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M. Zwaan

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Stability and Error Estimates for Sinc Interpolation

M. Zwaan

Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands.

In this paper we consider stability and error estimates for sinc interpolation. This interpolation technique can be used for interpolating a finite or countable number of data. There is a large amount of literature on the subject of error estimates for sinc interpolation. In this paper we derive estimates by means of operator norms and obtain new bounds for the amplitude error and the time jitter error, which apply for non uniform sampling.

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0. Introduction.

In this paper we consider stability and error estimates for the following problem. Given the data $\{g_i\} \in \ell^2(\mathbb{I})$ and time markers $\{t_i\}_{i\in \mathbb{I}}$, where g_i is the value of some function at time t_i. Here $\mathbb{I}$ is an index set which is equal to $\mathbb{Z}$ in the first two sections. In the last section conclusions are stated in the case that $\mathbb{I}$ is a finite subset of $\mathbb{Z}$. The space $\ell^2(\mathbb{I})$ is the Hilbert space of sequences of complex numbers $\{g_i\}$ such that

$$\|g\|_{\ell^2}^2 := \sum_{i\in I} |g_i|^2 < \infty.$$

The problem is to find a function f which lies in a Hilbert space of interpolating functions such that

$$\sqrt{\pi/r}\ f(t_i\ \pi/r) = g_i \quad \forall i \in \mathbb{I}. \tag{0.1}$$

The reason for the factors π/r and $\sqrt{\pi/r}$ will become clear in section 1. In this paper we consider stability and error estimates corresponding to problem (0.1).

The first error estimate we compute is the amplitude error, which is defined as follows. Suppose the data $\{g_i\}$ are perturbed to $\{g_i'\}$. The solution which corresponds to the perturbed problem is called f' and satisfies

$$\sqrt{\pi/r}\ f'(t_i\ \pi/r) = g_i' \quad \forall i \in \mathbb{I}. \tag{0.2}$$

The amplitude error is defined as the difference between f and f' in supremum norm,

$$e_{\text{amp}} = \|f - f'\|_{\text{sup}}.$$

A second error estimate is called the time jitter error, which is defined as follows. Suppose the measurement times $\{t_i\}$ are perturbed to $\{t_i'\}$. The solution that corresponds to the perturbed problem is again denoted by f' and satisfies

$$\sqrt{\pi/r}\ f'(t_i'\ \pi/r) = g_i \quad \forall i \in \mathbb{I}. \tag{0.3}$$

The time jitter error is the difference between f and f' in the supremum norm,

$$e_{\text{tj}} = \|f - f'\|_{\text{sup}}.$$

There is a lot of literature on error estimates in the case of uniform sampling (i.e. $t_i = i$, for $i \in \mathbb{Z}$). The purpose of this paper is to prove new error bounds which apply for non uniform sampling, by expressing the errors in terms of linear operators and to estimate their norms. In this way a rather transparent derivation of the error bounds is obtained. In literature alternate methods are being used, which are discussed in the last section.

In the next section we explain some preliminary notions. In section 2 estimates for the time jitter error and the amplitude error are given. In section 3 conclusions about stability of the problem (0.1) are stated in the case of finite index sets.

1. Preliminaries.

Most of the material in this section can be found in [9]. We introduce the Paley-Wiener space, $\mathbb{P}_r$, which consists of entire functions $f: \mathbb{C} \to \mathbb{C}$ that are of exponential type r, such that the restriction of f to the real line lies in $L^2(\mathbb{R})$.

Definition 1.1. $f: \mathbb{C} \to \mathbb{C}$, is called an entire function of exponential type r, if there is a real constant $\gamma \geq 0$ such that $|f(z)| \leq \gamma e^{r|Im z|}$.
The space $\mathbb{P}_r$ is given by $\left\{ f: \mathbb{C} \to \mathbb{C} \mid \quad f_{|R} \in L^2(\mathbb{R}), \text{ and } f \text{ is entire of exponential type } r \right\}$

Denote the Fourier transform of a function u by $\hat{u}$ and the inverse Fourier transform by $\check{u}$. Define the space $L^2_r(\mathbb{R})$ as the set of $L^2(\mathbb{R})$-functions that are zero almost everywhere outside the interval $[-r, r]$. There is a characterization of $\mathbb{P}_r$,

$$f \in \mathbb{P}_r \iff \exists u \in L^2_r(\mathbb{R}), \text{ such that } \check{u} = f.$$

Define the inner product $\langle\ ,\ \rangle_{P_r}$ on $\mathbb{P}_r$ by,

$$\langle f, g \rangle_{P_r} := \int_R f(x)\overline{g(x)}dx.$$

With Parseval's identity and the above characterization,

$$\langle f, g \rangle_{P_r} = \int_{[-r,r]} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

So the Fourier transform is an isometric isomorphism between $\mathbb{P}_r$ and $L^2_r(\mathbb{R})$. From this it follows that $\mathbb{P}_r$ is a Hilbert space as well.

A useful inequality is obtained in [8] p. 107,

$$\|f\|_{\sup} \leq \|f\|_{P_r}, \forall f \in \mathbb{P}_r. \tag{1.1}$$

Here the supremum norm is defined by $\|f\|_{\sup} := \sup_{t \in R}|f(t)|$.

$\{ \sqrt{r/\pi}\ \text{sinc}_r(. - i\ \pi/r)\}_{i \in I}$ is an orthonormal basis for $\mathbb{P}_r$, which is denoted by $\{h_i\}_{i \in I}$. Here the *sinc*-function is defined by

$$\text{sinc}_r(t) := \begin{cases} \dfrac{\sin(rt)}{rt}, & t \neq 0 \\ 1, & t = 0 \end{cases}.$$

A system $\{\varphi_i\}_{i \in I}$ is called a Riesz basis for $\mathbb{P}_r$ if there exists a bounded linear invertible operator T on $\mathbb{P}_r$ such that

$$T\varphi_i = h_i, \quad \forall i \in \mathbb{I}. \tag{1.2}$$

An operator T on a Hilbert space is called invertible if its inverse, denoted by T^{-1}, exists and is bounded. Note that an orthonormal basis is a Riesz basis. An example of a non-orthonormal Riesz basis is the following. Let $\{t_i\}_{i\in I}$ be a sequence of real numbers such that

$$|t_i - i| \leq \alpha < 1/4, \quad \forall i \in \mathbb{I},$$

and write $\varphi_i = \sqrt{r/\pi}\ \mathrm{sinc}_r(. - t_i\pi/r)$. Then $\{\varphi_i\}_{i\in I}$ is a Riesz basis for $\mathbb{P}_r$. Moreover the norms of the operators T and T^{-1} can be estimated,

$$\|T\| \leq \frac{1}{1-\lambda}, \quad \|T^{-1}\| \leq 1 + \lambda, \tag{1.3}$$

where

$$\lambda := 1 - \cos \pi\alpha + \sin \pi\alpha. \tag{1.4}$$

In the rest of this paper we take $\{t_i\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ as defined above. In the case that $\mathbb{I}$ is a finite index set and $\{t_i\}_{i\in I}$ is a sequence of distinct real numbers, the system $\{\varphi_i\}_{i\in I}$ is a Riesz basis for its linear span. The point evaluation can be written in terms of an inner product in the space $\mathbb{P}_r$,

$$\langle f, \varphi_i \rangle_{P_r} = \sqrt{\pi/r}\ f(t_i\ \pi/r), \quad \forall f \in \mathbb{P}_r. \tag{1.5}$$

Two systems of vectors $\{\psi_i\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ are called biorthogonal if

$$\langle \varphi_i, \psi_j \rangle_{P_r} = \delta_{ij}, \quad \forall i, j \in \mathbb{I}.$$

δ_{ij} is the Kronecker delta. If $\{\varphi_i\}_{i\in I}$ is a Riesz basis, then its unique biorthogonal system $\{\psi_i\}_{i\in I}$ also is a Riesz basis, which is given by

$$\psi_i = T^* h_i, \quad \forall i \in \mathbb{I}. \tag{1.6}$$

T^* is the adjoint of T.

By (1.5) the interpolation problem (0.1) is a special type of moment problem. That is, given a Riesz basis $\{\varphi_i\}_{i\in I}$ with a biorthogonal sequence $\{\psi_i\}_{i\in I}$ and a sequence of complex numbers $\{g_i\}_{i\in I}$ lying in $\ell^2(\mathbb{I})$, we want to find a function f which lies in $\mathbb{P}_r$ such that

$$\langle f, \varphi_i \rangle_{P_r} = g_i, \quad \forall i \in \mathbb{I}. \tag{1.7}$$

The unique solution to this problem is

$$f = \sum_{i\in I} g_i \psi_i. \tag{1.8}$$

To obtain an explicit formula for the system $\{\psi_i\}_{i\in I}$ in formula (1.8), we introduce the Gram matrix of a system of vectors $\{\varphi_i\}_{i\in I}$ as

$$G_{ij} = \langle \varphi_j, \varphi_i \rangle_{P_r}, \quad \forall i, j \in \mathbb{I}.$$

By (1.5) this reduces to

$$G_{ij} = \mathrm{sinc}_\pi(t_i - t_j). \tag{1.9}$$

If $\{\varphi_i\}_{i\in I}$ is a Riesz basis, then the Gram matrix is a bounded linear invertible operator on ℓ^2. The biorthogonal system $\{\psi_i\}_{i\in I}$ can be expressed in terms of the Gram matrix and in terms of $\{\varphi_i\}_{i\in I}$,

$$\psi_i = \sum_{j\in I} \overline{(G^{-1})}_{ij}\varphi_j, \quad \forall i \in \boldsymbol{I}. \tag{1.10}$$

The inverse of the Gram matrix is the matrix representation of the operator TT^* in terms of the orthonormal basis $\{h_i\}_{i\in I}$. We estimate the norm of G^{-1} by $\|G^{-1}\| = \|TT^*\| = \|T\|^2$ and

$$\|G^{-1}\|^{1/2} = \|T\| \le \frac{1}{1-\lambda}, \tag{1.11}$$

where λ is given by (1.4). Similarly

$$\|G\|^{1/2} = \|T^{-1}\| \le 1+\lambda. \tag{1.12}$$

2. The amplitude error and the time jitter error.

Our first objective is an estimate of the amplitude error. Let $\{g_i\}_{i\in I}$ and $\{g_i'\}_{i\in I}$ be the data, and the perturbed data respectively, both lying in $\ell^2(\boldsymbol{I})$. Throughout this section the index set $\boldsymbol{I}$ is equal to $\mathbb{Z}$. In order to derive an estimate for the amplitude error, we prove the following proposition, which holds in the case of arbitrary separable Hilbert spaces $\mathcal{H}$. A Riesz basis for a Hilbert space is defined in the same manner as (1.2), where $\{h_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$.

Proposition 2.1 . *Let $\{\varphi_i\}_{i\in I}$ be a Riesz basis in a Hilbert space $\mathcal{H}$ (cf. (1.2)), with biorthogonal system $\{\psi_i\}_{i\in I}$, and suppose $\{g_i\}, \{g_i'\} \in \ell^2(\boldsymbol{I})$. The following estimate holds,*

$$\|\sum_{i\in I}(g_i - g_i')\psi_i\|_{\mathcal{H}} \le \|G^{-1}\|^{1/2}\, \|g-g'\|_{\ell^2}.$$

Proof:

By (1.6) and (1.11) we obtain

$$\|\sum_{i\in I}(g_i - g_i')\psi_i\|_{\mathcal{H}} =$$

$$\|\sum_{i\in I}(g_i - g_i')T^*h_i\|_{\mathcal{H}} \le \|T\|\, \|\sum_{i\in I}(g_i - g_i')h_i\|_{\mathcal{H}} =$$

$$\|T\|\, \|g-g'\|_{\ell^2} = \|G^{-1}\|^{1/2}\, \|g-g'\|_{\ell^2}.$$

This proves the proposition. □

Note that this proposition also holds in the case of finite linearly independent system of vectors $\{\varphi_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$. The solution to (0.1) is called f, the solution to the perturbed problem (0.2) is denoted by f', which are given by $\sum_{i\in I} g_i\psi_i$ and $\sum_{i\in I} g_i'\psi_i$ respectively (cf. (1.8)). By (1.1) and Proposition 2.1 we obtain in the case that $\mathcal{H} = \mathbb{P}_r$,

$$e_{\text{amp}} \leq \|G^{-1}\|^{1/2}\ \|g - g'\|_{\ell^2}. \tag{2.1}$$

From this estimate it follows that the solution is stable for perturbation of the data, since G^{-1} is a bounded operator on $\ell^2(\mathbb{I})$. The norm of G^{-1} is estimated by $\|G^{-1}\| \leq \frac{1}{1-\lambda}$, where λ is given by (1.4). We see that the norm of G^{-1} in the case of uniform sampling is equal to 1. In the case of nonuniform sampling the norm of G^{-1} may become larger if α tends to 1/4. The problem (0.1) is called well conditioned if $\|G^{-1}\|$ is not too large (i.e. close to 1), otherwise it is called ill conditioned. In the case of uniform sampling ($\alpha = 0$) the problem is well conditioned for perturbation of the data and the problem is ill conditioned if α is close to 1/4.

Next we consider the time jitter error. Let $\{t_i\}_{i\in I}$ and $\{t_i'\}_{i\in I}$ be the sequences of exact, respectively perturbed time markers. The solution to the exact problem (0.1) is written as f and the solution to the perturbed problem (0.3) as f'. Define $\varphi_i' = \sqrt{r/\pi}\ \mathrm{sinc}_r(. - t_i'\pi/r)$ and suppose

$$|t_i' - i| \leq \alpha' < 1/4, \quad \forall i \in \mathbb{I}.$$

Then $\{\varphi_i'\}_{i\in I}$ is a Riesz basis for $\mathbb{P}_r$. Moreover there exists a bounded linear invertible operator T' such that

$$T'\varphi_i' = h_i, \quad \forall i \in \mathbb{I} \tag{2.2}$$

and

$$\|T'\| \leq \frac{1}{1-\lambda'}, \quad \|T'^{-1}\| \leq 1 + \lambda', \tag{2.3}$$

where

$$\lambda' := 1 - \cos\pi\alpha' + \sin\pi\alpha'. \tag{2.4}$$

The biorthogonal system of $\{\varphi_i'\}$ is denoted as $\{\psi_i'\}$, which can be computed by

$$\psi_i' = \sum_{j\in I} \overline{(G'^{-1})_{ij}}\varphi_j', \quad \forall i \in \mathbb{I}. \tag{2.5}$$

Here G' is the Gram matrix of the system $\{\varphi_i'\}$,

$$G_{ij}' = \langle\varphi_j', \varphi_i'\rangle = \mathrm{sinc}_\pi(t_i' - t_j').$$

Again we have a relation between T' and G',

$$\|G'^{-1}\|^{1/2} = \|T'\|. \tag{2.6}$$

The solutions f to problem (0.1) and f' to problem (0.3) are given by, $f = \sum_{i\in I} g_i\psi_i$, and $f' = \sum_{i\in I} g_i\psi_i'$.

In order to find an estimate for the time jitter error, we choose the following approach. We look for a perturbation operator V, such that $V\varphi_i = \varphi_i'$, for all $i \in \mathbb{I}$. If such an operator exists, then we have the relation $\psi_i = V^*\psi_i'$.

The following proposition expresses the difference between f and f' in terms of the norm of T' and $I - V$. This proposition holds for arbitrary separable Hilbert spaces $\mathcal{H}$.

Proposition 2.2 . *Let $\{\varphi_i'\}$ and $\{\varphi_i\}$ be two systems of vectors in a separable Hilbert space $\mathcal{H}$, such that there exists a bounded linear operator V with $V\varphi_i = \varphi_i'$, for all $i \in \boldsymbol{I}$. Let $\{\psi_i'\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be their respective biorthogonal sequences and assume that $\{g_i\} \in \ell^2(\boldsymbol{I})$. If $\{\varphi_i'\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ (cf. (2.2)), then*

$$\|\sum_{i\in I} g_i\psi_i - \sum_{i\in I} g_i\psi_i'\|_{\mathcal{H}} \le \|I - V\| \; \|T'\| \; \|g\|_{\ell^2}.$$

Proof:

$$\|\sum_{i\in I} g_i(\psi_i' - \psi_i)\|_{\mathcal{H}} = \|\sum_{i\in I} g_i(I - V^*)\psi_i'\|_{\mathcal{H}} \le \|I - V\| \; \|\sum_{i\in I} g_i\psi_i'\|_{\mathcal{H}} =$$

$$\|I - V\| \; \|\sum_{i\in I} g_i T'^* h_i\|_{\mathcal{H}} \le \|I - V\| \; \|T'\| \; \|\sum_{i\in I} g_i h_i\|_{\mathcal{H}} = \|I - V\| \; \|T'\| \; \|g\|_{\ell^2}.$$

which proves the estimate. □

In the following we prove existence of this operator V, and in addition, we obtain an estimate for the norm of $I - V$ in terms of the difference of t_i and t_i'. First we give two Lemma's .

Lemma 2.3 . *Let $\{\varphi_i\}_{i\in I}$ be a Riesz basis for a separable Hilbert space $\mathcal{H}$ (cf. (1.2)). Suppose $\{\varphi_i'\}_{i\in I}$ satisfies*

$$\sum_{i\in I} |\langle f, \varphi_i - \varphi_i'\rangle_{\mathcal{H}}|^2 \le C^2\|f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H},$$

where C is a constant. Then there exists a bounded linear operator V on $\mathcal{H}$ such that

$$V\varphi_i = \varphi_i', \quad \forall i \in \boldsymbol{I},$$

and $\|I - V\| \le \|T\|C$.

Proof:

Let $\{\varphi_i\}$ be a Riesz basis for $\mathcal{H}$, then $T\varphi_i = h_i$ and $\psi_i := T^*h_i$ is its biorthogonal sequence. Define the bounded linear operator W on $\mathcal{H}$ by

$$Wf = \sum_{i\in I} \langle f, \varphi_i - \varphi_i'\rangle_{\mathcal{H}} \; \psi_i.$$

Then,

$$\|Wf\|^2 = \|\sum_{i\in I} \langle f, \varphi_i - \varphi_i'\rangle_{\mathcal{H}} \; \psi_i\|_{\mathcal{H}}^2 \le$$

$$\|T\|^2 \, (\sum_{i\in I} |\langle f, \varphi_i - \varphi_i'\rangle_{\mathcal{H}}|^2) \le \|T\|^2 C^2 \|f\|_{\mathcal{H}}^2.$$

So $\|W\| \le C\|T\|$. The adjoint of W is

$$W^*f = \sum_{i\in I} \langle f, \psi_i\rangle_{\mathcal{H}} \; (\varphi_i - \varphi_i'),$$

and

$$(I - W^*)\varphi_i = \varphi_i'.$$

The result follows by taking $V = I - W^*$. □

Lemma 2.3 is a slight generalization of Schäfke's Theorem [8], where the system $\{\varphi_i\}_{i\in I}$ is assumed to be an orthonormal basis. The following Lemma is proven in [8] (p. 181, Lemma 3).

Lemma 2.4 . *Assume that $\{t_i\}_{i\in I}$ is a sequence of real numbers such that*

$$(\pi/r)\sum_{i\in I}|f(t_i\,\pi/r)|^2 \le D^2\|f\|^2_{P_r}, \quad \forall f\in \mathbb{P}_r,$$

where D is a constant. If $\{t_i'\}_{i\in I}$ is a sequence of real numbers which satisfies

$$|t_i - t_i'| \le \gamma, \quad \forall i\in \boldsymbol{I},$$

then

$$(\pi/r)\sum_{i\in I}|f(t_i\pi/r) - f(t_i'\pi/r)|^2 \le D^2(e^{\pi\gamma}-1)^2\|f\|^2_{P_r}, \quad \forall f\in \mathbb{P}_r.$$

An estimate for the time jitter error can now be derived, by means of a norm estimate for $I - V$.

Theorem 2.5 . *Let $\{t_i\}_{i\in I}$ and $\{t_i'\}_{i\in I}$ be sequences of real numbers which satisfy,*

$$|t_i - i| \le \alpha < 1/4, \quad \forall i\in \boldsymbol{I}, \tag{2.7}$$

$$|t_i' - i| \le \alpha' < 1/4, \quad \forall i\in \boldsymbol{I}, \tag{2.8}$$

and

$$|t_i - t_i'| \le \gamma \le \alpha+\alpha', \quad \forall i\in \boldsymbol{I}. \tag{2.9}$$

The time jitter error can be estimated by

$$e_{tj} \le (\|G'^{-1}\|\ \|G^{-1}\|\ \|G\|)^{1/2}\ (e^{\pi\gamma}-1)\ \|g\|_{\ell^2}.$$

Proof:

If $\{t_i\}$ satisfies the above estimate, then $\{\varphi_i\}$ is a Riesz basis for $\mathbb{P}_r$. Let $f\in\mathbb{P}_r$,

$$(\pi/r)\sum_{i\in I}|f(t_i\pi/r)|^2 = \|\sqrt{\pi/r}\ \sum f(t_i\pi/r)h_i\|^2_{P_r} =$$

$$\|\sqrt{\pi/r}\ \sum_{i\in I} f(t_i\pi/r)T^{*^{-1}}\psi_i\|^2_{P_r} \le \|T^{-1}\|^2\|\sum_{i\in I}\langle f,\varphi_i\rangle\psi_i\|^2_{P_r} = \|T^{-1}\|^2\|f\|^2_{P_r}.$$

So, the conditions of Lemma 2.4 are satisfied with $D = \|T^{-1}\|$, whence the following inequality is obtained

$$\sum_{i\in I}|\langle f,\varphi_i-\varphi_i'\rangle_{P_r}|^2 = \sum_{i\in I}(\pi/r)|f(t_i\pi/r) - f(t_i'\pi/r)|^2 \le$$

$$\|T^{-1}\|^2(e^{\pi\gamma}-1)^2\|f\|^2.$$

Hence the sequences $\{\varphi_i\}$ and $\{\varphi_i'\}$ satisfy the conditions of Lemma 2.3, with $C = \|T^{-1}\|(e^{\pi\gamma}-1)$. This implies the existence of a linear operator V on $\mathbb{P}_r$ such that $V\varphi_i = \varphi_i'$ and

$$\|I - V\| \le \|T\|\ \|T^{-1}\ \|(e^{\pi\gamma}-1).$$

Since the $\{t_i'\}_{i\in I}$ satisfy (2.8), the system $\{\varphi_i'\}_{i\in I}$ is a Riesz basis. By Proposition 2.2 we have that

$$\|f - f'\|_{P_r} \le (\|G'^{-1}\|\ \|G^{-1}\|\ \|G\|)^{1/2}\ (e^{\pi\gamma}-1)\ \|g\|_{\ell^2}.$$

The result follows by (1.1). □

A few remarks are in order. From this estimate we see that the solution f' is stable for perturbation of the time markers. By the norm estimates (1.11), (1.12) (2.3) and (2.6) we obtain

$$\|G\|^{1/2} \le 1 + \lambda,$$

$$\|G^{-1}\|^{1/2} \le \frac{1}{1-\lambda}$$

and

$$\|G'^{-1}\|^{1/2} \le \frac{1}{1-\lambda'}.$$

Here

$$\lambda := 1 - \cos \pi\alpha + \sin \pi\alpha$$

and

$$\lambda' := 1 - \cos \pi\alpha' + \sin \pi\alpha'.$$

In the case of uniform sampling (α is zero and α' is close to zero) the problem (0.1) is well conditioned for perturbation of the time markers. If we sampled nonuniformly, especially when α or α' is close to $1/4$, the problem may become ill conditioned for perturbation of the time markers.

This estimate for the time jitter error can also be obtained by means of the amplitude error [2,5], which is the approach below. Suppose we did measure the data $\{g_i\}_{i\in I}$ at the time markers $\{t_i\pi/r\}_{i\in I}$. Suppose that the sequence of measurement times is registered by our device as $\{t'_i\pi/r\}_{i\in I}$. The function we sampled is denoted by f, so $g_i = \sqrt{\pi/r}\ f(t_i\pi/r)$. The situation which is registered by our measuring device is false, since it says that the value of f at $t'_i\pi/r$ is equal to g_i. However, the true value of f at $t'_i\pi/r$ is $g'_i := (\sqrt{\pi/r})\ f(t'_i\pi/r)$. So we may consider $\{g'_i\}_{i\in I}$ as the exact data and $\{g_i\}_{i\in I}$ as the perturbed data at $\{t'_i\pi/r\}_{i\in I}$. With the above notation, we have

$$f = \sum_{i\in I} g_i\psi_i = \sum_{i\in I} g'_i\psi'_i, \tag{2.10}$$

and we define $f' = \sum_{i\in I} g_i\psi'_i$. The time jitter error is given by

$$e_{tj} = \|f - f'\|_{\sup} = \|\sum_{i\in I} g_i\psi_i - \sum_{i\in I} g_i\psi'_i\|_{\sup} = \|\sum_{i\in I} g'_i\psi'_i - \sum_{i\in I} g_i\psi'_i\|_{\sup}.$$

The estimate from Theorem 2.5 can now be derived by formula (2.1) as follows. Let $\{t_i\}$ and $\{t'_i\}$ satisfy the conditions of Theorem 2.5. By applying (2.1) with ψ'_i and T' in the role of ψ_i and T respectively,

$$e_{tj}^2 = \|f - f'\|_{\sup}^2 \le \|f - f'\|_{P_r}^2 = \|\sum_{i\in I}(g_i - g'_i)\psi'_i\|_{P_r}^2 \le$$

$$\|T'\|^2 \sum_{i\in I} |g_i - g'_i|^2 = \|G'\|\ (\pi/r) \sum_{i\in I} |f(t_i\pi/r) - f(t'_i\pi/r)|^2.$$

From the proof of Theorem 2.5, we know that the sequence $\{t_i\}_{i\in I}$ satisfies the condition of Lemma 2.4, with $D = \|T^{-1}\|$. Hence

$$e_{tj} \le \|G'\|^{1/2}\ \|T^{-1}\|\ (e^{\pi\gamma} - 1)\ \|f\|.$$

The desired estimate now follows by (2.10), (1.6), (2.6), (1.11) and (1.12).

This shows that the estimate of Theorem 2.5 can be proven by using (2.1).

3. Conclusions and Remarks

In this section we consider the solution of the problem (0.1) in the case that the index set $\mathbb{I}$ is a finite subset of $\mathbb{Z}$ and we state conclusions concerning the amplitude and time jitter error. If the time markers t_i are all distinct for all $i \in \mathbb{I}$, then the system $\varphi_i = \text{sinc}_T(. - t_i\pi/r)$ is a Riesz basis for its linear span cf. [9]. A solution to problem (0.1) is in this case $f = \sum_{i\in I} g_i\psi_i$. This solution is not unique, but it is the one with smallest norm among all solutions, the minimal norm solution in $\mathbb{P}_r$ to (0.1), see [1]. If both the t_i's and the t_i''s are all distinct, then the estimates of Formula (2.1) and Theorem 2.5 are valid. This implies that problem (0.1) in the case of finite index sets is stable for perturbation of the data and the time markers.

If the time markers t_i are lying close to each other, then the system φ_i may become effectively linearly dependent, from a numerical point of view and the matrix G may become singular. Hence the biorthogonal system $\{\psi_i\}_{i\in I}$ which is given by (1.10) cannot be computed. In such a case the algorithm to compute f breaks down. Then the problem (0.1) is ill conditioned for perturbation of the data and of the time markers.

We applied the above interpolation technique in magenetic resonance imaging (MRI), which is a diagnostic method to measure and display cross sections of a human organ, e.g. the beating human heart. A cross section of the heart has to be reconstructed at prescribed measurement times, which are called phases. But since the time markers at which data are measured do, in general, not coincide with the phases, an interpolation technique is used to obtain information at these phases.

Since there is a vast amount of literature on this subject, we want to make some remarks on error estimates. In the literature most of the bounds for the time jitter error and the amplitude error are given in the case of uniform sampling. Here the estimates are given in the case of non uniform sampling.

A bound for the amplitude error is derived in [3] via the truncation error (which is not considered here) and in [4] and [5] by writing the amplitude error as $e_{\text{amp}} = \sum_{i\in I}(g_i - g_i')\psi_i$ and using estimate (1.1). Here $\{g_i\}_{i\in I}$ are the true data and $\{g_i'\}_{i\in I}$ are the perturbed data. The time jitter error is derived in [2],[4] and [5] via the amplitude error and by applying the mean value theorem to the sampled function. In this paper new bounds for the amplitude and the time jitter error which apply in the case of non uniform sampling, are obtained by using the special structure of the Paley Wiener space (cf. Lemma 2.4) and by expressing the errors in terms of bounded linear operators. Explicit formula's for the error bounds are obtained by estimating the operator norms (cf. (1.11), (1.12) and (2.3)).

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