Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

A.M. Cohen, E.E. Shult

Affine polar spaces

Department of Analysis, Algebra and Geometry Report AM-R8909 July

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

Affine Polar Spaces

Arjeh M. Cohen

CWI,
Kruislaan 413,
1098 SJ Amsterdam,
The Netherlands
&

Ernest E. Shult †

Mathematics Department
Cardwell Hall
Kansas State University
Manhattan, KA 66502
USA.

ABSTRACT

Affine polar spaces are polar spaces from which a hyperplane (that is a proper subspace meeting every line of the space) has been removed. These spaces are of interest as they constitute quite natural examples of 'locally polar spaces'. A characterization of affine polar spaces (rank at least 4) is given as locally polar spaces whose planes are affine. Moreover, the affine polar spaces are fully classified in the sense that all hyperplanes of the fully classified polar spaces (rank at least 3) are determined.

Key Words & Phrases: polar spaces, buildings, incidence systems.
1980 Mathematics Subject Classification: 51B25, 22E20, 05B25.

Introduction.

In 1959, VELDKAMP [9] initiated the synthetic study of geometries induced on the set of absolute points, lines, planes, etc. with respect to a polarity, and named the subject polar geometry. After subsequent work of TITS [7], BUEKENHOUT & SHULT [2] and BUEKENHOUT & SPRAGUE [3] a somewhat larger class of point, line geometries emerged which could be characterized by the beautiful axiom

> If p is a point and L a line, then the set of points incident with L and collinear with p is either a singleton or the set of all points incident with L,

which we shall quote as the 'one or all' axiom. An incidence system $(P, \mathbf{L})$ [i.e., a pair consisting of a set P (of *points*) and set $\mathbf{L}$ (of *lines*) together with a relation between them, called incidence, such that each line is incident with at least 2 points] is called a *polar space* if the 'one or all' axiom is satisfied. An incidence system is called *nondegenerate* if no point is collinear with all others, and it is called *singular* if any two of its points are collinear. If X is a subset of the point set P of the incidence system $(P, \mathbf{L})$ and $L \in \mathbf{L}$, we denote by $X(L)$ the set of points in X incident to L, and by $\mathbf{L}(X)$ the set of all lines in $\mathbf{L}$ incident to at least two points of X. Thus, $\mathbf{L}(X) = \{L \in \mathbf{L} \mid |X(L)| > 1\}$. Restricting incidence of $(P, \mathbf{L})$, we can regard $(X, \mathbf{L}(X))$ as an incidence system. If each point incident to a line in $\mathbf{L}(X)$ belongs to X, we say that X is a *subspace* of $(P, \mathbf{L})$. A subspace of a polar space is again a polar space. The *singular rank* of an incidence system $(P, \mathbf{L})$ is the maximal number n (possibly ∞) for which there exists a chain of distinct subspaces $\varnothing \neq X_0 \subset X_1 \subset \cdots \subset X_n$ such that $(X_i, \mathbf{L}(X_i))$ is singular for each i $(0 \le i \le n)$, with the understanding that $n = -1$ if $X = \varnothing$. The *rank* of a polar space $(P, \mathbf{L})$ is the number $n+1$ where n is its singular rank. By definition, this number is 0 if $P = \varnothing$, and 1 if $P \neq \varnothing$ but $\mathbf{L} = \varnothing$. The main characterization

† The work was partially supported by the National Science Foundation, U.S.A.

Report AM-R8909
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

results hinted to above imply that if (P, L) is a nondegenerate polar space of finite (singular) rank ≥ 3, then it is one of a known list of examples (cf. BUEKENHOUT & SPRAGUE [3]). In this paper we shall limit ourselves to the situation in which all lines are *thick* (i.e., are incident to at least three points); the list of "thick" examples can be found in TITS [7]. (See also section 5 below.)

One of the main tools in Veldkamp's original approach is the notion of a *hyperplane*, a proper subspace with the property that every line is incident to (at least) one of its points; it did not reoccur in the subsequent papers quoted above. It is the goal of this paper to study the hyperplanes B of nondegenerate polar spaces (P, L) of finite rank $n \geq 3$ whose lines are thick, as well as to synthetically describe the incidence systems $(A, \mathrm{L}(A))$ induced on their complements $A = P \setminus B$.

Section 2 gives some properties of these hyperplane complements. The interest in these 'affine polar spaces' $(A, \mathrm{L}(A))$ arose from the abundance of properties analogous to those of the usual affine spaces, i.e., the geometries induced on the complements of hyperplanes in projective spaces. Notably, the fact that (classical) affine spaces Q are locally projective spaces (in the sense that, for each point $a \in Q$, the incidence system whose points are the lines of Q on a and whose lines are the planes on a is a projective space), corresponds to the property of $(A, \mathrm{L(A)})$ being a locally polar space. This draws attention to the question whether all spaces that are locally polar can be classified. The analogous question for projective spaces has given rise to various characterizations (see, e.g., TEIRLINCK [6]). Adopting a stronger notion of locally polar spaces (P, L), namely that $x^{\perp}$ rather than $\mathrm{L}_x := \{L \in \mathrm{L} \mid x \in L\}$ carry the structure of a polar space for each $x \in P$, JOHNSON & SHULT [5] have obtained a satisfactory characterization without any assumptions on rank, thickness of lines, or degeneracy. (For a review of other results in this direction, see [loc. cit.].) In Section 3, we characterize affine polar spaces by an axiom system in which the locally polar space axiom is prominent (cf. 3.1.iii). Some of the proofs involved are based on ideas of J.I. Hall as displayed in the characterization of 'locally cotriangular graphs' of HALL & SHULT [4]. In the remainder of that Section 3, properties are derived from this axiom system, which alleviate the proof, to be found in Section 4, that the system of Section 3 is indeed a characterizing axiom system.

In view of the classification of nondegenerate polar spaces of rank at least 3, the classification of affine polar spaces comes down to the determination of all hyperplanes in well-known polar spaces. This determination is carried out in the last section (§5).

1. Hyperplanes.

Throughout this section, (P, L) is a nondegenerate polar space, all of whose lines are thick. Since there is at most one line incident with any two points (cf. BUEKENHOUT & SHULT [2]), a line is uniquely determined by the set of all points incident to it. We shall thus frequently view members of L as subsets of P. Also, if x, y are collinear and distinct, we shall write xy to denote the line containing them. We recall that a *hyperplane* B of (P, L) is a proper subspace such that $B(L) \neq \emptyset$ for each line $L \in \mathrm{L}$.

If $X \subseteq P$, we write $X^{\perp}$ for the subset P of points collinear to each point of X, and $x^{\perp} = \{x\}^{\perp}$ if $x \in P$. Furthermore, $\langle X \rangle$ denotes the subspace of (P, L) generated by X. (It exists since the intersection of an arbitrary collection of subspaces is again a subspace.)

1.1. Lemma. *Let B be a hyperplane of (P, L).*

(i) *If (P, L) has rank at least 2, then B is a maximal proper subspace and the collinearity graph induced on $P \setminus B$ is connected of diameter at most 3.*

(ii) *If X is a subspace not contained in B, then $X \cap B$ is a hyperplane of $(X, \mathrm{L}(X))$.*

Proof. (i). Take x, $y \in P \setminus B$. We show that $y \in \langle B, x \rangle$, the subspace generated by B and X. If x and y are collinear, then $y \in xy = \langle x, (xy) \cap B \rangle$ and we are done.
Assume that x and y are noncollinear. If $t \in \{x,y\}^{\perp} \setminus B$, then applying the above argument to x and t, and once more to t and y (instead of x and y), we are done, again.
Thus we remain with the case where $\{x,y\}^{\perp} \subseteq B$. By nondegeneracy of (P, L), there are noncollinear v, $w \in \{x,y\}^{\perp}$ (cf. BUEKENHOUT & SHULT [2]). Since lines are thick, there is $u \in xv \setminus \{x,v\}$. By the 'one or all' axiom, there must be a point $z \in (yw \cap u^{\perp}) \setminus \{y,w\}$. Now x, u, z, y is a path in $P \setminus B$ and we can finish by applying the first paragraph three times. The conclusion is that $y \in \langle B,x \rangle$ for each $y \in P \setminus B$, whence $\langle B,x \rangle = P$.

(ii) is obvious from the definition. □

1.2. Remark. In (i) it is necessary to assume nondegeneracy, the subspace $B = L_1$ being a counterexample in the polar space $(Q, \{L_1, L_2, L_3\})$ where L_1, L_2, L_3 are lines meeting in a fixed point of $Q = L_1 \cup L_2 \cup L_3$. Also thickness is readily seen to be crucial.

The lemma implies that B can have at most one *deep point*, i.e., a point incident with no line of $\mathrm{L}(P \setminus B)$, as we shall see from the corollary below. A subspace X is called *nondegenerate* if $(X, \mathrm{L}(X))$ is nondegenerate i.e., $\mathrm{rad}\,(X) := X \cap X^{\perp} = \varnothing$, and *degenerate* otherwise.

1.3. Corollary. *Let B be a hyperplane of (P, L).*

(i) *If B is degenerate, then $B = b^{\perp}$ and $\mathrm{rad}\,(B) = \{b\}$ for a unique point $b \in P$.*

(ii) *Any deep point of B is in $\mathrm{rad}\,(B)$; in particular, there is at most one deep point in B.*

Proof. (i). Suppose $b \in \mathrm{rad}\,(B)$. Then $B \subseteq b^{\perp}$. But $b^{\perp}$ is a hyperplane, so by (i) of the above lemma, $B = b^{\perp}$. Nondegeneracy of (P, L) yields $b^{\perp\perp} = \{b\}$, whence $\mathrm{rad}\,(B) = B \cap b^{\perp\perp} = \{b\}$.
(ii). Suppose $d \in B$ is incident with no line of $\mathrm{L}(P \setminus B)$. Then by thickness every line on d must have another point in B. Thus $d^{\perp} \subseteq B$. Using $B \neq P$ and the above lemma again, we get $B = d^{\perp}$, and we can finish as before. □

The above corollary brings to light that, for each point $x \in P$, the subspace $x^{\perp}$ is a hyperplane. We now recall the construction of a linear space - i.e., a space in which every pair of points lie on a unique line - on the set H of all hyperplanes of (P, L) in which (P, L) can be embedded. The basic idea is caught in the following lemma.

1.4. Lemma. *Suppose (P, L) has rank ≥ 3, and let B_1, B_2 be distinct hyperplanes.*

(i) *If $x \in B_1 \setminus B_2$, then $B_1 = \langle x, B_1 \cap B_2 \rangle$.*

(ii) *If $p \in P \setminus (B_1 \cap B_2)$, then there is at most one hyperplane containing p and $B_1 \cap B_2$.*

Proof. Observe that B_1 and B_2 are polar spaces. (i). If B_1 is nondegenerate then, as B_1 contains a line, lemma 1.1 applies, yielding that $B_1 \cap B_2$ is a maximal subspace of B_1, and the assertion holds.
If B_1 is degenerate, then, by the corollary, $B_1 = d^{\perp}$ for some $d \in x^{\perp}$.
Suppose $d \notin B_2$. Since every line on d has a point in B_2, whence in $B_1 \cap B_2$, we have $B_1 = \langle B_1 \cap B_2, d \rangle$. Thus, we are done if $\langle B_1 \cap B_2, x \rangle$ contains d. But this is the case as either $x = d$ or xd is a line in B_1 containing x and a point of $B_1 \cap B_2$.
It remains to consider $d \in B_2$. Then every plane π on xd is in $\langle B_1 \cap B_2, x \rangle$ as it is spanned by x and $\pi \cap B_2$. If $y \in B_1 \setminus B_2$, then there is $t \in \{x, y, d\}^{\perp} \setminus B_2$. Applying the previous argument to the plane $\langle x, t, d \rangle$ and subsequently to $\langle t, y, d \rangle$, we find $y \in \langle B_1 \cap B_2, x \rangle$. Hence (i).
(ii). Suppose $p \in P \setminus (B_1 \cap B_2)$ and B is a hyperplane containing $B_1 \cap B_2$ and p. Then $B_1 \supset B_1 \cap B \supseteq B_1 \cap B_2$, and so, by (i), we must have $B_1 \cap B = B_1 \cap B_2$. Also, (i) applied to B_1 and B gives $B = \langle B_1 \cap B, p \rangle$. We conclude that $B = \langle B_1 \cap B_2, p \rangle$, whence B is the unique hyperplane containing $B_1 \cap B_2$ and p. □

This lemma implies that the pair (H, S) where S is the collection of all intersections $B_1 \cap B_2$ with $B_1, B_2 \in \mathrm{H}$, $B_1 \neq B_2$, becomes a linear incidence system if incidence of $B \in \mathrm{H}$ and $S \in \mathrm{S}$ is defined by $S \subseteq B$. This incidence system will be called the *Veldkamp space* of (P, L).

1.5. Lemma. *The map $x \mapsto x^{\perp}$ from P to H is an injective morphism from (P, L) to (H, S) mapping lines onto lines.*

Proof. In view of corollary 1.3 the map is injective. Now, let $L \in \mathrm{L}$, take two points $x, y \in L$, and let B be a subspace containing $x^{\perp} \cap y^{\perp} = L^{\perp}$. We have to show $B = z^{\perp}$ for some $z \in L$. By lemma 1.1 there exists $b \in B \setminus L^{\perp}$. Then, by the 'one or all' axiom, there is a unique point $z \in b^{\perp} \cap L$. According to the previous lemma there is at most one hyperplane containing b and $L^{\perp}$. But $z^{\perp}$ and B are such hyperplanes. Therefore $B = z^{\perp}$ as required. □

2. Hyperplane complements.

Throughout this section, $(P,\mathbb{L})$ is a non-degenerate polar space all of whose lines are thick, B is a hyperplane of it, and $A = P\setminus B$. We may define a derived incidence system $(A,\mathbb{L}(A))$ where incidence is that of $(P,\mathbb{L})$. We wish to examine some of the properties of $(A,\mathbb{L}(A))$ - enough to show that $(A,\mathbb{L}(A))$ carries with it sufficient information to recover $(P,\mathbb{L})$. More precisely we shall show that if $(A,\mathbb{L}(A))$ is embedded in a second polar space so that its complement there is also a hyperplane, then the embedding extends to an isomorphism of $(P,\mathbb{L})$ onto the second polar space.

First observe that, for each line $L \in \mathbb{L}(A)$, the set $P(L)\setminus A(L)$ is a singleton. If $(P,\mathbb{L})$ has rank at least three, each line of $(A,\mathbb{L}(A))$ lies on at least two affine planes. Any three pairwise collinear points of $(A,\mathbb{L}(A))$ lie on an affine plane. This implies $(A,\mathbb{L}(A))$ is a gamma space - i.e., a space in which, for every point p and line L, none, one or all points of L are collinear with p.
For each $L \in \mathbb{L}(A)$, set $\Delta(L) = \{a \in A \mid a^{\perp} \cap A(L) = \emptyset \text{ or } L\}$, where $A(L)$ denotes the set of all points in A incident to L. We define an equivalence relation on $\mathbb{L}(A)$ as follows: L_1 is *parallel to* L_2 if and only if $\Delta(L_1) = \Delta(L_2)$. We denote this relation by $L_1 \,/\!/\, L_2$. The symbol $[L]$ will denote the equivalence class of all lines in $\mathbb{L}(A)$ parallel to L.

2.1. Lemma. *Two lines of* $\mathbb{L}(A)$ *are parallel if and only if their intersections with* B *coincide. Moreover, if* $L \in \mathbb{L}(A)$, *each point of* $\Delta(L)$ *lies on a unique member of* $[L]$. *Thus* $\Delta(L)$ *is the disjoint union of the lines of* $[L]$, *regarded as point sets of* A.

Proof. The first assertion is a direct consequence of the 'one or all' property of polar spaces.
Suppose $x \in \Delta(L)$. Then x is collinear with the unique point b of B incident with L. Set $M = xb$. Then $M \in \mathbb{L}(A)$, as $x \in A$. Now, $\Delta(L) = \{y \in A \mid y \perp b\} = \Delta(M)$. Thus $M \in [L]$.
If $N \in \mathbb{L}(A)$ is a line on x distinct from M, it meets B in a point $c \neq b$ so the choice of a point $y \in b^{\perp} \cap A\setminus(x^{\perp} \cup c^{\perp})$ (possible since lines are thick and $(P,\mathbb{L})$ is nondegenerate!) leads to $y \in \Delta(M)\setminus\Delta(N)$. Hence M is the unique member of $[L]$ on x. □

Let $\mathbb{L}(A)/\!/$ denote the collection of parallel classes $[L]$ on $\mathbb{L}(A)$. A direct consequence of the previous lemma is

2.2. Corollary. *There is a 1–1 correspondence* $f: B\setminus \mathrm{rad}\, B \to \mathbb{L}(A)/\!/$, *which takes each point* $b \in B\setminus\mathrm{rad}\, B$ *to the parallel class* $f(b) := \{L \in \mathbb{L}(A) \mid b \in B(L)\}$.

Proof. From the previous lemma, we have that $L,M \in f(b)$ implies $\Delta(L) = \Delta(M)$. Assume, in addition, that $\Delta(L) = \Delta(N)$ for some line $N \in \mathbb{L}(A)$. Let a be the unique point of B lying in N. Then, for $y \in b^{\perp} \cap A$, we have $y \in \Delta(N)$ so $y \in a^{\perp} \cap A$. As this argument is reversible, $b^{\perp} \cap A = a^{\perp} \cap A$. Now each point of $A(N)$ lies in $b^{\perp}$ and since N is thick, a itself lies in $b^{\perp}$. If $a^{\perp} \cap A = \emptyset$, then both a and b are deep points of B, whence, by corollary 1.3, $a{=}b$. Assume $a^{\perp} \cap A \neq \emptyset$. Then the set of lines and planes on b form a non-degenerate polar space. Yet, if $b \neq a$, then ab lies in the radical of this residue space, an absurdity. Thus $b{=}a$. This means that the set $f(b)$ is a full parallel class of $(A,\mathbb{L}(A))$. □

2.3. Lemma. *Assume* $(P,\mathbb{L})$ *has rank three or more. If* b_1 *and* b_2 *are two points of* B *which lie on a line* L *not containing a point of* $\mathrm{rad}\, B$, *then there exists an affine plane* π *in* A *such that the restriction* $f|_L : L \to \{[L] \mid L \in \mathbb{L}(\pi)\}$ *of* f *as given in the previous corollary is a 1-1 correspondence.*
Conversely, for each affine plane π *in* A, *the elements* $f^{-1}([L])$ *for* $L \in \mathbb{L}(\pi)$ *comprise a line of* B *not containing a point of* $\mathrm{rad}\, B$.

Proof. First assume $L \in \mathbb{L}(B)$ and $B(L) \cap \mathrm{rad}\, B = \emptyset$. Since $(P,\mathbb{L})$ has rank three or more, there exists a projective plane T on L, and $(T\setminus B, \mathbb{L}(T\setminus B))$ is an affine plane whose parallel classes are (via f) in 1-1 correspondence with the points of the line $\pi \cap B \in \mathbb{L}(B)$, the so-called 'line at infinity'. Thus if $L \in \mathbb{L}(\pi)$ and b is the 'point at infinity' in $B(L)$, then $f(b) = [L]$ by the previous corollary.
Conversely if π is an affine plane then, letting $\{b\} = B(L)$, we see that the union T of $P(L)$ over all $L \in \mathbb{L}(\pi)$ is a projective plane (since it is clearly generated by any two of the lines L). It follows that $\{f^{-1}[L] \mid L \in \mathbb{L}(\pi)\}$ coincides with the line $B(T)$. This line clearly contains no point of $\mathrm{rad}\, B$ since no point of $B(\pi)$ is in $\mathrm{rad}\, B$. □

For $L,M \in \mathbb{L}(A)$, set $[L]\sim[M]$ if there are $L', M' \in \mathbb{L}(A)$ with $L \parallel L'$, $M \parallel M'$, and $\langle L',M'\rangle$ a projective plane in $(P,\mathbb{L})$.

2.4. Lemma. *Again assume $(P,\mathbb{L})$ has rank at least three. Then two points b_1 and b_2 of $B\setminus \mathrm{rad}\, B$ are collinear by a line disjoint from $\mathrm{rad}\, B$ if and only if $f(b_1)\sim f(b_2)$ (i.e., if and only if there exist lines $L,M \in \mathbb{L}(A)$ such that $A(L) \subseteq \Delta(M)$, $L \in f(b_1)$, and $M \in f(b_2)$).*

Proof. Suppose first that b_1 and b_2 are collinear by a line R in $\mathbb{L}(B)$ disjoint from $\mathrm{rad}\, B$. Then by the previous lemma, there exists an affine plane π containing 2 lines L and M which are not parallel and for which $f(b_1) = [L]$ and $f(b_2) = [M]$. It follows that $A(L) \subseteq \Delta(M)$.
Conversely, assume lines L and M exist with $L \in f(b_1)$, $M \in f(b_2)$, and $A(L) \subseteq \Delta(M)$. Then for each point p of A lying in L, we have $p^{\perp} \cap A(M) = \varnothing$ or $A(M)$. In either case p is collinear with the point b_2 comprising $B(M)$. Thus $A(L) \subseteq b_2^{\perp}$. Since L is thick, $|A(L)| \geq 2$, so $b_2^{\perp}$ contains the point b_1 of $B(L)$. Thus b_2 is collinear with b_1. Finally if the line R on b_1 and b_2 contains a deep point of B, no point of A could be collinear with both b_1 and b_2. But we have just seen that the points of $A(L)$ are collinear with both b_1 and b_2. Thus R contains no point of $\mathrm{rad}\, B$. □

It will be convenient to write $\Delta([L])$ rather than $\Delta(L)$; since there is no ambiguity, we shall sometimes do so.

Let $\equiv$ be the relation on the set of parallel classes of $\mathbb{L}(A)$ defined by $[L]\equiv[M]$ if and only if $\Delta([L]) \cap \Delta([M]) = \varnothing$.

2.5. Lemma. *Suppose, for two lines L, M of $\mathbb{L}(A)$, we have $[L]\equiv[M]$. Then there is a unique partition $A = \dot{\bigcup}_{\sigma \in I} \Delta(L_\sigma)$ with L, $M \in \{L_\sigma\}_{\sigma \in I}$. The points $f^{-1}([L])$ and $f^{-1}([M])$ of B are collinear by a line R and the set $\{f^{-1}([L_\sigma])\}_{\sigma \in I}$ coincides with $B(R)\setminus \mathrm{rad}\, B$. Moreover, if $(P,\mathbb{L})$ has rank three or more, then $\mathrm{rad}\, B \subseteq B(R)$.*

Proof. Suppose $\Delta(L) \cap \Delta(M) = \varnothing$ for L, $M \in \mathbb{L}(A)$ and set $B(L) = \{b\}$, $B(M) = \{c\}$. Then each point of $A(L)$ is collinear with exactly one point of $A(M)$ and vice versa, forcing a 1-1 correspondence $A(L) \rightarrow A(M)$ defined by collinearity. Since $(P,\mathbb{L})$ is a polar space, $b \neq c$ and b and c are a collinear pair of points. Let R be the line in $\mathbb{L}$ on b and c. Clearly $R \in \mathbb{L}(B)$. Suppose a point x in A were collinear with two points of R. Then $x \in b^{\perp} \cap c^{\perp}$ and as $f(b) = [L]$ and $f(c) = [M]$ this means $x \in \Delta(L) \cap \Delta(M)$, a contradiction.
On the other hand, the polar space property forces $x^{\perp} \cap B(R) \neq \varnothing$ and so we see that each point of A is collinear with exactly one point of $B(R)\setminus \mathrm{rad}\, B$. Since collinearity of $x \in A$ with $r \in B(R)\setminus \mathrm{rad}\, B$ means $x \in \Delta(f(r))$, we have a partition

$$A = \dot{\bigcup_{r \in B(R)\setminus \mathrm{rad}\,(B)}} \Delta(f(r)). \tag{2.1}$$

If $(P,\mathbb{L})$ has rank at least three, then by the previous lemma, R must contain a deep point of B.

It remains to show that this partition of A is the unique such one containing $\Delta(L)$ and $\Delta(M)$ as components. Suppose instead there were a second partition

$$A = \Delta(L) \cup \Delta(M) \cup \dot{\bigcup_{\tau \in J}} \Delta(L_\tau).$$

Since this partition is assumed to differ from that in (2.1), there exists at least one value $\tau \in J$ such that $\Delta(L_\tau)$ is not one of the components of (2.1). Set $\{y\} = f^{-1}([L_\tau]) = L_\tau \cap B$, $y \notin R$. As argued for b and c alone, $\Delta(L_\tau) \cap \Delta(L) = \varnothing$ implies b is collinear with y via some line R' distinct from R. But similarly $\Delta(L_\tau) \cap \Delta(M) = \varnothing$ implies y collinear with c whence $R' \subseteq R^{\perp}$, so $\langle R,R'\rangle$ is a plane. This means $(P,\mathbb{L})$ has rank at least three. Then by the above, R' and R both contain a deep point. But by corollary 1.3, there is only one deep point. Thus $R \cap R' \supseteq \{b,d\}$ with $b \neq d$. This implies $(P,\mathbb{L})$ is not linear, defying the non-degeneracy of $(P,\mathbb{L})$ by well-known arguments. □

As an immediate consequence, we have

2.6. Corollary. *Suppose (P,L) has rank at least 3 or* $\mathrm{rad}\, B \neq \emptyset$. *Then the reflexive closure of $\equiv$ is an equivalence relation. If X is an $\equiv$-class on $\mathrm{L}/\,\|$ of size at least 2, then there is a line $R \in \mathrm{L}(B)$ with* $B(R) = \mathrm{rad}\, B \cup \{f^{-1}([N]) \mid [N] \in X\}$.

2.7. Proposition. *For $i = 1,2$, let (P_i,L_i) be a non-degenerate polar space. Let B_i be a hyperplane of (P_i,L_i). Set $A_i = P_i \setminus B_i$. Suppose $\phi : (A_1,\mathrm{L}_1(A_1)) \rightarrow (A_2,\mathrm{L}_2(A_2))$ is an isomorphism of incidence systems. Then ϕ can be uniquely extended to an isomorphism $\phi : (P_1,\mathrm{L}_1) \rightarrow (P_2,\mathrm{L}_2)$.*

Proof. For $L \in \mathrm{L}_i(A_i)$, let $[L]$ be the parallel class in $(A_i,\mathrm{L}_i(A_i))$ containing L - i.e. all lines L' of $\mathrm{L}_i(A_i)$ such that $\Delta(L') = \Delta(L)$. Then, for each $i = 1,2$, there are bijective mappings $f_i : B_i \setminus \mathrm{rad}\, B_i \rightarrow \mathrm{L}_i(A_i)/\,\|$ from the set of non-deep points of B_i to the set of parallel classes on $\mathrm{L}_i(A_i)$, $i = 1,2$.

Obviously ϕ, being an isomorphism, maps parallel classes on $\mathrm{L}_1(A_1)$ to parallel classes on $\mathrm{L}_2(A_2)$, and commutes with the 'functor' $\Delta : \mathrm{L}_i(A_i) \rightarrow \mathrm{P}(A_i)$, the power set of A_i, $i = 1,2$. Since the property of (P_i,L_i) having rank at least three can be recognized in $(A_1,\mathrm{L}_1(A_1))$ by the property that each line lies in an affine plane, it follows that

$$(P_1,\mathrm{L}_1) \textit{ has rank at least three if and only if } (P_2,\mathrm{L}_2) \textit{ does.} \tag{2.2}$$

If (P_1,L_1) is a generalized quadrangle, then the presence of a deep point in B_1 can be recognized by the fact that the reflexive closure of the relation $\equiv$ (defined for (P_i,L_i) as above for (P,L)) is an equivalence relation on the set of parallel classes $\mathrm{L}_1(A_1)/\,\|$. Otherwise, B_1 is a non-degenerate generalized quadrangle.

On the other hand, if (P_1,L_1) has rank at least three, the presence of a deep point in B_1 is indicated by the appearance of two lines L, M in $\mathrm{L}_1(A_1)$ with $[L] \equiv [M]$. Thus

$$B_1 \textit{ has a (unique) deep point if and only if } B_2 \textit{ does}. \tag{2.3}$$

We now extend $\phi : A_1 \rightarrow A_2$ to $\hat{\phi} : P_1 \rightarrow P_2$ as follows. First $\hat{\phi}$ restricted to A_1 is ϕ. If $b \in B_1 \setminus \mathrm{rad}\, B_1$ is a non-deep point of B_1, set $\hat{\phi}(b) = f_2^{-1}([\phi(L_b)])$ where L_b is any representative of the parallel class $f_1(b)$. Put another way, since Δ is 'functorial', there is an induced map $\bar{\phi} : \mathrm{L}_1(A_1)/\,\| \rightarrow \mathrm{L}_2(A_2)/\,\|$. Then $\hat{\phi}(b) = f_2^{-1} \cdot \bar{\phi} \cdot f_1(b)$. Finally, if d_1 is a deep point of B_1, then d_1 is unique via lemma 1 and, by (2.3), B_2 has a unique deep point d_2, and we write $\hat{\phi}(d_1) = d_2$.

From the above it is clear that, if there is an extension of ϕ as stated, it must coincide with $\hat{\phi}$. Thus uniqueness follows and it remains to show that $\hat{\phi} : P_1 \rightarrow P_2$ induces a mapping $\hat{\phi} : \mathrm{L}_1 \rightarrow \mathrm{L}_2$ via incidence. We already have $\hat{\phi} : \mathrm{L}_1(A_1) \rightarrow \mathrm{L}_2(A_2)$, a bijective mapping preserving incidence since ϕ was an isomorphism $(A_1, \mathrm{L}_1(A_1)) \rightarrow (A_2, \mathrm{L}_2(A_2))$.

Suppose, first, (P_1,L_1) is a generalized quadrangle and B_1 has a deep point d_1. Then the reflexive closure of the relation $\equiv$ on $\mathrm{L}_1(A_1)$ is an equivalence relation, and so the reflexive closure of $\equiv$ on $\mathrm{L}_2(A_2)/\,\|$ is also an equivalence relation. Since each line of $\mathrm{L}_i(B_i)$ is formed by taking d_i together with $f_i^{-1}([L])$ where $[L]$ ranges over a fixed $\equiv$-class on $\mathrm{L}_i(A_i)/\,\|$, we see ϕ induces a bijection $\mathrm{L}_1(B) \rightarrow \mathrm{L}_2(B)$ and we are done.

Next, suppose (P_1,L_1) and (P_2,L_2) are still both generalized quadrangles but the reflexive closure of $\equiv$ is not an equivalence relation on $\mathrm{L}_i(A_i)/\,\|$. Then still it is true that whenever $\Delta(L) \cap \Delta(M) = \emptyset$ there is a line R on $f_1^{-1}([L]) = b$ and $f_1^{-1}([M]) = c$ in $\mathrm{L}_1(B)$ (containing no deep point) and a unique partition

$$A_1 = \dot{\bigcup_{r \in R}} \Delta(f_1(r)).$$

Then

$$A_2 = \phi(A_1) = \dot{\bigcup_{r \in R}} \phi(\Delta(f_1(r)) = \dot{\bigcup_{r \in R}} \Delta(\bar{\phi}(f_1(r)) = \dot{\bigcup_{r \in R}} \Delta(f_2(\hat{\phi}(r)) \tag{2.4}$$

is a partition on A_2 containing $\Delta(f_2(\hat{\phi}(b)))$ and $\Delta(f_2(\hat{\phi}(c)))$ as components. By lemma 2.5, $\hat{\phi}(b)$ and $\hat{\phi}(c)$ are collinear by a line R_2 in $\mathrm{L}_2(B)$ and there is a partition

$$A_2 = \dot{\bigcup_{r' \in R_2}} \Delta(f_2(r')) \tag{2.5}$$

also containing $\Delta(f_2(\hat{\phi}(b)))$ and $\Delta(f_2(\hat{\phi}(c)))$ as components. But by lemma 2.5 such a partition is unique subject to containing these two components and so the right side of (2.5) is the same partition as the one in the expression after the last equal sign of (2.5). This means $\hat{\phi}(R) = R_2$.
Now assume (P_1, L_1) has rank at least three. Then the reflexive closure of $\equiv$ is an equivalence relation whose classes of size at least 2 represent lines in $\mathrm{L}_1(B_1)$ on a deep point d_1 of B_1. Then, just as in the first part of the proof when (P_1, L_1) was a generalized quadrangle, $\hat{\phi}$ takes the point-shadows of lines of $\mathrm{L}_1(B_1)$ lying on a deep point of B_1 to the point-shadows of lines of $\mathrm{L}_2(B_2)$ lying on a deep point of $\mathrm{L}_2(B_2)$ - i.e. ϕ induces a 1-1 mapping of all lines on d_1 in L_1 to all lines of L_2 on a deep point d_2.
There remain the lines of $\mathrm{L}_1(B_1)$ contained in $B \setminus \mathrm{rad}\, B$. Since (P_1, L_1) has rank at least three, such a line R has as its points the set $\{f_1^{-1}([L]) \mid L \in \mathrm{L}(\pi)\}$ for some affine plane π of $(A_1, \mathrm{L}_1(A_1))$ (cf. lemma 2.3). Then $\hat{\phi}(f_1^{-1}([L])) = f_2^{-1}([\phi L])$, so $\hat{\phi}(R) = \{f_2^{-1}([L_2]) \mid L_2 \in \mathrm{L}(\phi\pi)\}$ where $\phi\pi$ is an affine plane of $(A_2, \mathrm{L}_2(A_2))$. By lemma 2.3 once more, $f_2^{-1}([\phi(L)])$, $\phi(L) \in \phi(\pi)$, now ranges over all points of a line R_2 of $\mathrm{L}_2(B_2)$ not containing a deep point of B_2. This means $\hat{\phi}(R) = R_2$. As (P_2, L_2) has rank three, this procedure is reversible and so $\hat{\phi}$ induces a bijective mapping of the lines of B_1 not containing a deep point to the lines of B_2 not containing a deep point. It is now clear the $\hat{\phi}$ induces a complete bijective mapping $\mathrm{L}_1 \to \mathrm{L}_2$ via point shadows and so $\hat{\phi}$ is an isomorphism $(P_1, \mathrm{L}_1) \to (P_2, \mathrm{L}_2)$ extending ϕ. □

3. Affine polar spaces.

We consider here the following axioms concerning a connected incidence system (P, L):

(3.1.i) Any two collinear points lie on a unique line (thus, L can be viewed as a collection of subsets of P); any three pairwise collinear points lie in a plane, i.e., a singular subspace of singular rank 2.

(3.1.ii) There exists a plane. The points and lines incident with (i.e., contained in) any fixed plane form an affine plane.

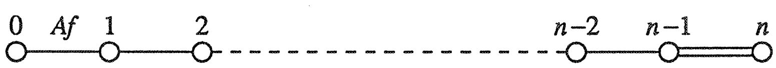
(3.1.iii) If $p \in P$ and π is a plane such that p is not contained in π, then $p^{\perp} \cap \pi$ is either empty, is the set of points on a line or is the set of points in π. Moreover, $x^{\perp} \subseteq y^{\perp}$ implies $x = y$ for any two points x and y.

By Π we shall denote the set of planes of (P, L). Note that any three pairwise collinear points that are not contained in a line lie in a unique plane. For, by (3.1.i), (P, L) is a gamma space, and (3.1.ii) implies that (L_x, Π_x) - the incidence system of lines and planes on $x \in P$ - is a non-degenerate polar space. Thus, if $y,z \in x^{\perp}$ are such that xy and xz are distinct lines and $z \perp y$, then $\langle x,y,z \rangle$ is the unique line (member of Π_x) on the collinear points xy and xz of L_x.

Also, any line lies in a plane. For, if $L \in \mathrm{L}$, then by connectedness of the incidence system, we may assume there is $M \in \mathrm{L}$ with $L \cap M \neq \emptyset$ and a plane $\pi \in \Pi$ with $M \subset \pi$; invoking (3.1.iii), we see that there is a line M' contained in $\pi \cap L^{\perp}$. Now, the previous paragraph shows that $\langle L, M' \rangle$ is a plane on L.

3.1. Remark. The triple (P, L, Π) satisfying (3.1) is a connected geometry of points, lines, and planes. This leads to an alternate description of the geometry, in terms of a diagram. For incidence systems (P, L) with finite singular rank n, hypothesis (3.1) is equivalent to the following

(3.2.i) Γ is a residually connected geometry with diagram



having at least three nodes (i.e., $n \geq 2$).

(3.2.ii) The residue of an object of type 0 is a building - i.e. a non-degenerate polar space of rank n.

(3.2.iii) If $P = \{$ objects of type 0$\}$ and $\mathrm{L} = \{$ objects of type 1$\}$, then the incidence system induced on (P, L) is a gamma space.

To see that (3.2) implies (3.1) is straightforward, the triple (P, L, Π) being the set of objects of type 0,1,2, respectively.
Assume (3.1). Then (3.1.iii) and the fact that planes are linear spaces imply that (P, L) is a gamma space.

We next argue that each point lies on some plane and each such plane has a fixed universal order $q \geq 2$ independent of the plane or the point. Since (P, L) is connected and Π is not empty it suffices to show that if a point x lies on a plane of order q and y is a point collinear with x then y lies on a plane of order q. Suppose x lies on plane π. If $y \in \pi$ we are done, so suppose $y \notin \pi$. Since $y^{\perp} \cap \pi$ contains x, by (3.1.ii), $y^{\perp} \cap \pi$ is a line or is π. In any event, there is a line L in $y^{\perp} \cap \pi$ lying on x. Then by (3.1.iii) as $|L| > 2$ (from π being an affine plane), we see that $\langle L, y \rangle$ generates a plane on y. Thus every point lies on a plane. Now consider the sets of lines and of planes on a point p - i.e. the residue geometry at p. Thus we regard the lines on p as Points and the planes on p as Lines. Then by (3.1.iii), two Points L and M on p are Collinear if and only if $L^{\perp} \subseteq M$ (so $\langle L, M \rangle$ generates an affine plane). Moreover, if π is a plane on p and L is a line on p with $L \cap \pi = \{p\}$ it follows that $L^{\perp} \cap \pi$ is either a line on p, or includes all lines on p within π - i.e. one or all Points incident with the Line P. Thus the residue geometry at a point obeys the fundamental 'one or all' polar space axiom of [2]. Moreover, the last part of (3.1.ii) implies this residue geometry has no radical. Thus by the theorem of Buekenhout & Shult [2], the residue geometry is a non-degenerate polar space of rank at least two, and so is a building. Since (P, L) is a gamma space, the subspaces of the residue geometry of a point correspond to singular subspaces whose planes are affine and whose point residues are projective. Hence, including singular subspaces of rank i $(0 \leq i \leq n)$ as objects of our geometry, we obtain a diagram geometry with diagram that of (3.1.i). It is clearly residually connected, and all parts of (3.2) hold.

Note that if Γ satisfies (3.1), the polar space comprising the residue of a point has thick lines. This is because for each Line of $Res(x) := (\mathrm{L}_x, \Pi_x)$ there is $\pi \in \Pi_x$ such that all lines through x incident with π have $q+1$ points. Since q, if finite, is at least two, any Line of $Res\ (x)$ must have at least $1+q \geq 3$ points. Thus all Lines of $Res\ (x)$ are thick. This is important in applying the results of section 2.

For the remainder of this section, we assume that Γ satisfies (3.1). Since members of L and Π are determined by their point shadows (i.e., the sets of points incident with them), we may and shall regard them as sets of points. We shall now derive properties of the affine polar space (P, L).

3.2. Lemma. *For any two points x and y, the polar spaces Res (x) and Res (y) are isomorphic.*

Proof. First, assume $d(x,y) = 2$, that is x and y are at distance 2. The lines on x are of two types: the set $\mathrm{L}_A(x,y)$ of lines which meet $y^{\perp}$ and the set $\mathrm{L}_B(x,y)$ of lines which do not meet $y^{\perp}$.
Similarly, by (3.1.ii), there are two sets of planes on x, the set $\Pi_A(x,y)$ of planes meeting $y^{\perp}$ at a line and the set $\Pi_B(x,y)$ of planes which do not contain a point of $y^{\perp}$. Now each plane in $\Pi_A(x,y)$ meets $y^{\perp}$ at a line L and carries exactly one line on x parallel to L. This means, $(\mathrm{L}_B(x,y), \Pi_B(x,y))$ is a hyperplane of the polar space $Res(x)$. Similarly $(\mathrm{L}_B(y,x), \Pi_B(y,x))$ is a hyperplane of the polar space $Res\ (y)$.
For every line M in $\mathrm{L}_A(x,y)$, there is a corresponding line $\phi(M) = \langle M \cap y^{\perp}, y \rangle$ in $\mathrm{L}_A(y,x)$. Moreover, for each plane π in $\Pi_A(x,y)$, there is a corresponding plane $\phi(\pi) = \langle \pi \cap y^{\perp}, y \rangle$. These mappings preserve incidence and have both left and right inverses. Thus we have an isomorphism

$$\phi : (\mathrm{L}_A(x,y), \Pi_A(x,y)) \to (\mathrm{L}_A(y,x), \Pi_A(y,x)).$$

Since the polar spaces $Res\ (x)$ and $Res\ (y)$ are both thick, by proposition 2.5, ϕ can be extended to an isomorphism $\hat{\phi} : Res\ (x) \to Res\ (y)$. Thus the conclusion of the lemma holds when x and y are at distance 2 from one another.
Next suppose x and y are collinear, and set $L = xy$. Since $Res(x)$ is a non-degenerate polar space of rank at least 2 there is a plane π on L. Then choose $z \in \pi \backslash L$. Again since $Res(z)$ ia a non-degenerate polar space there is a plane π_1 on z not lying in $\pi^{\perp}$ and intersecting π at the line L' on z parallel to L. Then, for any point $w \in \pi_1 \backslash L'$, we have $d(w,x) = 2 = d(w,y)$. Then, from the argument of the previous case, $Res(x) \cong Res(w) \cong Res(y)$. Thus we see that if x and y are collinear, then $Res(x) \cong Res(y)$.
Finally, since (P, L) is connected, the isomorphism holds for any x, y in P. □

3.3. Lemma. *The collinearity graph of $\Gamma = (P, \mathrm{L})$ has diameter at most 3. If x and y are at distance 3, all lines on x contain a point at distance 2 from y.*

Proof. Assume $d(x,y) = 3$. Then there exists a point z collinear with x and at distance 2 from y. The lines and planes on z which meet $y^{\perp}$ at a point or line respectively, form the incidence system $(\mathrm{L}_A(z,y), \Pi_A(z,y))$, and complements the hyperplane $\overline{B} = (\mathrm{L}_B(x,y), \Pi_B(z,y))$ of $Res(z)$ for which the line

$L = zx$ is a deep point. By lemma 2.1, since $Res(z)$ has thick lines, L is the unique deep point of $\bar{B}$ and $\bar{B} = (\bar{L})^{\perp}$. Thus $\mathbf{L}_B(z,y)$ is simply the set of all lines on z lying within $x^{\perp}$, and every point of $\bar{B}$ distinct from L is adjacent to a line of $\mathbf{L}_A(z,y)$. This means, that each point $r \in x^{\perp} \cap z^{\perp}$ not on the line L, r is collinear with a point of $y^{\perp}$ (since zr lies in a plane with a line of $\mathbf{L}_A(z,y)$ carrying a point of $y^{\perp}$). Thus we see that if L in $Res(x)$ contains a point z at distance 2 from y, then the same holds for any line M of $Res(x)$ with $M \subseteq L^{\perp}$. Since $Res(x)$, being a non-degenerate polar space of rank at least two, is connected, we see that *every* line of $Res(x)$ carries a point at distance 2 from y. It follows that no pair of points of Γ are at distance 4, and, since Γ is connected, Γ has diameter at most 3. $\square$

3.4. Definition. We define an equivalence relation " $//$ " on $\mathbf{L}$ as follows. For each line L in $\mathbf{L}$, set

$$\Delta(L) = \{p \in P \mid p^{\perp} \cap L = L \text{ or } \varnothing\}.$$

We write $L_1 // L_2$ if and only if $\Delta(L_1) = \Delta(L_2)$ and say L_1 is *parallel* to L_2, for any two lines L_1 and L_2 of $\mathbf{L}$. Manifestly, " $//$ " is an equivalence relation on $\mathbf{L}$. For each line L of $\mathbf{L}$, we let $[L]$ be the equivalence class of $\mathbf{L}$ containing line $\mathbf{L}$ - i.e. the set of all lines parallel to L.

Note that if π is a plane, and L_1 and L_2 belong to the same parallel class in the ordinary sense of parallelism for an affine plane, then $\Delta(L_1) = \Delta(L_2)$, and so L_1 and L_2 are parallel in the sense of the previous two paragraphs. For, if $p \in \Delta(L_1)$ then by (3.1.ii), $p^{\perp} \cap \pi$ is either empty, or is a line, or is π. In the first and last cases $p \in \Delta(L_2)$. If $p^{\perp} \cap \pi = M \in \mathbf{L}$, then clearly, as $p \in \Delta(L_1)$, either $M = p^{\perp} \cap L_1 = L_1$ or $M \cap L_1 = p^{\perp} \cap L_1 = \varnothing$. In any case, M is parallel (in the ordinary sense) to L_1 and hence to L_2. Thus, in all cases, $p \in \Delta(L_2)$.

This shows that " $//$ " contains at least the transitive extension on $\mathbf{L}$ of the relation of being parallel lines within an affine plane. In the next two lemmas it will be seen that parallelism is *precisely* this extension.

3.5. Lemma. *Suppose $y \in P$ and $L \in \mathbf{L}$. Then, for at least one line $L_0 \in [L]$, the intersection $y^{\perp} \cap L_0$ is non-empty.*

Proof. Let y, L be such that there is no plane π on L with $y^{\perp} \cap \pi \neq \varnothing$. By the previous lemma, there is a path $y \perp t \perp x$ with $x \in L$.
Take a plane ρ_1 on tx. As $H_1 = y^{\perp} \cap \rho_1$ contains t but not x, it is a line, so $\sigma_1 = \langle H_1, y \rangle$ is a plane. Similarly $K_1 = L^{\perp} \cap \rho_1$ is a line (observe that $t \notin L^{\perp}$ since otherwise $\langle t, L \rangle$ defies the hypothesis) and $\pi_1 = \langle K_1, L \rangle$ a plane. If $H_1 \cap K_1$ contains a point, say w, then $w \in \pi_1 \cap y^{\perp}$, contradicting the hypothesis. Thus $H_1 \cap K_1 = \varnothing$, and so $H_1 // K_1$.
Now choose $x_1 \in K_1 \backslash \{x\}$. Take t_1 to be the point on H_1 and on the line in ρ_1, through x_1 parallel to tx, and take y_1 in σ_1 such that $N_1 = yy_1$ is parallel to H_1 and t_1y_1 is parallel to ty. Denote by L_1 the line in π_1 on x_1 parallel to L. If $x_1 \perp y_1$, the $x_1 \in (t_1y_1)^{\perp}$, so $x_1 \in \Delta(t_1y_1) = \Delta(ty)$ whence, as $x_1 \perp t$, we have $x_1 \in (ty)^{\perp}$, and so $x_1 \in y^{\perp} \cap \pi_1$, a contradiction. Thus $d(x_1, y_1) = 2$.
From now on assume that there is no line $L_0 // L$ with $y^{\perp} \cap L_0 \neq \varnothing$. This implies the earlier assumption on L that there is no plane π on L with $y^{\perp} \cap \pi \neq \varnothing$. Suppose μ is a plane on L_1 with $\mu \cap y_1^{\perp} \neq \varnothing$. Then, as $x_1 \in \mu \backslash y_1^{\perp}$, we have that $\mu \cap y_1^{\perp}$ is a line. If this line is a parallel of L_1, then $\langle y_1, \mu \cap y_1^{\perp} \rangle$ contains a parallel L_0 of L with $y_1 \in y^{\perp} \cap L_0$, a contradiction. So $\mu \cap y_1^{\perp}$ is a line of μ meeting L_1 in a point, say z_1. Now $z_1 \in \Delta(K_1) = \Delta(N_1)$ and $z_1 \in y_1^{\perp}$ imply $z_1 \in N_1^{\perp}$ so $z_1 \perp y$, again a contradiction. Hence there is no plane ν on L_1 such that $y_1^{\perp} \cap \nu \neq \varnothing$. We repeat the construction of the previous paragraph to get H_2, K_2, N_2, L_2 as H_1, K_1, N_1, L_1, now starting from L_1, x_1, y_1, t_1 instead of L, x, y, t. We choose the plane $\rho_2 = \langle H_2, K_2 \rangle$ on t_1x_1 in such a way that $y_1^{\perp} \cap \rho_2$ and $y^{\perp} \cap \rho_2$ are distinct lines. (In $Res(t_1)$, this simply means that $H_1 \notin H_2^{\perp}$. Then $N_2 // H_2 // K_2$ (as before).
Since $y^{\perp} \cap \rho_1 = H_1$ and $y^{\perp} \cap \rho_2 \neq H_2$, we see (from a look at $Res(t_1)$) that $N_1 \not\subseteq N_2^{\perp}$. Therefore, $y^{\perp} \cap N_2 = \{y_1\}$, whence $y \notin \Delta(N_2) = \Delta(K_2)$. Consequently, $y^{\perp} \cap K_2$ is a point, and so $y^{\perp} \cap \pi_2 \neq \varnothing$. But now there is a line $L_2 // L_1 // L$ in π_2 on K_2, the final contradiction. $\square$

3.6. Corollary. *If $y \in \Delta(L)$, then y lies on a line of $[L]$. In other words, $\Delta(L)$ is the point-set union of the lines of $[L]$.*

Proof. Suppose $L_0 \in [L]$ and $L_0 \subseteq y^{\perp}$. If $y \in L_0$ we are done. Otherwise we may form the plane $P = \langle y, L_0 \rangle$ and find a parallel of L_0 in y and again we are done. So we must assume no line of $[L]$ lies in $y^{\perp}$. But since $y \in \Delta(L)$, this means $y^{\perp} \cap L_0 = \varnothing$ for each $L_0 \in [L]$, contrary to the previous lemma. □

3.7. Lemma. *No two lines of $[L]$ meet at a point.*

Proof. Suppose $L_1, L_2 \in [L]$ and $L_1 \cap L_2 = \{p\}$. Then, for any point $q \in L_1 \setminus \{p\}$, we have $q \in \Delta(L_1) = \Delta(L_2)$ and $q \in p^{\perp}$, so $q \in L_2^{\perp}$. Thus $L_1 \subseteq L_2^{\perp}$ and there is a plane π containing L_1 and L_2. Let y be any point of $p^{\perp}$ not in π. Then $y^{\perp} \cap \pi$ is a line on π or is all of π. In the former case, if $y^{\perp} \cap \pi$ is a line distinct from L_1 then $y \notin \Delta(L_1)$ and so $y^{\perp} \cap \pi$ is also distinct from L_2. But similarly if $y^{\perp} \cap \pi$ contains L_1 it must also contain L_2 and hence all of π. Thus π corresponds to a Line of $Res(p)$ containing two Points L_1 and L_2 whose 'perps' in $Res(p)$ are identical. But as $Res(p)$ is a non-degenerate polar space, it is well known that this implies $L_1 = L_2$, against our assumption $L_1 \cap L_2 = \{p\}$. The proof is complete. □

Now we see from corollary 3.5 and lemma 3.6 that $\Delta(L)$ is the union of disjoint lines from $[L]$. Moreover, if L_1 and L_2 are two distinct lines of $[L]$, either $L_1^{\perp} \cap L_2 = \varnothing$ or $L_1 \subseteq L_2^{\perp}$. In the latter case L_1 and L_2 are parallel lines within the same affine plane $\langle L_1, L_2 \rangle$. For, if $L_1 \subseteq L_2^{\perp}$ and $x \in L_2$, then $x \notin L_1$ (by lemma 3.6) and so $\pi = \langle L_1, x \rangle$ is a plane, and so x carries a line L' in π parallel to L_1. But then L_2 and L' are lines of $[L]$ lying on x, so $L' = L_2$ by lemma 3.6 again. Thus $\langle L_1, L_2 \rangle$ is the plane π.

It is clear that, for $L \in \mathbb{L}$, we may form a linear incidence system whose points are the lines of $[L]$ and whose lines are the sets of lines of $[L]$ lying in planes generated by two mutually perpendicular members of $[L]$. We denote this incidence system by the symbol $\Delta(L)/L$.

3.8. Lemma. *Assume $L \in \mathbb{L}$ and x is a point not in $\Delta(L)$. Then the lines on x which do not meet $\Delta(L)$ and the planes on x which do not meet $\Delta(L)$ form a hyperplane $B(x, [L])$ of $Res(x)$.*

Proof. Suppose π is a plane on x containing a point z of $\Delta(L)$. Then, by corollary 3.5, z lies on a line L_1 in $[L]$. Since x is not in $\Delta(L) = \Delta(L_1)$, $x^{\perp} \cap L_1 = \{z\}$ and so $L_1^{\perp} \cap \pi$ is a line M not on x. But $M \subseteq \Delta(L)$. Any further point of $\pi \cap \Delta(L_1)$, since it lies in $z^{\perp}$ must lie in $L_1^{\perp}$ whence $\Delta(L) \cap \pi = \Delta(L_1) \cap \pi = L_1^{\perp} \cap \pi = M$.
Thus every line on x lying in π meets $\Delta(L)$ (at a point of M) except one, namely, the line M_1 on x parallel to M (in π). As, obviously, $M_1 \cap \Delta(L_1) = \varnothing$, we have thus seen that every Line of $Res(x)$ has exactly one or all of its Points represented by lines on x not meeting $\Delta(L)$. It follows that the lines on x not meeting $\Delta(L)$ represent a hyperplane $B(x, [L])$ of $Res(x)$. □

We also consider the incidence system $A(x, [L]) = (\mathbb{L}_A(x, [L]), \Pi_A(x, [L]))$ of all lines on x which meet $\Delta(L)$ and all planes on x which meet $\Delta(L)$ at a line. Then $A(x, [L])$ and $B(x, [L])$ together form $Res(x)$.

3.9. Corollary. *If x and y are two points not lying in $\Delta(L)$, then $B(x, [L]) \cong B(y, [L])$ as incidence systems. Moreover, $A(x, [L]) \cong \Delta(L)/L$.*

Proof. For each line L_1 of $[L]$ there is a unique point $x(L_1)$ with $\{x(L_1)\} = x^{\perp} \cap L_1$ and a unique line $\phi_x(L_1) := x(x(L_1))$ on x meeting $\Delta(L)$ at a point of L_1. If L_1 and L_2 lie in $[L]$, then $L_1 \subseteq L_2^{\perp}$ if and only if $x(L_1) \in x(L_2)^{\perp}$ if and only if $\phi_x(L_1) \subseteq (L_2)^{\perp}$. Thus $\phi_x : [L] \to \mathbb{L}_A(x, [L])$ is a 1-1 correspondence preserving collinearity and as $A(x, [L])$ is a gamma space which is partial linear, the "$\perp\perp$" operation on pairs of distinct collinear points defines lines, and so ϕ_x extends to an isomorphism $\phi : \Delta(L)/L \to A(x, [L])$ of linear incidence systems.
Similarly, $\Delta(L)/L \cong A(y, [L])$ and so $f = \phi_y \phi_x^{-1}$ is an isomorphism $f : A(x, [L]) \to A(y, [L])$. By proposition 2.7, f extends to an isomorphism $Res(x) \to Res(y)$, whose restriction to the complementary hyperplane $B(x, [L])$ defines the required isomorphism $B(x, [L]) \to B(y, [L])$ between hyperplanes. □

3.10. Lemma. *Let L be a line and x a point in $P \setminus \Delta(L)$. Suppose M is a line on x, representing a deep point of $B(x, [L])$ in $Res(x)$. Then*

(i) $\Delta(L) \cap \Delta(M) = \varnothing$.

(ii) *There is no line in $[M]$ on which there is a plane meeting $\Delta(L)$.*

Proof. By hypothesis, no plane on M meets $\Delta(L)$ (non-trivially). Let $[M]_0$ represent those lines M' of $[M]$ for which no plane on M' meets $\Delta(L)$. Thus $M \in [M]_0$.
If $u \in \Delta(L) \cap \Delta(M)$, then u carries a line $M' \in [M]$, and any plane on M' will contain u, so meets $\Delta(L)$ non-trivially, whence $[M] \setminus [M]_0$ is nonempty. Thus (i) will be a consequence of (ii).
As for (ii), suppose, by way of contradiction, that $[M] \setminus [M]_0$ is nonempty. Since $\Delta(M)/M$ is connected (cf. corollaries 3.9, 3.8, and lemma 1.1(i)), there is a line $M_1 \in [M]_0$ lying in a plane π_1 with a line $M_2 \in [M] \setminus [M]_0$. Then M_2 lies in a plane π_2 which meets $\Delta(L)$ at a line N. Since $\pi_1 \cap \Delta(L) = \varnothing$, by hypothesis, we see N is parallel to $M_2 = \pi_1 \cap \pi_2$ in π_2. Since N lies in $\Delta(L)$ and $N \notin [L]$, it corresponds to a plane ρ in $\Delta(L)$ representing a line of $\Delta(L)/L$. (Observe that $\rho = \langle N, L_1 \rangle$ for each $L_1 \in [L]$ meeting N at a point.) Take a point $y \in M_1$. The fact $M_1 \cap \Delta(L) = \varnothing$ implies that $y^\perp \cap \rho$ is a line M_y not in $[L]$. Moreover, if $q \in M_y \cap N$, then $y^\perp \cap \pi_2$ contains M_2 and q and hence all of π_2. Thus $N \subseteq y^\perp$ and so $\pi_3 = \langle y, N \rangle$ is a plane and π_3 contains a line N' on y parallel to N. From the above, $N \in [M]$, so $N' \in [M]$, and, by lemma 3.7, $N' = M_1$. But then π_3 is a plane containing M_1 and meeting $\Delta(L)$ in N, contradicting $M_1 \in [M]_0$.
Thus M_y must be parallel to N. Now choose any point z on M_y. Then $z^\perp \cap M_1$ is not M_1 as $M_1 \in [M]_0$, so $z^\perp \cap M_1 = \{y\}$ and $z^\perp \cap \pi_1$ is a line S not parallel with M_1. Thus $S \cap M_2 = \{y_2\}$. Then $y_2^\perp \cap \rho$ includes both N and z not lying on M. Thus $y_2^\perp$ contains all of ρ. But since ρ contains lines from $[L]$ we have $y_2 \in \Delta(L)$, so π_1 meets $\Delta(L)$ non-trivially against our choice of M_1 as lying in $[M]_0$. This contradiction completes the proof. □

3.11. Remark. Quite clearly the converse of lemma 3.9 holds - that is

If $\Delta(L) \cap \Delta(M) = \varnothing$ and $x \in \Delta(M)$ and M' is the unique (see lemma 3.6) line of $[M]$ lying on x, then M' represents a deep point of the hyperplane $B(x, [L])$ of $Res(x)$.

Proof. Clearly as $M' \cap \Delta(L) = \varnothing$, the line M' represents a point of $B(x, [L])$. But if M' were not a deep point, there would be a plane π on M' meeting $\Delta(L)$ at a point y. But then y carries a line M'' parallel to M' so $y \in \Delta(M) \cap \Delta(L)$, a contradiction. □

3.12. Definition. We denote by $\bar{\mathbf{L}}$ the collection of parallel classes of lines of $\mathbf{L}$. We define the relation $\equiv$ on $\bar{\mathbf{L}}$ by asserting $[L] \equiv [M]$ if and only if $\Delta(L) \cap \Delta(M) = \varnothing$. Considering that $L, L' \in [L]$ if and only if $\Delta(L) = \Delta(L')$, this relation is certainly well defined.

3.13. Lemma. *The reflexive closure of the relation $\equiv$ is an equivalence relation on $\bar{\mathbf{L}}$.*

Proof. Assume $\Delta(L) \cap \Delta(M) = \varnothing = \Delta(M) \cap \Delta(N)$ for lines L, M, N, and that $\Delta(L) \cap \Delta(N)$ contains a point u. Then there exist lines L' and N' on u belonging to $[L]$ and $[N]$, respectively, and, by the above remark, L' and N' both represent deep points of the hyperplane $B(u, [M])$ of $Res(u)$. As $Res(u)$ is a nondegenerate thick polar space, corollary 1.3 forces $L' = N'$ and hence $\Delta(L) = \Delta(N)$. This proves the assertion. □

3.14. Corollary. *Suppose $\Delta(L_1) \cap \Delta(L_2) = \varnothing$ for two lines L_1 and L_2. Then the sets $\Delta([L])$ for $[L]$ running over the members of the $\equiv$-class of $[L_1]$, form a partition of P.*

Proof. Denote by X the $\equiv$-class of $[L_1]$. Clearly, for $[M], [N] \in X$, we have $\Delta(M) \cap \Delta(N) = \varnothing$ so it remains only to show that each $p \in P$ lies in some $\Delta(L)$ for $[L] \equiv [L_1]$. Choose $x \in \Delta(L_1)$ and $y \in \Delta(L_2)$. Then $B(y, [L_1])$ contains a deep point - some line parallel to L_2 - see remark 3.8. But, by corollary 3.9, $B(y, [L_1]) \cong B(p, [L_1])$ and so the latter contains a deep point in $Res(p)$ represented by the line L. By the above lemma, $\Delta(L) \cap \Delta(L_1) = \varnothing$ and so $[L] \equiv [L_1]$ as required. □

We now wish to define a second relationship on $\bar{\mathbf{L}}$. Write $[L] \sim [M]$ if some line L' of $[L]$ and some line M' of $[L]$ lie together in a plane. Note that in this case $\Delta(L) \cap \Delta(M)$ is not empty (it contains every point of the aforementioned plane, for example). The next several lemmas concern this relation $\sim$.

3.15. Lemma. *For any line M, the subset $\Delta(M)$ is a subspace of (P,L). If L is a line with $L \subseteq \Delta(M)$, then* $[L] \sim [M]$.

Proof. Suppose $L \cap \Delta(M)$ contains two distinct points u_1, u_2. If u_1 and u_2 lie on a line M' of $[M]$ then $L = M'$ by (3.1.i).
Thus we may assume u_1 and u_2 are not collinear by a line of $[M]$, so $L \notin [M]$. Now, by corollary 3.6, there exist lines $M_i \in [M]$ on u_i for both $i = 1,2$. Since $L \subset \Delta(M)$, in the space $\Delta(M)/M$, the points M_1 and M_2 are collinear, and so $\langle M_1, M_2 \rangle$ is a clique of the collinearity graph containing L. But as $\langle M_1, M_2 \rangle$ is a plane carrying a member of $[M]$ lying on u_2, and u_2 lies on a unique member of $[M]$ (cf. lemma 3.6), we see that $\langle M_1, M_2 \rangle$ *is* this plane. Since $\langle L, M_1 \rangle$ is now a plane, $[L] \sim [M]$, as required. □

3.16. Lemma. *Let L and M be lines such that $L \cap \Delta(M) = \emptyset$ and $\Delta(L) \cap \Delta(M) \neq \emptyset$. Then $[L] \sim [M]$.*

Proof. There exists a point $u \in \Delta(L) \cap \Delta(M)$ and by corollary 3.6 u lies on a line M' of $[M]$ and a line L' of $[L]$. Since $diam(\Delta(L)/L) \leq 3$ (for it is a polar space minus a hyperplane, see corollaries 3.9, 3.8, and lemma 1.1(i)); there exist planes π_1, π_2, and π_3 such that π_1 contains L', $\pi_1 \cap \pi_2 = L_1 \in [L] \setminus \{L'\}$, $\pi_2 \cap \pi_3 = L_2 \in [L], L_3 \subseteq \pi_3, L_3 \neq L_2, L_3 \in [L]$ so that the line L is either L_3, L_2 or L_1.
Now $(M')^{\perp} \cap \pi_1$ is a line N_1. If $N_1 \in [L]$ we are done since $\langle N_1, M' \rangle$ is a plane. So we may assume N_1 is not parallel to L_1 and so $N_1 \cap L_1$ is a point $p_1 \in L_1$. Now in the plane $\langle N_1, M' \rangle$ and on the point p_1 there is a line M_1 parallel to M'. As $p_1 \in \Delta(M)$, the assumption $L \cap \Delta(M) = \emptyset$ yields $L \neq L_1$. So $L = L_3$ or L_2. Now $(M_1)^{\perp} \cap \pi_2 = N_2$ is a line. Again if $N_2 \in [L]$ we are done since $\langle N_2, M_1 \rangle$ is a plane and $M_1 \in [M]$. Thus we may assume N_2 is not parallel to L_2 and hence $N_2 \cap L_2$ is a point p_2 which lies in $\Delta(M)$. Again $L \cap \Delta(M) = \emptyset$ yields $L \neq L_2$, whence $L = L_3$. On p_2 there is a line M_2 parallel to M_1, so $M_2 \in [M]$, and again $(M_2)^{\perp} \cap \pi_3$ is a line N_3. If $N_3 \in [L]$ we are done as $\langle N_3, M_2 \rangle$ is a plane. Thus N_3 is not parallel to $L_3 = L$ and so $\emptyset \neq N_3 \cap L_3 \subseteq L \cap \Delta(M) = \emptyset$, a contradiction. This completes the proof.
□

3.17. Lemma. *Let L and M be lines. Assume $|L \cap \Delta(M)| \geq 1$ and $[L] \sim [M]$. Then $L \subseteq \Delta(M)$.*

Proof. Since $[L] \sim [M]$ there exists a plane π containing $L_1 \in [L]$ and $M_1 \in [M]$.
Suppose, by way of contradiction, L does not lie in $\Delta(M)$. Then $|L \cap \Delta(M)| = 1$ by hypothesis and the previous lemma. Choose $w \in L \setminus \Delta(M)$. Then $w^{\perp} \cap M_1$ is a point p, and so $w^{\perp} \cap \pi$ is a line N not parallel to M_1. If N were not parallel to L_1 we would have $|w^{\perp} \cap L_1| = 1$ against $w \in L \subseteq \Delta(L_1)$. Thus N is parallel to L_1. Then there is a line N' on w in the plane $\langle w, N \rangle$ parallel to N. Thus $N' \in [L]$ and since it lies on w, $L = N'$. Thus $L^{\perp} \cap \pi = N$. Recall there is a point $v \in L \cap \Delta(M)$. Now $v^{\perp} \cap \pi$ contains N, whence a point p of M_1, and so, as $v \in \Delta(M)$, we must have $M \subseteq v^{\perp}$. This means $v^{\perp} \cap \pi \supseteq \langle N, M_1 \rangle = \pi$. Taking $q \in M_1 \setminus \{p\}$, we see that $q \in \Delta(L)$ and $v \in q^{\perp} \cap L$, whence $q \in L^{\perp} \subseteq w^{\perp}$. Therefore, $w^{\perp} \supseteq M_1$, so $w \in \Delta(M)$, contrary to assumption. This completes the proof. □

The final lemma of this section combines both relations $\equiv$ and $\sim$ on $\overline{L}$.

3.18. Lemma. *Let L, M, and R be lines with $[R] \sim [L] \equiv [M]$. Then either $[R] = [L]$ or $[R] \sim [M]$.*

Proof. We may assume that R and L intersect at a point x in some plane π. Then $R \subseteq \Delta(L)$, whence (as $[L] \equiv [M]$) we have $R \cap \Delta(M) = \emptyset$, and in view of lemma 3.16 we may assume $[R] \equiv [M]$ (otherwise the proof is complete). Now R and L are deep points of the hyperplane $B(x,[M])$ in $Res(x)$. But deep points are unique (cf. corollary 1.3) so $L = R$. and the proof is complete. □

4. Characterization of affine polar spaces.

Starting with an affine polar space, i.e. an incidence system (P,L) satisfying (3.1), we form a new geometry $(\underline{P}, \underline{L})$ whose definition depends upon the following case division: (i) All $\equiv$-classes on $\overline{L}$ have cardinality one, and (ii) some $\equiv$-class on $\overline{L}$ contains at least two elements.

The points of $\underline{P}$ are of two or three kinds

$\underline{P}_1$: the points of P;

$\underline{P}_2$: the elements of $\overline{L}$, that is, the parallel classes $[L]$, $L \in L$;

in the event of case (ii)

$\underline{P}_3$: the symbol ∞ (otherwise we disregard $\underline{P}_3$).

The lines also, are of two or three kinds

$\underline{L}_1$: for each $L \in \mathbb{L}$, the set $L \cup \{[L]\}$;

$\underline{L}_2$: the set $\{[L] \mid L \in \mathbb{L}(\pi)\}$ for each plane π;

$\underline{L}_3$: the sets $\{\infty\} \cup X$ for each $\equiv$-class X of $\underline{L}$ of cardinality at least 2 (of course if case (ii) fails, $\underline{L}_3$ is empty).

Incidence on $(\underline{P},\underline{L})$ is defined by containment.

4.1. Theorem. *The incidence system* $(\underline{P},\underline{L})$ *is a nondegenerate polar space of rank at least 3. If we put* $\underline{P}_0 = \underline{P}_2 \cup \underline{P}_3$ *and* $\underline{L}_0 = \underline{L}_2 \cup \underline{L}_3$*, then* $(\underline{P}_0,\underline{L}_0)$ *is a hyperplane of* $(P,\mathbb{L})$.

Proof. To show that $(P,\mathbb{L})$ is a polar space it suffices to verify the basic 'one or all' axiom for non-incident point-line pairs of $\underline{P}\times\underline{L}$. This task will be easier if we review first what $\mathbb{L}$-collinearity means for points.

First, two points p,q of $\underline{P}_1$ are collinear if they are collinear in $(P,\mathbb{L})$. Second, a point $p \in \underline{P}_1$ and a point $[L]$ of $\underline{P}_2$ are collinear if and only if $p \in \Delta(L)$. Third, two elements $[L]$ and $[M]$ of $\underline{L}$ are collinear if and only if $[L]\equiv[M]$ or $[L] \sim [M]$. Fourth, ∞ is collinear with all elements of $\underline{L}_2 \cup \underline{L}_3$.

CASE 1. *point p; line* $M \cup \{[M]\} \in \underline{L}_1$. If $p \in \Delta(M)$ then, by corollary 3.4, p lies on a line $M' \in [M]$ and $M' \cup \{[M]\}$ is a line on p, so p is collinear with $[M]$. But as $p \in \Delta(M)$, by definition, $p^{\perp} \cap M = \varnothing$ or M. Thus p is $\underline{L}$-collinear to just $[M]$ or to all of $M \cup \{[M]\}$.

If $p \notin \Delta(M)$, then $|p^{\perp} \cap M| = 1$ and p is not collinear with $[M]$ so p is $\underline{L}$-collinear with exactly one point of $M \cup \{[M]\}$ in this case.

CASE 2. *point* $p \in P$*; line* $\{[L] \mid L \in \mathbb{L}(\pi)\} \in \underline{L}_2$. First assume $p^{\perp} \cap \pi = \varnothing$. Then, for all $L \in \mathbb{L}(\pi)$, we have $p^{\perp} \cap L = \varnothing$ so $p \in \Delta(L)$, whence p is $\underline{L}$-collinear with $[L]$. Similarly, if $p^{\perp} \supseteq \pi$, then $p \in \Delta(L)$ for all $L \in \mathbb{L}(\pi)$ and the same conclusion holds.

In the remaining case, $p^{\perp} \cap \pi$ is a line N. Then, by definition of $\Delta(L)$, we have, for $L \in \mathbb{L}(\pi)$, that $p \in \Delta(L)$ holds if and only if $[L] = [N]$, so p is $\underline{L}$-collinear with exactly one point of the line.

CASE 3. *point* $p \in P$*; line* $\underline{A} = (\{\infty\} \cup X) \in \underline{L}_3$. Here, by corollary 3.14, the sets $\Delta([L])$, $[L] \in X$, partition P, hence p is $\underline{L}$-collinear with precisely one point of $\underline{A}$.

CASE 4. *point* $[N] \in \underline{P}_2$*; line* $M \cup \{[M]\} \in \underline{L}_1$. Since the point and line are not incident, we have $[N] \neq [M]$.

Assume $[N]$ is not $\underline{L}$-collinear with any point of M. Then $M \cap \Delta(N) = \varnothing$. If $\Delta(M) \cap \Delta(N) = \varnothing$, then $[M]$ and $[N]$ are collinear by a line of $\underline{L}_3$. So assume $\Delta(M) \cap \Delta(N) \neq \varnothing$. We have now attained the hypotheses of lemma 3.16 and so $[M] \sim [N]$ - i.e., they are $\underline{L}_2$-collinear. So far, we have shown that

$$\text{If } [N] \text{ is collinear with no point of } M\text{, it is collinear with } [M]. \tag{4.1}$$

Next assume $[N]$ is collinear with two points of M. Then $|N' \cap \Delta(M)| \geq 2$ for some $N' \in [N]$, and so, as $\Delta(M)$ is a subspace (cf. lemma 3.15), $N \subseteq \Delta(M)$ and so $[N] \sim [M]$. This means that

$$\text{If } [N] \text{ is collinear with two points of } M\text{, it is collinear with all points of } M \cup [M]. \tag{4.2}$$

It remains to consider the case where $[N]$ is collinear with $[M]$ and a point of M. Then $M \cap \Delta(N) \neq \varnothing$. In particular $\Delta(M) \cap \Delta(N) \neq \varnothing$ and so $[N] \sim [M]$. We now have the hypotheses of lemma 3.17 and so $M \subseteq \Delta(N)$. This shows

$$\text{If } [N] \text{ is collinear with } [M] \text{ and a point of } M\text{, then it is collinear with all points of } M \cup \{[M]\}. \tag{4.3}$$

The assertions (4.1), (4.2), and (4.3), put together, complete case 4.

CASE 5. *point* $[N] \in \underline{P}_2$*; line* $\underline{A} = \{[L] \mid L \in \mathbb{L}(\pi)\} \in \underline{L}_2$. If a line $N' \in [N]$ meets π non-trivially then $(N')^{\perp} \cap \pi$ is either a line R or all of π. In the former case $N' \subseteq \Delta(R)$ and $[R]$ is the unique point $[L]$ of $\underline{A}$ with $N' \subseteq \Delta(L)$. Therefore $[N]\sim[R]$ and, for $L \in \mathbb{L}(\pi)$ with $[L] \neq [R]$, neither $[N]\equiv[L]$ nor $[N]\sim[L]$ holds (cf. lemma 3.17). Thus $[N]$ is collinear with exactly one member of $\underline{A}$, namely $[R]$. In the latter case, $N' \subseteq \Delta(L)$ for all $L \in \mathbb{L}(\pi)$ and, by 3.15, $[N]$ is collinear with all members of $\underline{A}$.

Thus we may assume no member of $[N]$ meets π non-trivially - i.e. $\pi \cap \Delta(N) = \emptyset$. This means each point of π is collinear with exactly one point of N' for each $N' \in [N]$. There are thus only two situations which can arise

(a) There exists a unique $x_0 \in N$ with $x_0^\perp \supseteq \pi$ and $x^\perp \cap \pi = \emptyset$ for all $x \in N \setminus \{x_0\}$.

(b) For each $x \in N$, the intersection $x^\perp \cap \pi = M_x$ is a line, and the lines M_x all belong to one parallel class of π.

Consider first case (a). Here, the lines of L on x_0 lying in the linear space $\langle x_0, \pi \rangle$ form a projective plane in $Res(x_0)$ and those meeting π are the complement of a hyperplane of this projective plane. Since this hyperplane is a projective line of $Res(x_0)$ it means that there is an affine plane π_1 of $\langle x_0, \pi \rangle$ lying on x_0 and containing no line on x_0 meeting π non-trivially. In particular, $\pi_1 \cap \pi = \emptyset$. Now for each line L in π, the subspace $\langle x_0, L \rangle$ is a plane, and it carries a line L' on x_0 parallel to L. But L' is a line on x_0 in $\langle x_0, \pi \rangle$ not meeting π and so, from the construction of π_1 we see that L' lies in π_1. Thus π_1 is a plane with $\{[L'] \mid L' \in \mathrm{L}(\pi)\} = \underline{A}$, so we may replace the plane π by π_1 in our proof. But now we have the situation in which N meets π non-trivially which was covered at the beginning of this case, so we are done.
Next we consider case (b). Clearly $M_x \in \underline{A}$ for each $x \in N$. If $x, y \in N$, then $y^\perp \cap M_x = M_x$ or $\emptyset$. Thus, $N \subseteq \Delta(M_x)$, whence, by lemma 3.16, $[N] \sim [M_x]$. Consequently, $[N]$ is $\underline{L}$-collinear with each $[M_x]$.
Next assume $[N] \equiv [L_0]$ for some L_0 in π. For any $L \in \mathrm{L}(\pi)$, we have $[L] \sim [L_0] \equiv [N]$, and so, by lemma 3.18, we see that $[N] \sim [L]$. Thus $[N]$ is collinear to all points of $\underline{A}$.
Therefore we may assume $\Delta(N) \cap \Delta(L) \neq \emptyset$ for any $L \in \mathrm{L}(\pi)$. Since, for $L \in \mathrm{L}(\pi)$ and $x \in N$ with $[L] \neq [M_x]$, we have $N \cap \Delta(L) = \emptyset$ (as $x^\perp \cap L = M_x \cap L$ is a point), it follows from lemma 3.16 that $[N] \sim [L]$. Thus $[N]$ is collinear with each member of $\underline{A}$. This completes case 5.

CASE 6. *point* $[N] \in \underline{P}_2$; *line* $\{\infty\} \cup X \in \underline{L}_3$. Assume $[N] \sim [L]$ for some $[L] \in X$. Then, for any $[M] \in X$ different from $[L]$, we have $[N] \sim [L] \equiv [M]$, and so, by lemma 3.18 once again, $[N] \sim [M]$. Thus, as $[M]$ was arbitrary in $X \setminus \{[L]\}$, we see that $[N]$ is $\underline{L}$-collinear with all points on the line $\underline{A}$.
Otherwise, $[N] \not\sim [L]$ for any $[L] \in X$, which means - as $[N] \not\equiv [L]$ for any $[L] \in X$ by the hypothesis $[N] \notin \underline{A}$ - that $[N]$ is $\underline{L}$-collinear only with ∞ on $\underline{A}$. This finishes case 6.

CASE 7. *point* ∞; *line* $M \cup \{[M]\} \in \underline{L}_1$. Then ∞ is adjacent only to $[M]$ so the 'one or all' rule holds.

CASE 8. *point* ∞; *line* $\{[L] \mid L \in \mathrm{L}(\pi)\} \in \underline{L}_2$. The point ∞ is collinear to all $[L]$.

CASE 9. *point* ∞; *line* $(\{\infty\} \cup X) \in \underline{L}_3$. In this case ∞ is incident with the line, so there is nothing to prove.

This establishes that $(\underline{P}, \underline{L})$ is a polar space. Nondegeneracy of $(\underline{P}, \underline{L})$ follows from the fact that, for each $x \in P$, the space $Res(x)$ is non-degenerate. Finally, $(\underline{P}_0, \underline{L}_0)$ is clearly a hyperplane of $(\underline{P}, \underline{L})$. □

5. Classification of hyperplanes

According to the fundamental result of Tits & Veldkamp (cf. TITS [7], Theorem 8.22), a polar space of rank ≥ 3 whose planes are Desarguesian is isomorphic to the polar space associated with a nondegenerate polarity, the polar space associated with a nondegenerate pseudo-quadratic form or the Grassmannian whose points are the lines of a 3-dimensional projective space $\mathbb{P}^3$ and whose lines are the pencils (X, π), where X is a projective point of the projective plane π, consisting of all projective lines on X in π.

We shall deal with the latter case first.

The Grassmannian of lines in $\mathbb{P}^3$. Let $\mathbb{P}^3$ denote the projective space of rank 3 over some (skew) field. Setting P for the set of lines of $\mathbb{P}^3$ and L for the set of pencils (X, π), where X is a point of $\mathbb{P}^3$ incident to the plane π of $\mathbb{P}^3$, we obtain a polar space (P, L) of rank 3 if incidence of the line $l \in P$ with the pencil (X, π) is given by $X \in l \subseteq \pi$.

5.1. Proposition. *If B is a hyperplane of (P, L), then either $B = l^\perp$ for some $l \in P$, or there is a symplectic polarity on $\mathbb{P}^3$ (i.e., a polarity with the property that all points of $\mathbb{P}^3$ are absolute) such that B coincides with its absolute lines.*

Proof. Suppose B contains a plane of (P, L). Then, up to duality, we may assume B contains all lines $l \in P$

contained in a plane π of $\mathbb{P}^3$. Take a point X of $\mathbb{P}^3$ outside π, and consider the plane of $(P, \mathbb{L})$ consisting of all lines of $\mathbb{P}^3$ on X. There must be a line $(X, \pi') \in \mathbb{L}$ all of whose P-points belong to B. The plane π' meets π in a line, say l. Now any $\mathbb{P}^3$-line meeting l belongs to B. For, if m is such a line meeting l in Z, say, consider the plane π'' on Z containing both X and m. Since the P-points $\pi'' \cap \pi$ and $\pi'' \cap \pi'$ both belong to $B \cap (Z, \pi'')$, the whole P-line (Z, π''), whence m, belongs to B. Thus, $B = l^{\perp}$.
Next, suppose B has rank 2. Then, in each plane π of $\mathbb{P}^3$, there is a unique point $\sigma(\pi)$ of $\mathbb{P}^3$ such that $(\sigma(\pi), \pi) \in \mathbb{L}(B)$. Also, each point X of $\mathbb{P}^3$, lies in a unique plane $\sigma(X)$ of $\mathbb{P}^3$ such that $(X, \sigma(X)) \in \mathbb{L}(B)$. It is readily seen that σ defines a symplectic polarity, and that $B = \{l \in P \mid \sigma(l) = l\}$, where $\sigma(l)$ is the line $\sigma(X_1) \cap \sigma(X_2)$ whenever $l = X_1X_2$ for X_1, X_2 distinct points of $\mathbb{P}^3$. □

The embedding of $(P, \mathbb{L})$ in the Veldkamp space is a "synthetic version" of the well-known Plücker embedding of the polar space $(P, \mathbb{L})$ in the Klein quadric, and is also valid if the division algebra is non-commutative. In the finite case, over $\mathbb{F}_q$, have $(q^2+1)(q^2+q+1)$ points in P and $q^2(q^3-1)$ hyperplanes corresponding to symplectic polarities, together accounting for the $(q^6-1)/(q-1)$ points of the Veldkamp space.

Projective embeddings. Now we suppose $(P, \mathbb{L})$ has an embedding (W, π, ϕ), that is, a thick projective space W with polarity π such that ϕ: $P \to W \backslash W^{\perp\pi}$ is an injection mapping lines to lines of W_π, the polar space of totally isotropic points and lines with respect to π, and such that $W = [\phi P]$. (Here we adopt the terminology and much of the notation of Tits [7], § 8.5.) We recall that a *morphism* μ: $(\overline{W}, \overline{\pi}, \overline{\phi}) \to (W, \pi, \phi)$ of embeddings of $(P, \mathbb{L})$ is a morphism μ: $\overline{W} \to W$ of projective spaces such that $\overline{\pi} = \mu^* \pi$ and $\phi = \mu\overline{\phi}$ (where $\mu^*\pi(x,y) = \pi(\mu x, \mu y)$ for $x, y \in \overline{W}$), and that an embedding (W, π, ϕ) is called dominant if every morphism of embeddings to it is an isomorphism.

By Tits [7], § 8.6 and 8.7, the embeddable polar space $(P, \mathbb{L})$ has a dominant embedding (W, π, ϕ), and $(P, \mathbb{L}) \cong W_\pi$ or W_κ, where κ is a projective pseudo-quadratic form in W with associated polarity π. In particular, up to polar space isomorphism, we may assume $(P, \mathbb{L}) = W_\pi$ or W_κ, and $\phi = id_P$.

5.2. Proposition. *Let $(P, \mathbb{L})$ be a nondegenerate polar space of finite rank ≥ 3 of the form W_π or W_κ, where W is a projective space and π is a polarity of W and κ is a projective pseudo-quadratic form with associated polarity π. Suppose that (W, π, id_P) is a dominant embedding. If H is a hyperplane of $(P, \mathbb{L})$, then $[H]$, the projective subspace of W spanned by H, is a hyperplane of W, and $H = [H] \cap P$.*

Proof. If $[H] \neq W$, then, for $x \in P \backslash H$, by lemma 2.1, $\langle H, x \rangle = P$, so $[H, x] = [P] = W$, showing that $[H]$ is a hyperplane. Furthermore, $[H] \cap P$ is proper subspace of P containing H, and so, by the same lemma, $H = [H] \cap P$, as required.
Therefore, assume $[H] = W$. Then (W, π, id_H) is an embedding of H. By Tits [7], § 8.6 (and the observation $\mathrm{rk}\, H \geq \mathrm{rk}\, P - 1 \geq 2$), there exists a morphism μ of embeddings of H from a dominant embedding $(\overline{W}, \overline{\pi}, \overline{\phi})$ to $W, \pi, id_H)$. (Thus $\mu\overline{\phi} = id_H$ and $\mu^* \pi = \overline{\pi}$). Since, by assumption $H \neq P$, theorem 8.6 and corollary 8.7 of [7] show that $P = W_\kappa$, and $\overline{\phi} H = \overline{W}_{\overline{\kappa}}$, where $\overline{\kappa}$ is a projective pseudo-quadratic form with associated polarity $\overline{\pi}$. Now $\mu^* \kappa$ (cf.[7], 8.4.1) and $\overline{\kappa}$ are both projective pseudo-quadratic forms with associated polarity $\overline{\pi} = \mu^* \pi$, and if $\overline{x} \in \overline{W}$ satisfies $\overline{\kappa}(\overline{x}) = 0$ (in the obvious interpretation that $\overline{q}(\overline{x}) = 0$ for any pseudo-quadratic form $\overline{q}$ representing $\overline{\kappa}$), then $\overline{x} \in \overline{\phi} H$, so $\mu(\overline{x}) \in H \subseteq P$, whence $\mu^* \kappa(\overline{x}) = \kappa(\mu(\overline{x})) = 0$. Thus, as $[\overline{H}] = \overline{W}$, (by [7] , 8.2.5), we have $\overline{\kappa} = \mu^* \kappa$. If $x \in P$, taking $\overline{x} \in \overline{W}$ with $x = \mu(\overline{x})$, we get $\overline{\kappa}(\overline{x}) = \mu^* \kappa(\overline{x}) = \kappa(x) = 0$, and so $x = \mu(\overline{x}) \in \mu(\overline{\phi} H) = H$, showing $P = H$, a contradiction. Hence the proposition. □

Non-embeddable polar spaces. The classification of polar spaces possessing planes non-Desarguesian planes has also been completed by Tits [7], see § 9.1. In this case, the planes are defined over a division Cayley algebra C. Conversely, for each division Cayley algebra C there is a unique nondegenerate polar space of rank at least 3 whose planes are defined over C. It has rank 3 and is not embeddable in a projective space.
Throughout the remainder of this section, we let $(P, \mathbb{L})$ be a nondegenerate polar space of finite rank 3 whose planes are defined over a division Cayley algebra C. Denote by n the norm map from C to k, the center of C. Then by [loc. cit.], $(P, \mathbb{L})$ has rank 3, is uniquely determined up to isomorphism, and for any

two noncollinear $x,y \in P$ the subspace $\{x,y\}^{\perp}$ is isomorphic to the dual Q^* of the generalized quadrangle Q associated with the quadratic form $C \oplus k^4 \to k$ defined by

$$(x_0; x_1,x_2,x_3,x_4) \mapsto n(x_0) - x_1x_3 + x_2x_4$$

where $x_0 \in C$ and $x_1,x_2,x_3,x_4 \in k$.

Let E be the algebraic k-group of linear transformations of $C \oplus k^4$ that is the direct product of the anisotropic orthogonal group (type D_4) over k on C (acting trivially on the direct summand k^4) and the group $GL(2,k)$ (an algebraic k-group of type A_1T_1, where T_1 indicates a 1-dimensional torus) acting trivially on C and on k^4 via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x_1,x_2,x_3,x_4) = (ax_1 + bx_2, cx_1 + dx_2, (dx_3 + cx_4)/(ad-bc), (bx_3 + ax_4)/(ad-bc))$$

Then E induces a group of automorphisms of Q, whence of Q^*, of type $D_4A_1T_1$, fixing the lines $(0;*,*,0,0) := \{(0;x_1,x_2,0,0) \mid x_1,x_2 \in k\}$ and $(0;0,0,*,*)$ of Q.

The following result, derived by study of Q, establishes nonexistence of certain kinds of hyperplane of Q^*.

5.3. Lemma. *Let Q^* be as above.*

(i) *Each hyperplane of Q^* of rank 2 is of the form $p^{\perp}$ for some point p of Q^*.*

(ii) *There is no E-invariant ovoid on Q^* containing $\{x,y\}^{\perp}$, where $x = (0;*,*,0,0)$ and $y = (0;0,0,*,*)$.*

Proof. Let H^* be a hyperplane Q^*. Then its dual H is a set of points and lines of Q with the following properties.

(a) If l, m are lines of H meeting in a point $p \in Q$, then p belongs to H.

(b) If p is a point of H, then any line of Q on p belongs to H.

(c) Every point of Q belongs to at least one line of H.

As a direct consequence of (a) and (b), we obtain:

(d) If p, q are noncollinear points of H, then $p^{\perp} \cap q^{\perp}$ is entirely contained in H.

(i). Now suppose H^* has rank 2. Then it contains a line, and so H contains a point. Assume that H^* contains no deep point. Then, for each line l of H, there is $m \in H$ with $l \cap m = \varnothing$. We claim that H contains a quadrangle. For take a point p_1 of H and a line l_1 containing p_1. Then, by (b), l_1 belongs to H. As we have just seen, there is a line l_3 in B with $l_1 \cap l_3 = \varnothing$. By the 'one or all' axiom, there is a line l_2 on p_1 with $l_2 \cap l_3 = \{p_2\}$ for some point p_2. Then l_2 and p_2 belong to B in view of (b) and (a). Again by the above, there is a line l_4 in B disjoint from l_2. Now, letting p_3, p_4 be the unique points of $p_2^{\perp} \cap l_4$, $p_1^{\perp} \cap l_4$, respectively, we obtain the quadrangle with points p_1, p_2, p_3, p_4 fully contained in H, as claimed. Since the automorphism group of Q is (Moufang and hence) transitive on the set of quadrangles in Q, there is no loss in assuming that $p_1 = (0;1,0,0,0)k$, $p_2 = (0;0,1,0,0)k$, $p_3 = (0;0,0,1,0)k$, and $p_4 = (0;0,0,0,1)k$ belong to H. Take $a,b \in C$ with $a \neq b$ and $n(a) = n(b) = 1$. (This choice is possible.) Then $p_a = (a;0,-1,0,1)k$ and $p_b = (b;0,-1,0,1)k$ are distinct points of Q contained in $p_1^{\perp} \cap p_3^{\perp}$ and so, by (d), belong to H. Moreover, they are noncollinear, so for each $\lambda \in k$, the point $q_\lambda = (0;1,\lambda,\lambda^2,\lambda)k$, being contained in $p_a^{\perp} \cap p_b^{\perp}$, also belongs to H. Finally, p_3 and q_λ are non collinear, so $(0;0,0,\lambda,1)k$ being in $p_3^{\perp} \cap q_\lambda^{\perp}$, belongs to H for each $\lambda \in k$. We conclude that all the points of the line $p_3p_4 = \{(0;0,0,\lambda,\mu)k \mid \lambda, \mu \in k\}$ of Q are in H, so that p_3p_4 is a deep point of H^*. This contradicts the assumption, and so ends the proof of (i).

(ii). Now suppose H^* is an ovoid containing $\{x,y\}^{\perp}$. Then H is a spread of Q. Notice that the part of the spread corresponding to the subset $\{x,y\}^{\perp}$ of H^* covers all points $(x_0;x_1,x_2,x_3,x_4)$ of Q having $x_0 = 0$. Suppose z were the line of H on the point $p := (1;1,0,1,0)$ of Q. Then z contains a point $q = (q_0;q_1,q_2,q_3,q_4)$ of Q with $p \perp q$. Now p_0 and q_0 are linearly independent over k (for otherwise, z would contain a Q-point of $(0;*,*,*,*)$, and so meet a member of $\{x,y\}^{\perp\perp}$). Replacing q by a $q+\lambda p$ for a suitable $\lambda \in k$, we may assume, $(q_0 \mid 1) = 0$. The stabilizer E_p of p in E contains the orthogonal group of type B_3, so there is an E_p-conjugate $q' = (q'_0; q_1, q_2, q_3, q_4)$ of q with $q'_0 \neq q_0$. Thus, if H^* is E-invariant, the line on p and q' also belongs to the spread H, contains p and is distinct from z. This is absurd. Hence the lemma. □

It is immediate from part (i) of the above lemma that if H is a hyperplane of (P,L) of rank 3, then $H = x^{\perp}$ for some $x \in P$. Thus, it remains to classify nondegenerate hyperplanes of rank 2.

5.4. Lemma. *Let H be a nondegenerate hyperplane of rank 2. For each $a \in H$, the residue H_a of H at a is an ovoid in the generalized quadrangle P_a with the property that, if xa and ya are noncollinear points of P_a (where $x, y \in a^{\perp}\backslash\{a\}$) with $|\{xa, ya\}^{\perp} \cap H_a| > 1$, then $\{xa, ya\}^{\perp} \subseteq H_a$. Moreover, either $\{x,y\}^{\perp} \subseteq H$ or $\{x,y\}^{\perp} \cap H = \{x,y,a\}^{\perp}$.*

Proof. For x,y,a as indicated, take u,v to be points of $H \cap a^{\perp}\backslash\{a\}$ such that ua, va are distinct points of $\{xa, ya\}^{\perp} \cap H_a$. Since H has rank 2, H_a contains no lines. In particular, the points u,v are noncollinear. The subspace $\{x,y\}^{\perp} \cap H$ is a hyperplane of the generalized quadrangle $\{x,y\}^{\perp}$ (it is of rank 2 as it contains the line ua) and, by the previous lemma, either has shape $\{x,y,b\}^{\perp}$ for some $b \in \{x,y\}^{\perp}$ or coincides with $\{x,y\}^{\perp}$. In the latter case, we have $\{x,y\}^{\perp} \subseteq H$ and we are done, so assume the former. Then $b \in \{x,y,u,a,v\}^{\perp} = \{a\}$, so $\{x,y\}^{\perp} \cap H = \{x,y,a\}^{\perp}$, as required. □

The following lemma states that every point of the Veldkamp space lies on a *secant*, that is a line with at least two points of the form $x^{\perp}$ for some $x \in P$.

5.5. Lemma. *Let H be a nondegenerate hyperplane of (P,L) of rank 2. For each quadrangle $V \subset H$, and any two $x,y \in V^{\perp}\backslash H$, we have $\{x,y\}^{\perp} = x^{\perp} \cap H = y^{\perp} \cap H$.*

Proof. Clearly, x and y are noncollinear. Let $a \perp u \perp b \perp v \perp a$ be the circuit in V. The points xa, ya of P_a are noncollinear and satisfy $\{xa,ya\}^{\perp} \cap H_a \supseteq \{ua, va\}$. By the previous lemma, $\{x,y\}^{\perp} \not\subseteq H$ implies $\{x,y\}^{\perp} \cap H = \{x,y,a\}^{\perp}$. But then, likewise we have $\{x,y\}^{\perp} \cap H = \{x,y,b\}^{\perp}$, contradicting $a \in \{x,y,a\}^{\perp}\backslash b^{\perp}$. Hence $\{x,y\}^{\perp} \subseteq H$.
Suppose $h \in x^{\perp} \cap H$. If $z \in y^{\perp} \cap hx\backslash\{h\}$, then $z \in \{x,y\}^{\perp} \subseteq H$ and $h \in H$, so $x \in hz \subseteq H$, contradicting $x \notin H$ (for $x \in H$ would imply $\mathrm{rk} H = 3$). Hence $h \in \{x,y\}^{\perp}$, proving $\{x,y\}^{\perp} = x^{\perp} \cap H$. The remainder follows by symmetry in x and y. □

5.6. Lemma. *Let H be a nondegenerate hyperplane of (P,L) of rank 2. If $a,b \in H$ are distinct, then $\{a,b\}^{\perp\perp} \subseteq H$.*

Proof. If $a \perp b$, then $\{a,b\}^{\perp\perp}$ is the line on a and b, and so belongs to H by the definition of subspace. Otherwise, let $c \in \{a,b\}^{\perp\perp}$ and take distinct $u,v \in \{a,b\}^{\perp} \cap H$. There are $x,y \in \{a,b,u,v\}^{\perp}\backslash H$. By the above lemma, $\{x,y\}^{\perp} \subseteq H$. Thus, $c \in \{a,b\}^{\perp\perp} \subseteq \{x,y\}^{\perp} \subseteq H$, as required. □

We shall exploit the following description of (P,L). Recall that k is the center of the Cayley division algebra C and that $n: C \to k$ is the norm map. By proposition 7 of Tits [8], there is an algebraic group G of adjoint type E_7 over k whose anisotropic kernel is isogenous to $SO(C,n)'$ (the commutator subgroup of the group of all linear transformations the 8-dimensional k-vector space C that leave invariant the norm n). Adopting the labeling of Bourbaki [1] for the nodes of the Dynkin diagram of type E_7, the only maximal parabolic subgroups of G containing a fixed Borel subgroup are P_1, P_6, and P_7, the indices indicating the nodes of the diagram to which they correspond. According to Tits [7] the building whose polar space is (P,L) can be viewed as the rank 3 geometry whose points, lines, and planes are the cosets of respectively P_1, P_6, and P_7 in G, and in which two elements are incident if and only if they intersect (nonemptily).

A *root group of* G is a subgroup G-conjugate to the center of the unipotent radical of P_1. It is isomorphic to k^+. (The full unipotent radical R is unipotent of dimension 33 over k; its commutator subgroup coincides with the root group.) Denote by P_o the G-class of all root groups. For any two $x,y \in P_o$, we have either $[x,y] = 1$ or $\langle x,y \rangle \cong SL(2,k)$. We shall write $x \perp y$, and say x and y are collinear if $[x,y] = 1$. This definition is justified by the choice of lines in the following lemma.

5.7. Lemma. *The polar space $(P,\mathbf{L})$ of rank 3 defined over C is isomorphic to the space $(P_o,\mathbf{L}_o)$ whose lines are the sets $\{x,y\}^{\perp\perp}$ for any two distinct $x,y \in P_o$ with $[x,y]=1$.*

Proof. By what has been said above, P can be viewed as $\{P_1^g \mid g \in G\}$, and $\mathbf{L}$ as the collection of all $\{P_1^{g_1}, P_1^{g_2}\}^{\perp\perp}$ for $g_i \in G$ $(i=1,2; P_1^{g_1} \neq P_1^{g_2})$, where $P_1^{g_1} \perp P_1^{g_2}$ if and only if there is $h \in G$ with $P_1^{g_1} \cap P_6^h \neq \emptyset$ and $P_1^{g_2} \cap P_6^h \neq \emptyset$. Now if $x \in P_o$, then $N_G(x) \supseteq P_1^g$ for some $g \in G$ (and hence coincides with it); setting $f(x) = P_1^g$, we obtain a well-defined bijection $f: P_o \to P$. Moreover, $x,y \in P_o$ commute if and only if they are both in a conjugate of P_6. Hence the collinearity graphs of $(P_o,\mathbf{L}_o)$ and $(P,\mathbf{L})$ are isomorphic. Since $(P_o,\mathbf{L}_o)$ is built from its collinearity graph in the same way that $(P,\mathbf{L})$ can be obtained from its collinearity graph, the lemma follows. □

We shall now identify $(P,\mathbf{L})$ and $(P_o,\mathbf{L}_o)$. This enables us to consider P as a collection of (root) subgroups of G, and to consider G as a group of automorphisms of $(P,\mathbf{L})$.
Some more notation: for $S \subseteq G$ and $X \subseteq P$, we write $S \cap X := \{x \in X \mid x \subseteq S\}$ and $G(X)$ for the group generated by all $x \in X$.

/

5.8. Lemma. *Let X be a subspace of $(P,\mathbf{L})$. Then*

(i) *$G(X) \cap P$ is a union of $G(X)$-orbits; any two points from different orbits commute.*

(ii) *Suppose $\{u,v\}^{\perp\perp} \subseteq X$ for any two non-collinear $u,v \in X$. Then $y \in X$ implies that the full $G(X)$-orbit $y^{G(X)}$ of y belongs to X.*

Proof. (i). If, for $x,y \in G(X) \cap P$, we have $[x,y] \neq 1$, then, as the subgroup of G generated by x and y is isomorphic to $SL(2,k)$, we have $x \in y^{G(\{x,y\})}$. In particular, $x \in y^{G(X)}$. Thus, any two elements of $G(X) \cap P$ either commute or are $G(X)$-conjugate, whence (i).
(ii). Suppose $y \in X$. By induction on the length of an element of $G(X)$ expressed as a product of elements from root groups, it suffices for the proof of the last statement of the lemma to show that $y^\xi \in X$ for any $\xi \in x \in X$. If $[\xi,y]=1$, this is trivial, so suppose the contrary. Then $\{x,y\}^{\perp\perp} \subseteq X$ by the assumption on X. On the other hand, $G(\{x,y\})$ stabilizes this subspace (for, it stabilizes $(\langle x,y\rangle \cap P)^\perp$ which, by lemma 5.7, coincides with $\{x,y\}^\perp$), so $y^\xi \in \{x,y\}^{\perp\perp}$, whence $y^\xi \in X$. □

5.9. Proposition. *Let H be a hyperplane of $(P,\mathbf{L})$. Then $H = x^\perp$ for some $x \in P$.*

Proof. Suppose H is a nondegenerate hyperplane of P of rank 2. By lemma 5.5 there are non-collinear $x,y \in P$ with $\{x,y\}^\perp \subseteq H$. Take $u \in \{x,y\}^\perp$ and consider $O = (H \cap u^\perp)/u$ of $u^\perp/u$. The latter space is isomorphic to Q^* and its subspace O must be a hyperplane of rank 1, so is an ovoid, and contains $\{xu,yu\}^\perp$. By lemmas 5.6 and 5.8(ii) H is $G(H)$-invariant, so O is $C_{G(H)}(u)$-invariant. In particular, $C_G(\{x,y\})(u)$ induces an algebraic k-group of automorphisms of $u^\perp/u$ of type $D_4A_1T_1$ fixing the points xu and yu, and stabilizing O. But then lemma 5.3(ii) asserts that no such O exists, a contradiction. Hence the proposition. □

5.10. Remark. Part of the difficulties in establishing the above result arise from ignorance: we do not know whether, apart from the obvious algebraic subgroups, there are other overgroups in G of the algebraic subgroup $G(\{x,y\}^\perp)$ of type D_6. (Here x and y are as in the above proposition.) If the classification of all Moufang generalized quadrangles were available, another way to circumvent this problem would be to use that then the overgroup $G(H)$ of the hypothetical nondegenerate hyperplane H of $(P,\mathbf{L})$ containing $\{x,y\}^{\perp\perp}$ is known:

Lemma. *Any nondegenerate hyperplane H of rank 2 is a Moufang generalized quadrangle (cf. p. 274 of* TITS *[7]) and the group of all root automorphisms U_α (α a root of H; notation of [loc. cit.]) belongs to $G(H)$.*

Proof. Suppose $a \perp u \perp b \perp v \perp a$ are the points of a quadrangle in H. Then the points and lines of this quadrangle form an apartment of H. To verify that H is Moufang, we shall consider the representatives $\pi = \{u, ua, a, av, v\}$ and $\lambda = \{ub, u, ua, a, av\}$ of the two kinds of roots, and verify that $G(\{a,u,v\} \cup ua)$ contains the subgroup U_α for both $\alpha = \pi, \lambda$.
Assume $\alpha = \pi$. Then, by definition, U_α is the kernel of the action of the pointwise stabilizer in $\mathrm{Aut}(H)$ of

$\{u,a,v\} \cup ua \cup va$ on the set of all lines through $a^{\perp}$. Assume $b' \in \{u,v\}^{\perp} \cap H$. It suffices to find an element in $U_\alpha \cap G(ua)$ moving b to b'. The subgroup $C_G(u,v)$ of G centralizing u and v contains the orthogonal group D of type D_6 over k of Witt index 2; the root groups a,b, and b' belong to it and are 'classical root groups' of the classical group D (centers of the unipotent radical of the stabilizer of a line in the classical embeddable generalized quadrangle Q associated with D). In particular, there is $A \in a \in D$, fixing $\{a\}^{\perp}$ pointwise, such that $b^A = b'$. But then $A \in a \subseteq U_\alpha \cap G(a)$, as required.
$\alpha = \lambda$. Then U_α is the kernel of the action of the pointwise stabilizer of ua on the set of all lines passing through u or a. Assume $b' \in ub \setminus \{u\}$ and $v' \in av \setminus \{a\}$ are collinear. We need to find $B \in U_\alpha$ moving bv to $b'v'$. By taking $x,y \in \{a,b,u,v\}^{\perp}$, and considering $C_G(x,y)$ (again an algebraic k-group of type D_6 corresponding to the 12-dimensional orthogonal group over k of Witt index 2), an element $B \in G(ua) \cong (k^+)^{10}$ can be found that fixes every line on a and on u lying in $\{x,y\}^{\perp}$, and satisfies $b^B = b'$ and $v^B = v'$. As $ua \subseteq H$ and B also fixes x and y, we have $B \in U_\alpha \cap G(ua)$. Hence the lemma. □

We end the remark by indicating how we could use the above lemma to prove that every hyperplane of (P, L) is of the form $x^{\perp}$ if the classification of Moufang generalized quadrangles were available: Suppose H is a hyperplane wothout a deep point. Then, as in the proof of proposition 5.9 we can show that H is a non-degenerate generalized quadrangle and that there exist non-collinear $x,y \in P \setminus H$ with $\{x,y\}^{\perp} \subseteq H$. Now, $D := G(\{x,y\}^{\perp})$ is an algebraic k-group of type D_6 (Witt index 2) contained in H. But then, by the (*assumed*) classification of Moufang generalized quadrangles, $G(H)$ is an algebraic k-group which is an overgroup in G. The only maximal proper algebraic k-groups which are overgroups of D are parabolics of type D_6 and $DC_G(D) \cong D.SL(2,k)$. If $G(H)$ is a parabolic R of type D_6, we get $H \subseteq R \cap P = z^{\perp}$ for some root z in the center of the unipotent radical of R, and if $G(H) \subseteq DC_G(D)$, then $H \subseteq G(H) \cap P \subseteq (DC_G(D)) \cap P = \{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp}$. In both cases, according to lemma 5.8, the rank of H must be 3, a contradiction. Since also $G(H) = G$ leads to the contradiction $H = P$, this again establishes the nonexistence of rank 2 hyperlpanes in (P, L).

Summarizing the three propositions in this section, we get

5.11. Theorem. *Every hyperplane of a polar space of rank at least 3 that is not of the form $x^{\perp}$ for some point x, arises from a suitable embedding of the polar space in a projective space by intersecting it with a hyperplane of that projective space.*

6. References

1. Bourbaki, N., *Groupes et algèbres de Lie, Chap. IV, V, VI*, Hermann, Paris (1968).
2. Buekenhout, F. and E.E. Shult, *On the foundations of polar geometry*, Geom. Dedicata 3 (1974) 155-170.
3. Buekenhout, F. and A. Sprague, *Polar spaces having some line of cardinality two*, J. Combinatorial Th. A 33 (1982) 223-228.
4. Hall, J.I. and E.E. Shult, *Locally cotriangular graphs*, Geom. Dedicata 18 (1985) 113-159. Zbl 578.05058
5. Johnson, P. and E.E. Shult, *Local characterizations of polar spaces*, Geom. Dedicata (1989) 127-151.
6. Teirlinck, L., *On linear spaces in which every plane is either projective or affine*, Geometriae Dedicata 4 (1975) 39-44.
7. Tits, J., *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math., Vol. 386, Springer, Berlin (1974).
8. Tits, J.L., *Classification of algebraic semisimple groups*, pp. 33-62 in: *Algebraic groups and discontinuous subgroups, Boulder 1965*, Proceedings of Symp. in Pure Math., vol. IX, Amer. Math. Soc., Providence, 1966.
9. Veldkamp, F.D., *Polar Geometry I-IV*, Indag. Math. 21 (1959) 512-551.