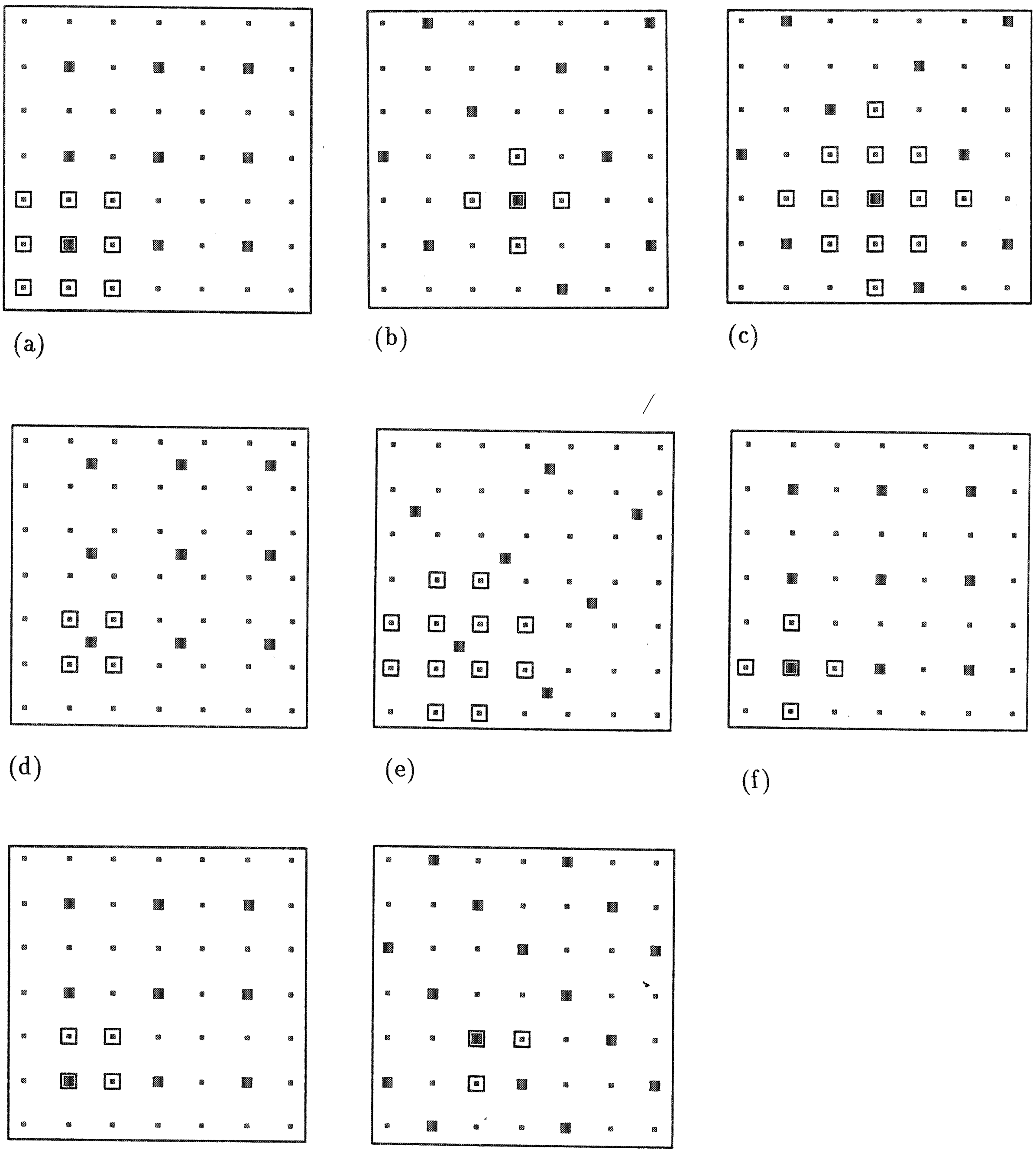


Figure 1. *Some examples of regular sampling strategies. The set* $X = \{ \cdot \}$, *the set* $S = \{ \blacksquare \}$, *and the sampling element* $K = \{ \square \}$.
In each case the origin of the sample element coincides with an element of S.
(a)-(f) *Examples of symmetric sampling elements.* (f) *Example where the covering assumption is not satisfied.* (g) *Example of a shape-symmetric sampling element.* (g) *Example of a non-symmetric sampling element.*

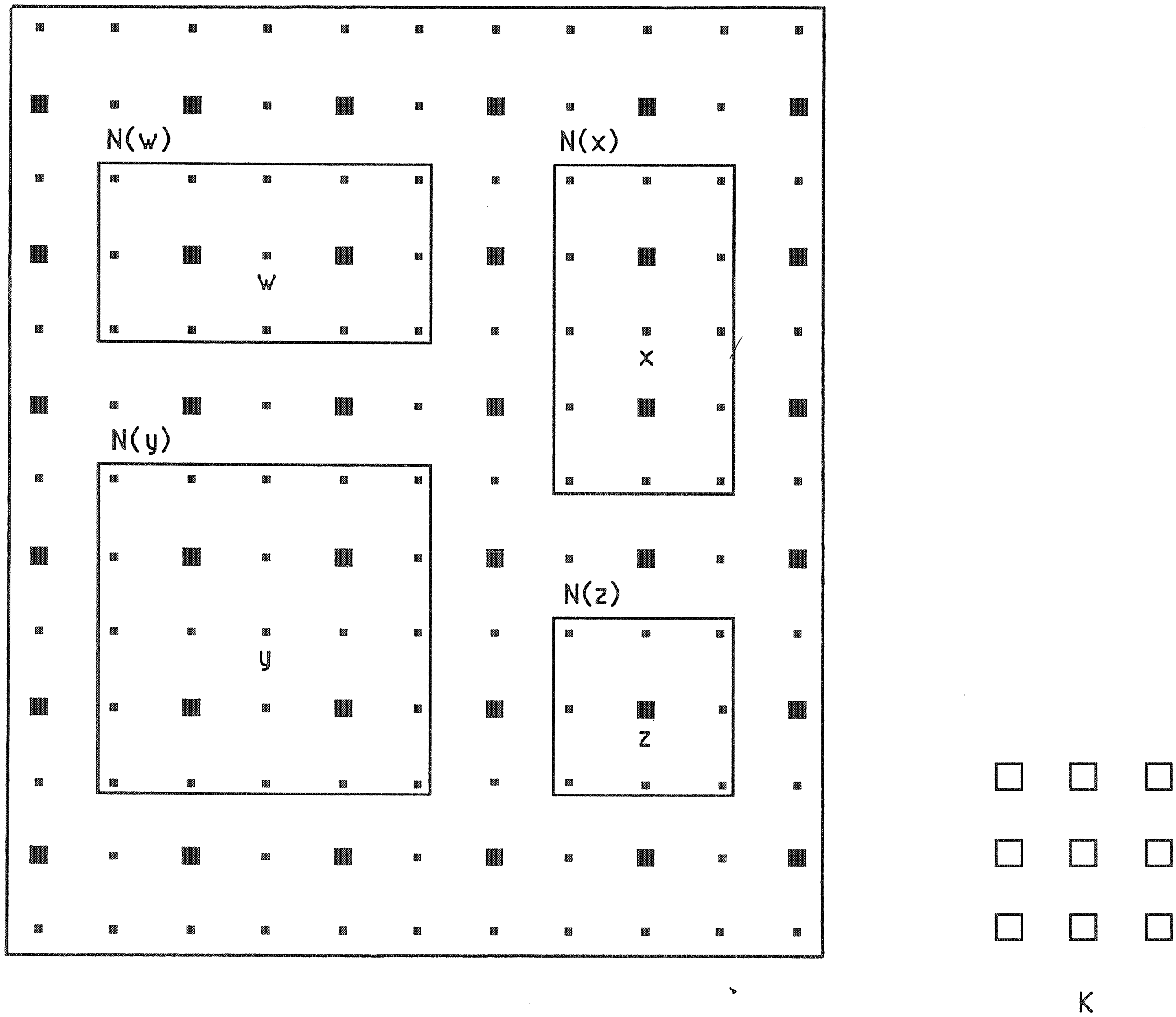
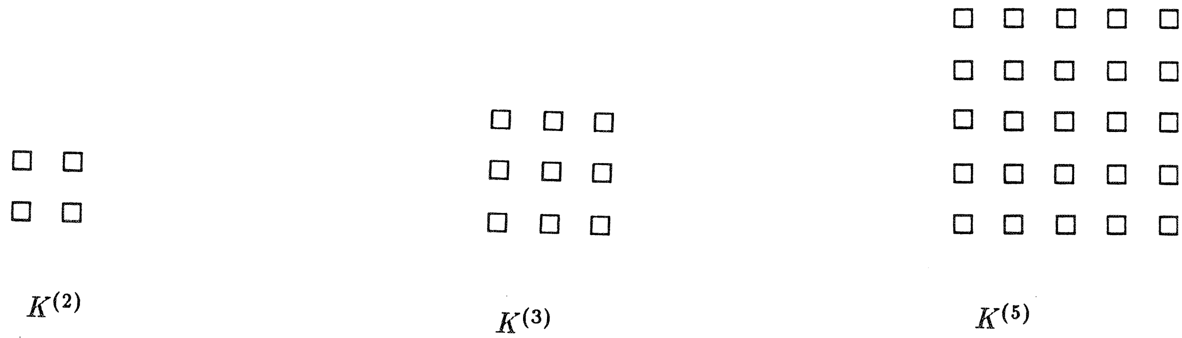
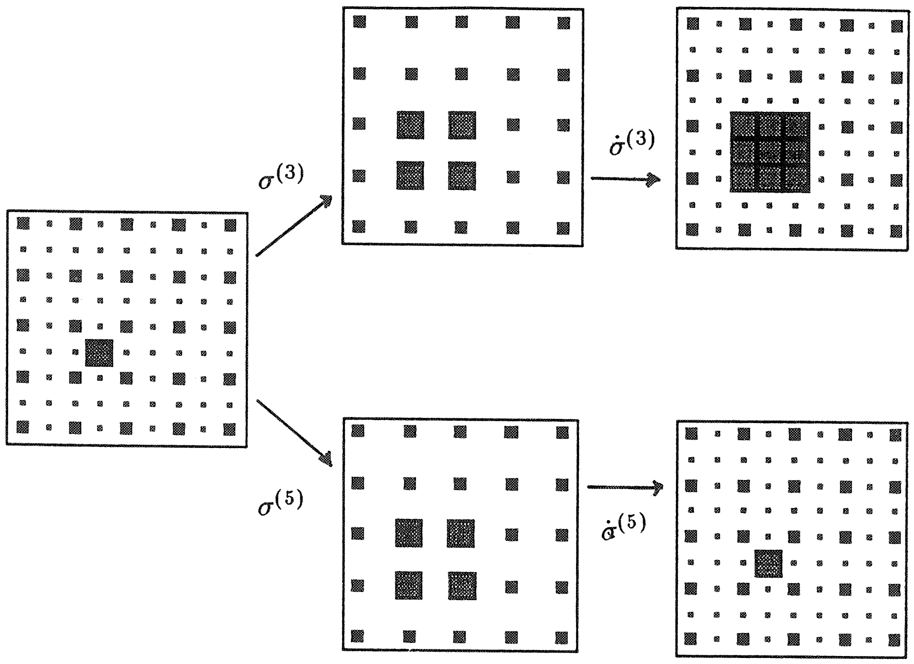


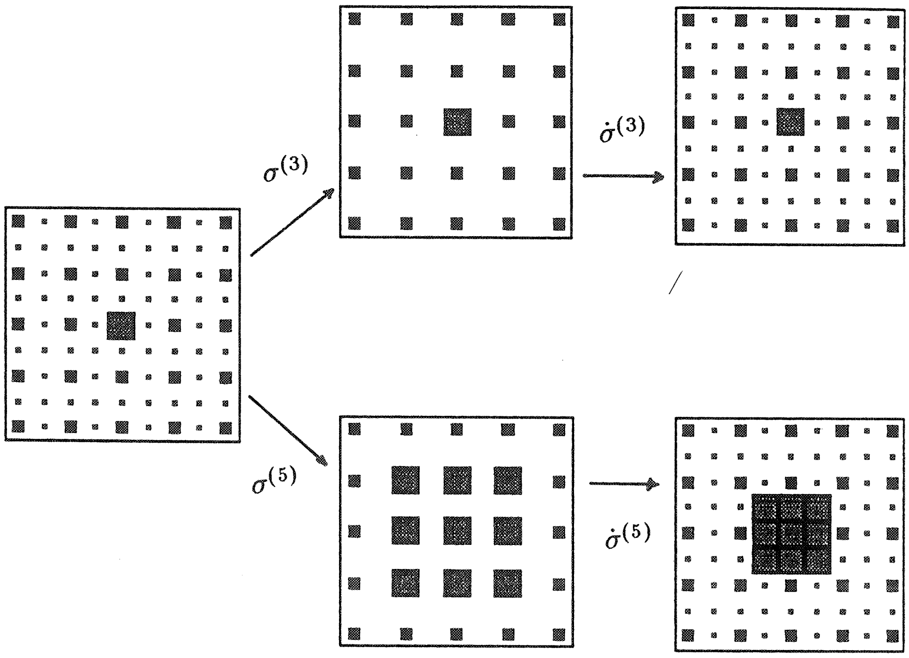
Figure 2. *The size and shape of the K-neighbourhood $N(x)$ of an element x depends on the actual choice of x. All symbols have the same meaning as in Figure 1.*



(a)

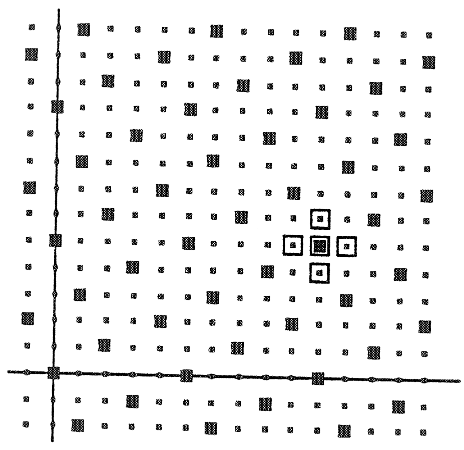


(b)



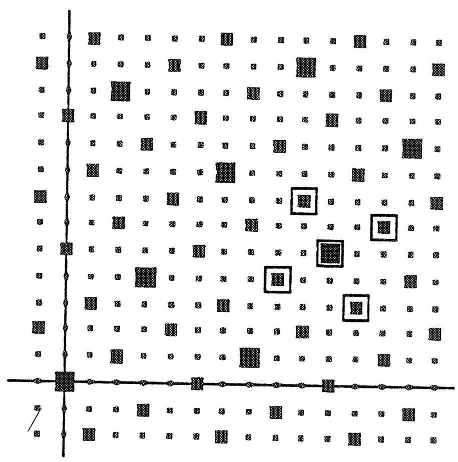
(c)

Figure 3. *The effects of different reconstruction operators. The same symbols as in Figure 1 have been used. The images consist of grey-shaded pixels.*
(a) *One shape-symmetric ($K^{(2)}$) and two symmetric ($K^{(3)}$ and $K^{(5)}$) sampling elements.*
(b) *If x has odd coordinates then the reconstruction with $K^{(5)}$ returns $\{x\}$ whereas the reconstruction with $K^{(3)}$ returns $K_x^{(3)}$.*
(c) *If x has even coordinates ($x \in S$) then the reconstruction with $K^{(5)}$ returns $K_x^{(3)}$ whereas the reconstruction with $K^{(3)}$ returns $\{x\}$.*

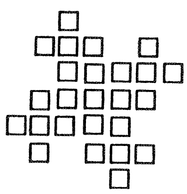


$S^{(0)} = \overline{\mathbb{Z}} = \{ \cdot , \blacksquare \}$, $S^{(1)} = \{ \blacksquare \}$,

$K^{(0)} = \{ \square \}$.

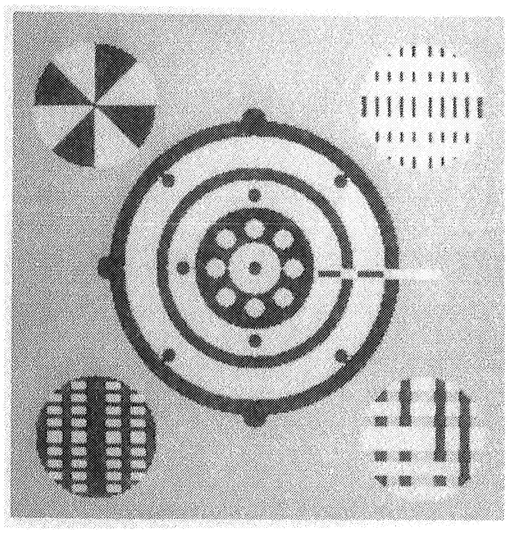


$S^{(2)} = \{ \blacksquare \}$, $K^{(1)} = \{ \square \}$.

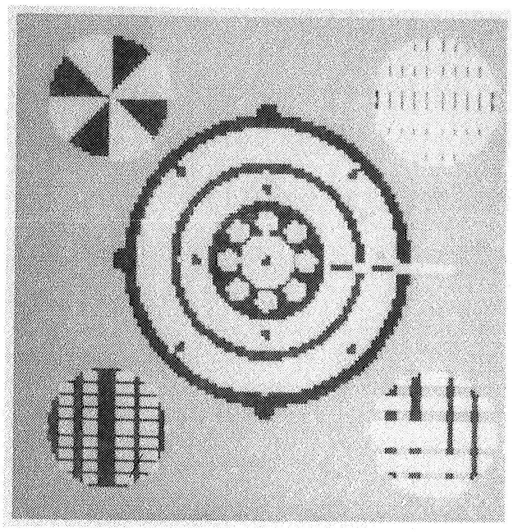


$K^{(0)} \oplus K^{(1)}$

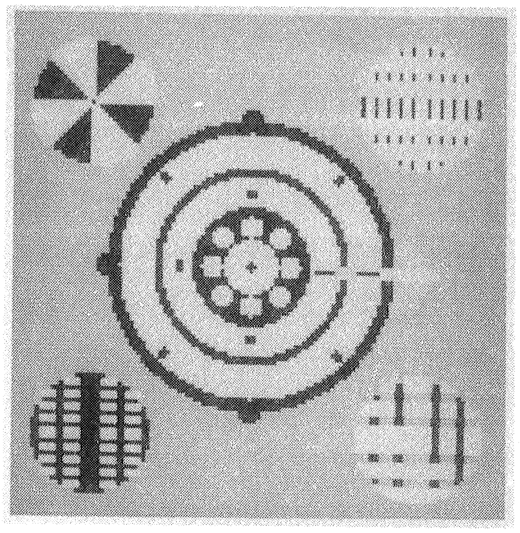
Figure 4. *An example of repeated sampling for the sampling strategy depicted in Figure 1(b). The group isomorphism $i : \overline{\mathbb{Z}} \rightarrow S$ is defined by $s_1 = i(1,0) = (2,1)$ and $s_2 = i(0,1) = (-1,2)$. Note that s_1 and s_2 are chosen orthogonal.*



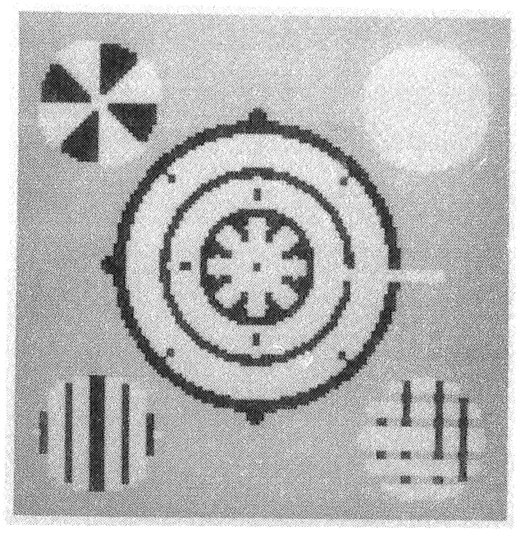
(a)



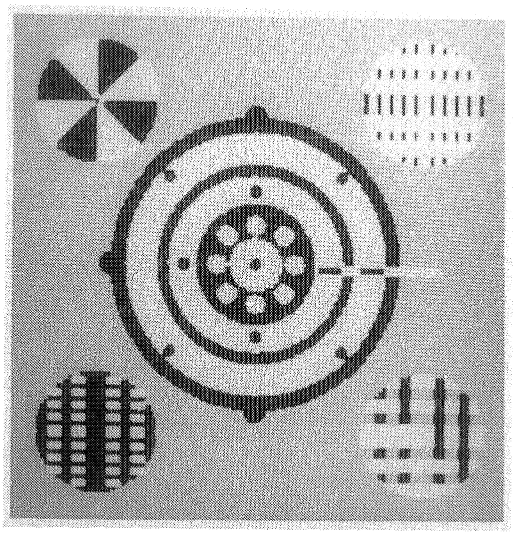
(b)



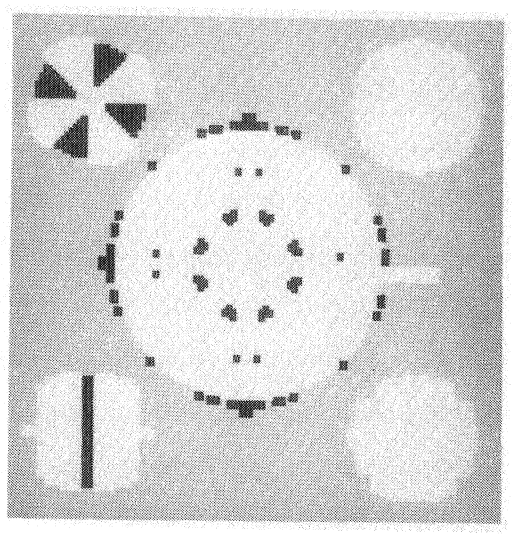
(c)



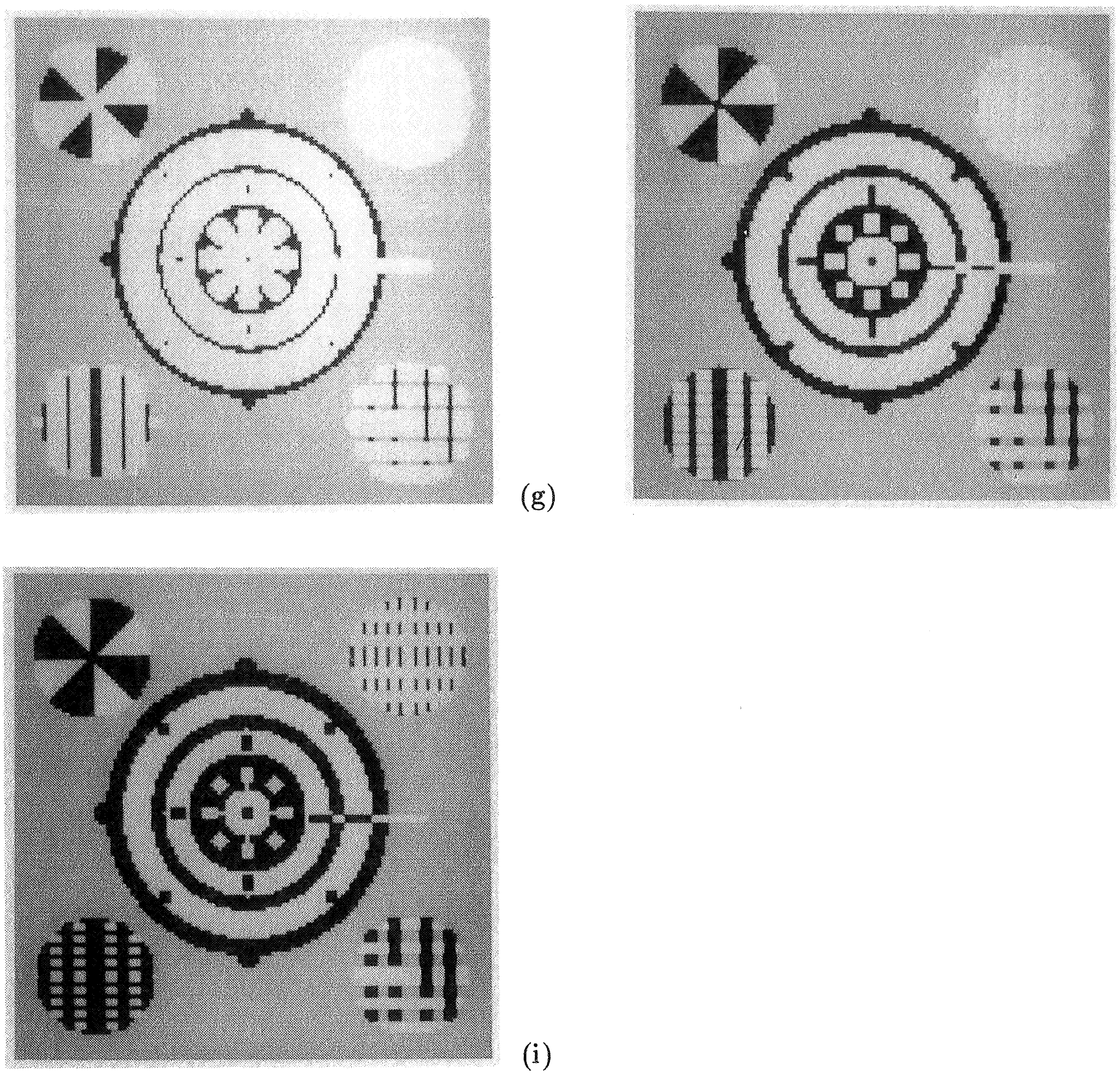
(d)



(e)



(f)



(g) (h) (i)

Figure 5. *The effects of different reconstruction operations (compare Example 4.6). In all examples a regular sampling grid was used with the group isomorphism $i : \overline{\mathbb{Z}} \to S$ defined by $i(1,0) = s_1 = (2,0)$ and $i(0,1) = s_2 = (0,2)$. For the sampling elements $K^{(2)}, K^{(3)}, K^{(5)}$, see Figure 3(a).*

(a) Original image.

(b) Reconstruction with $\rho^{(2)}$.

(c) Reconstruction with $\rho^{(3)}$.

(d) Reconstruction with $\rho^{(5)}$.

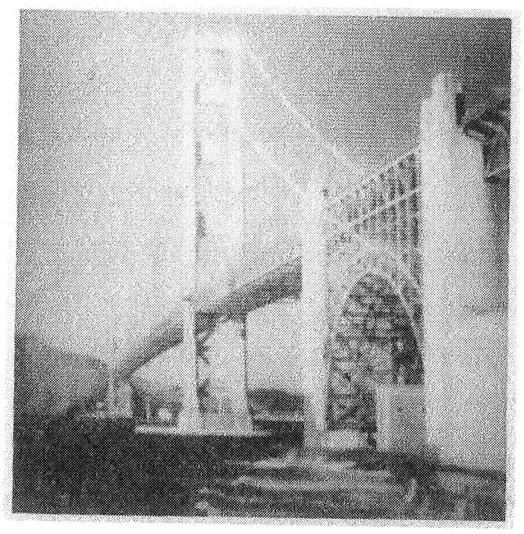
(e) The intersection of $\rho^{(2)}, \rho^{(3)}$ and $\rho^{(5)}$.

(f) Reconstruction with $\rho_k^{(5)}$.

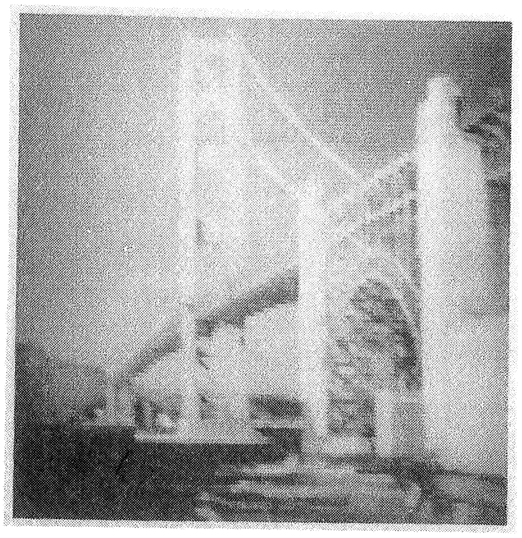
(g) Reconstruction with $\rho_a^{(5)}$.

(h) Haralick's opening-sampling and dilation-reconstruction with $K^{(3)}$.

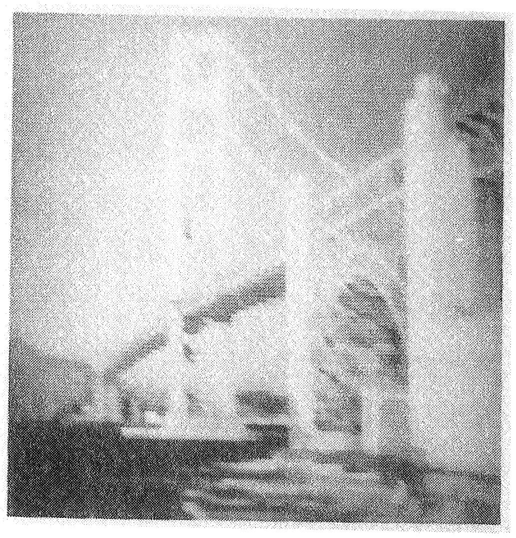
(i) Haralick's closing-sampling and closing-reconstruction with $K^{(3)}$.



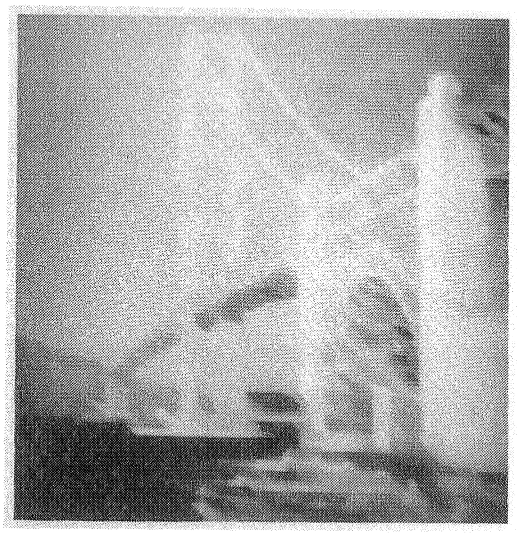
(a)



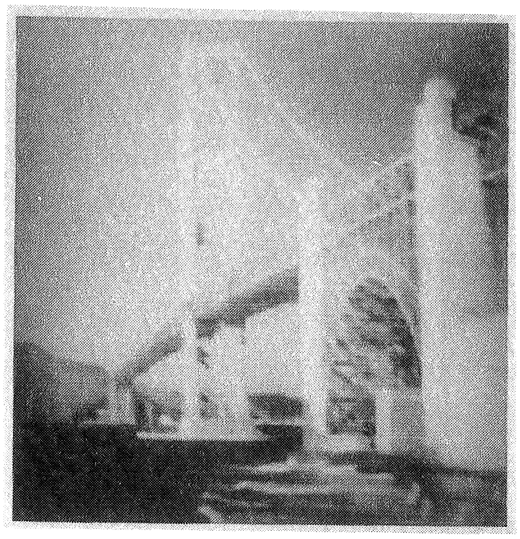
(b)



(c)

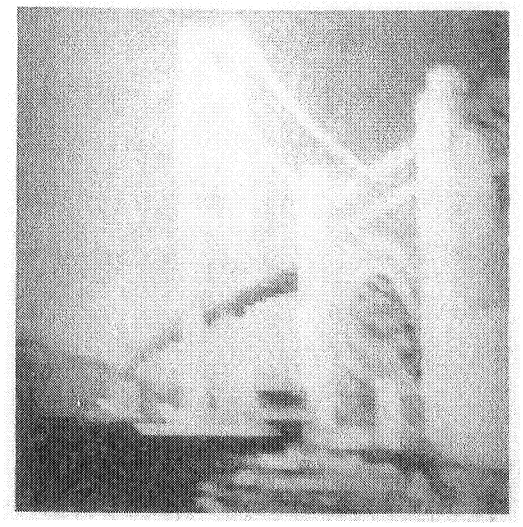


(d)

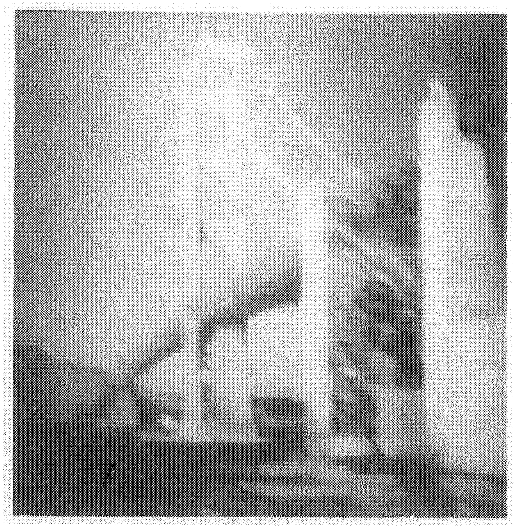


(e)

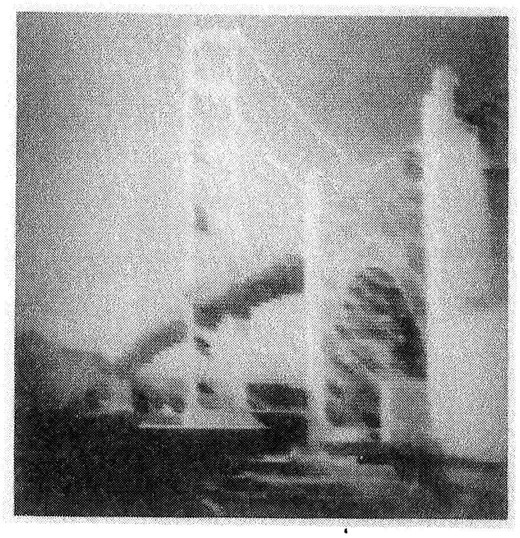
(f)



(g)



(h)



(i)

Figure 6. *As Figure 5 for the red component of a colour picture of the Golden Gate Bridge in San Francisco.*

(a) Original image.

(b) Reconstruction with $\rho^{(2)}$.

(c) Reconstruction with $\rho^{(3)}$.

(d) Reconstruction with $\rho^{(5)}$.

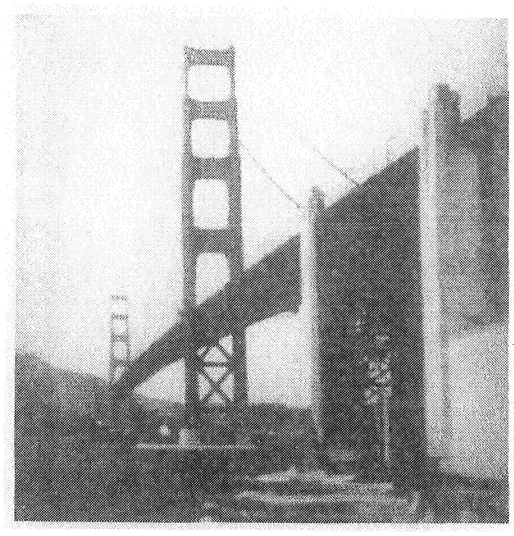
(e) The intersection of $\rho^{(2)}, \rho^{(3)}$ *and* $\rho^{(5)}$.

(f) Reconstruction with $\rho_k^{(5)}$.

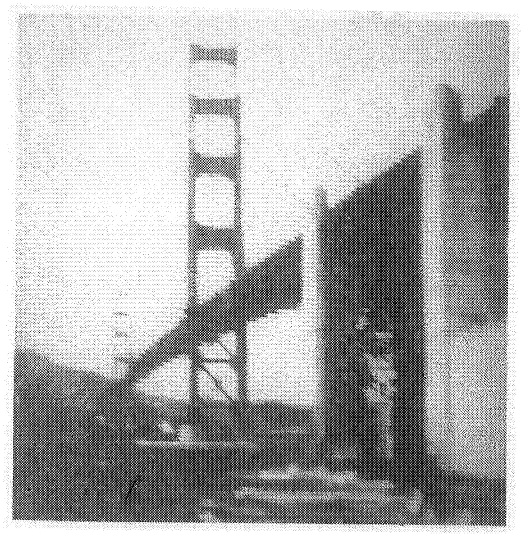
(g) Reconstruction with $\rho_a^{(5)}$.

(h) Haralick's opening-sampling and dilation-reconstruction with $K^{(3)}$.

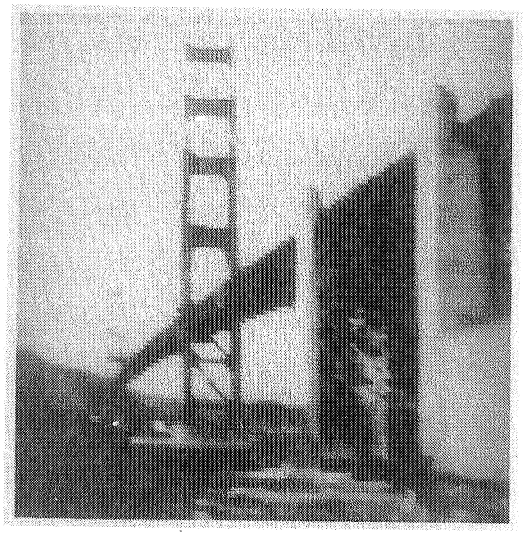
(i) Haralick's closing-sampling and closing-reconstruction with $K^{(3)}$.



(a)



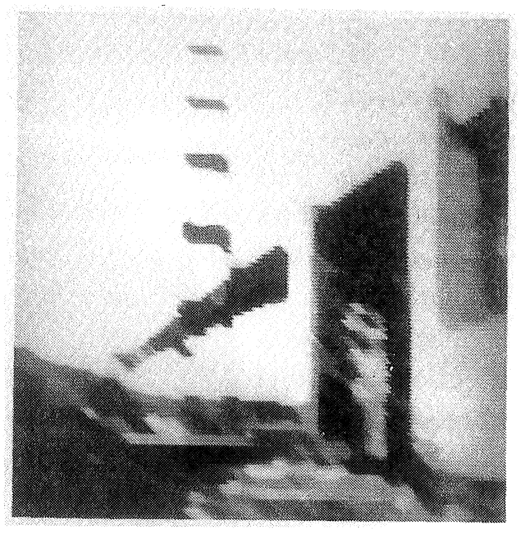
(b)



(c)

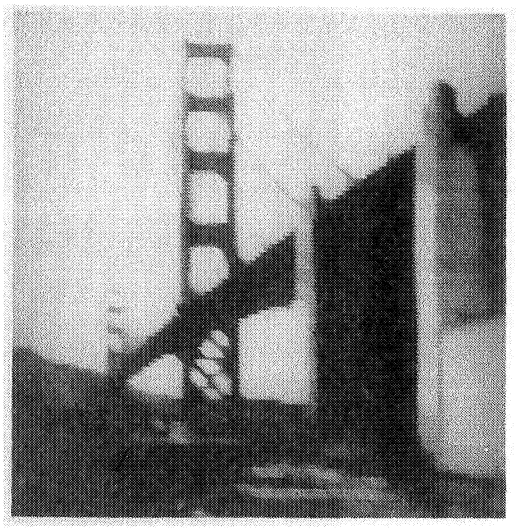
(d)

(e)



(f)

(g)



(h)

(i)

Figure 7. *As Figure 5 for the blue component of a colour picture of the Golden Gate Bridge in San Francisco.*
(a) Original image.
(b) Reconstruction with $\rho^{(2)}$.
(c) Reconstruction with $\rho^{(3)}$.
(d) Reconstruction with $\rho^{(5)}$.
(e) The intersection of $\rho^{(2)}, \rho^{(3)}$ *and* $\rho^{(5)}$.
(f) Reconstruction with $\rho_k^{(5)}$.
(g) Reconstruction with $\rho_a^{(5)}$.
(h) Haralick's opening-sampling and dilation-reconstruction with $K^{(3)}$.
(i) Haralick's closing-sampling and closing-reconstruction with $K^{(3)}$.

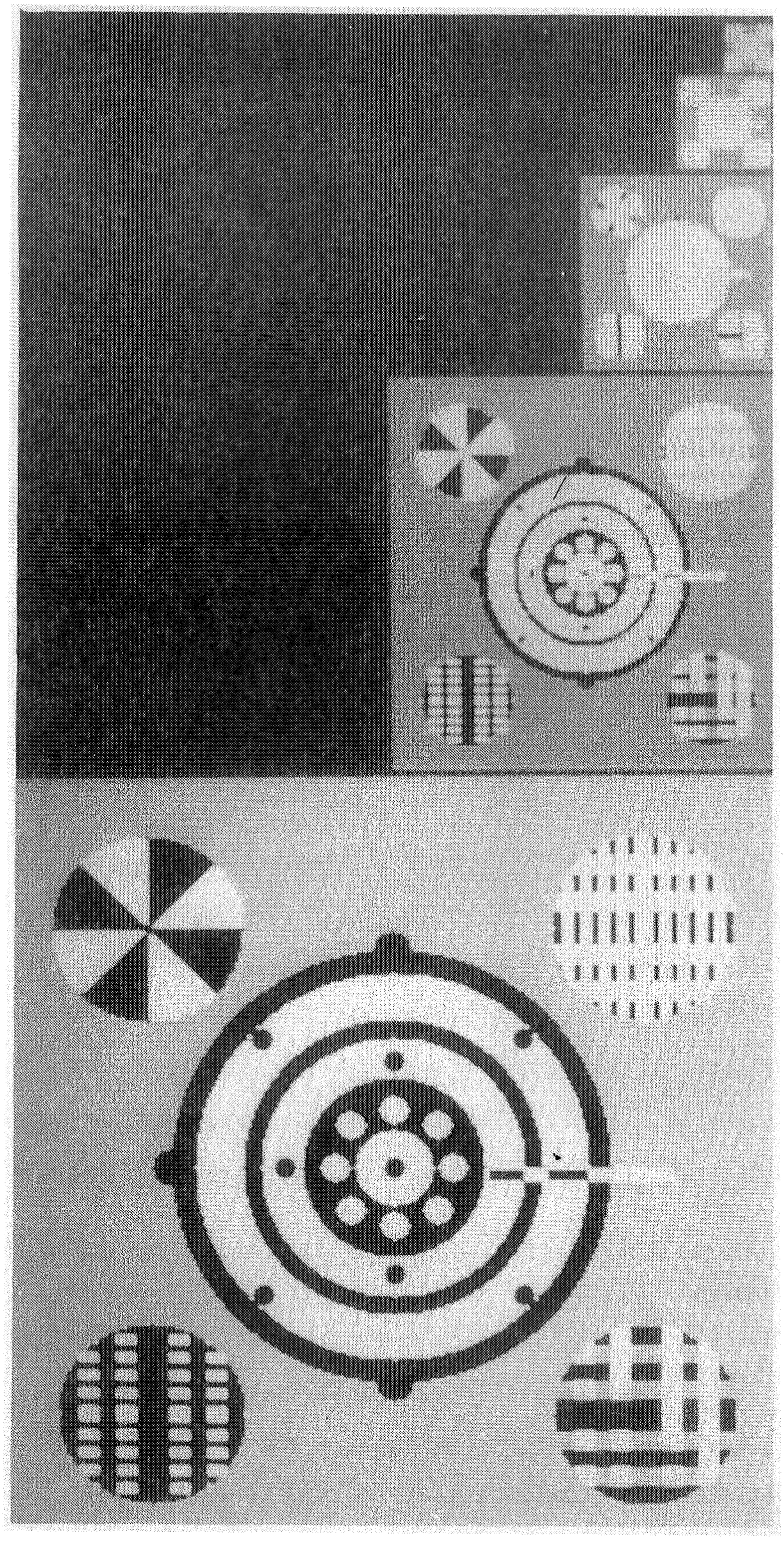


Figure 8. *An example of a morphological pyramid obtained by repeated application of the dilation sampling with sampling element $K^{(3)}$. The group isomorphism $i : \overline{\mathbb{Z}} \to S$ is defined by $i(1,0) = s_1 = (2,0)$ and $i(0,1) = s_2 = (0,2)$.*