Local and Global Order Reduction of some LOD Schemes

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1. Introduction

In this paper we shall discuss unconditional convergence of some simple splitting methods for the numerical solution of initial boundary value problems for partial differential equations (PDEs). Discretization in space of such PDE problems leads to large systems of ordinary differential equations (ODEs)

$$\dot{u}(t)=f(t,u(t)) \quad (0<t\leq T), \quad u(0)=u_0 \tag{1.1}$$

where the vector function f contains discretized space derivatives. The boundary conditions are also incorporated in f. Assume that f can be decomposed into two more simple functions f_1 and f_2,

$$f(t,v)=f_1(t,v)+f_2(t,v)\ , \tag{1.2}$$

as it is often the case for PDE probems with two space dimensions.

Standard implicit methods to approximate (1.1) require the solution of large systems of algebraic equations involving the whole function f. A well-known method is the implicit midpoint rule

$$u_{n+1}=u_n+\tau f(t_n+\tfrac{1}{2}\tau,\ \tfrac{1}{2}u_n+\tfrac{1}{2}u_{n+1}) \quad (n=0,1,2,\ldots)\ , \tag{1.3}$$

also called the Crank-Nicholson method in PDE literature. The vectors u_n approximate the exact solution u of (1.1) at $t_n=n\tau$, $\tau>0$ being the stepsize in time. Method (1.3) is of 2-d order in the classical ODE sense.

In terms of computational effort it can be more attractive to exploit the splitting (1.2). Yanenko [13] introduced the following Locally One Dimensional (LOD) method, which is based on the implicit midpoint rule,

$$u_{n+\frac{1}{2}}=u_n+\tau f_1(t_n+\tfrac{1}{2}\tau,\ \tfrac{1}{2}u_n+\tfrac{1}{2}u_{n+\frac{1}{2}})\ , \tag{1.4a}$$

$$u_{n+1}=u_{n+\frac{1}{2}}+\tau f_2(t_n+\tfrac{1}{2}\tau,\ \tfrac{1}{2}u_{n+\frac{1}{2}}+\tfrac{1}{2}u_{n+1}) \tag{1.4b}$$

(for $n=0,1,2,\ldots$). The vector $u_{n+\frac{1}{2}}$ is an intermediate vector, as in Runge-Kutta methods, to which we do not attach physical meaning. If f_1 and f_2 have a more simple structure than f, the computation of u_{n+1} from (1.4) can be done more efficiently than from (1.3). However, the LOD method (1.4) will have 2-d order only if f_1, f_2 are linear and commuting; in more general situations it will have merely 1-th order, due to lack of symmetry, [6].

Symmetry can be restored by interchanging after each step f_1 and f_2. This idea, which originates with Marchuk [8], leads to the following modification,

$$u_{n+\frac{1}{2}} = u_n + \tau f_1(t_n + \tfrac{1}{2}\tau,\ \tfrac{1}{2}u_n + \tfrac{1}{2}u_{n+\frac{1}{2}}), \tag{1.5a}$$

$$u_{n+1} = u_{n+\frac{1}{2}} + \tau f_2(t_n + \tfrac{1}{2}\tau,\ \tfrac{1}{2}u_{n+\frac{1}{2}} + \tfrac{1}{2}u_{n+1}), \tag{1.5b}$$

$$u_{n+\frac{3}{2}} = u_{n+1} + \tau f_2(t_{n+1} + \tfrac{1}{2}\tau,\ \tfrac{1}{2}u_{n+1} + \tfrac{1}{2}u_{n+\frac{3}{2}}), \tag{1.5c}$$

$$u_{n+2} = u_{n+\frac{3}{2}} + \tau f_1(t_{n+1} + \tfrac{1}{2}\tau,\ \tfrac{1}{2}u_{n+\frac{3}{2}} + \tfrac{1}{2}u_{n+2}), \tag{1.5d}$$

(for $n=0,2,4,\ldots$). This method is again of 2-d order, and it requires the same amount of work as (1.4).

At first sight method (1.5) seems superior to (1.4). As we shall see this conclusion is not justified. The reason for this is that the classical order concept for ODEs, to which we referred to until now, is the order of consistency for *nonstiff* ODEs where f satisfies a Lipschitz condition with moderate Lipschitz constant L and τL is assumed to be sufficiently small. In our situation, where (1.1) originates from a PDE problem, the Lipschitz constant L will contain negative powers of the meshwidth in space h. As a consequence, L will be very large for fine space grids and the classical convergence theory cannot be applied. In fact, the order of the discretization errors may be affected by small meshwidths h, a phenomenon called *order reduction*. The failure of the classical order concept will be demonstrated by means of a simple linear parabolic model problem. It should be noted that both LOD methods (1.4) and (1.5) are stable for much more general problems, see [11].

Order reduction for PDE problems does not occur exclusively with the LOD methods, but also with the implicit midpoint rule and other Runge-Kutta methods [1], [3], [7], [10], [12] and the Peaceman-Rachford ADI method [5]. An analysis similar to the one that will be presented here can be given for other splitting methods as well. We will consider the LOD methods (1.4), (1.5) since various aspects of order reduction show up for these two methods in a relatively simple way.

2. Preliminaries

The discretization errors of the LOD schemes will be analyzed for parabolic model problems on the unit rectangle $\Omega=(0,1)^2$

$$\frac{\partial}{\partial t}U(x,y,t) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]U(x,y,t) + G(x,y,t) \quad (\textit{for } (x,y)\in\Omega,\ 0<t\leq T),$$
$$U(x,y,0) \textit{ given } (\textit{for } (x,y)\in\Omega),\ U(x,y,t)=0 \quad (\textit{for } (x,y)\in\partial\Omega,\ 0<t\leq T). \tag{2.1}$$

Note that we deal here with homogeneous Dirichlet boundary conditions. The LOD schemes can also be applied for time dependent boundary conditions, but it is less clear then how natural boundary conditions for the intermediate vectors $u_{n+\frac{1}{2}}$ should be obtained (see [9] for a discussion in case no source term G is present).

Standard discretization in space with finite differences on a uniform space mesh Ω_h with meshwidth $h=1/m+1$ in both directions yields the ODE system

$$\dot{u}(t) = A\,u(t) + g(t) \quad (0<t\leq T),\ u(0) \textit{ given} \tag{2.2}$$

of dimension $M=m^2$. The vector $u(t)$ has components $u_{ij}(t)$ $(1\le i,j\le m)$ approximating $U(ih,jh,t)$, and $A=A_1+A_2$ as given by the stencils

$$A \cong h^{-2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix},\ A_1 \cong h^{-2}\begin{bmatrix} & 0 & \\ 1 & -2 & 1 \\ & 0 & \end{bmatrix},\ A_2 \cong h^{-2}\begin{bmatrix} & 1 & \\ 0 & -2 & 0 \\ & 1 & \end{bmatrix}. \tag{2.3}$$

The matrix A is penta-diagonal, whereas A_1, A_2, which approximate $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$, respectively, are both tri-diagonal. These matrices are all symmetric, negative definite and they commute with each other. The source term g in (2.2) is the restriction of G to the space grid; it can be distributed as

$$g_1(t)=\theta g(t),\ g_2(t)=(1-\theta)g(t) \tag{2.4}$$

with $\theta\in[0,1]$ (for example $\theta=1/2$). We thus obtain a system (1.1), (1.2) with

$$f_j(t,v)=A_jv+g_j(t) \quad (j=1,2)\,. \tag{2.5}$$

Let $\|\cdot\|$ denote the discrete L_2-norm, i.e.,

$$\|v\|=[h^2\sum_{i,j=1}^{m}|v_{ij}|^2]^{1/2} \quad (\textit{for } v=(v_{ij})\in\mathbf{R}^M)\,.$$

The function $f=f_1+f_2$ satisfies a Lipschitz condition with constant $L\approx 8h^{-2}$ (the most negative eigenvalue of A is approximately $-8h^{-2}$). If $h>0$ is small we thus have a very large Lipschitz constant and the classical convergence theory is not valid anymore. In fact, as we shall see, the convergence behaviour in time for $h\downarrow 0$ will be different from that for fixed $h=h_0$ bounded away from zero. The main difference between these two cases is that terms like $\|Av\|$ need not be bounded for $h\downarrow 0$, even if v is the restriction to the space grid of a smooth function $V(x,y)$. For example, if $e\in\mathbf{R}^M$ is the vector with all components equal to 1, then $(Ae)_{ij}=0$ for gridpoints (ih,jh) in the interior of the grid, but $(Ae)_{ij}=-h^{-2}$ on the gridpoints adjacent to $\partial\Omega$; consequently $\|Ae\|\sim h^{-3/2}$.

By $\|\cdot\|$ we shall also denote the induced spectral norm for $M\times M$ matrices. Let $r(z)=(1-\frac{1}{2}z)^{-1}(1+\frac{1}{2}z)$ be the stability function of the implicit midpoint rule. Since A_1, A_2 are negative definite it can be easily shown that for arbitrary $\tau>0$

$$\|(I-\tfrac{1}{2}\tau A_j)^{-1}\|\le 1,\quad \|r(\tau A_j)\|\le 1 \quad (j=1,2)\,. \tag{2.6}$$

Let $U_h(t)$ be the restriction to the space grid Ω_h of the exact PDE solution, and let

$$\alpha_h(t)=\dot U_h(t)-f(t,\,U_h(t)) \quad (0\le t\le T)\,. \tag{2.7}$$

This α_h measures the error due to space discretization. For our model problem it is $O(h^2)$. Further we introduce the notation

$$F_j(t)=f_j(t,U_h(t)) \quad (0\le t\le T)\,. \tag{2.8}$$

Note that $\|F_j(t)\|$ will be bounded uniformly in h, provided U is a smooth solution, despite the fact that f_j contains negative powers of h.

We shall be concerned with bounds for the full global errors

$$\| U_h(t_n)-u_n \| \leq C\tau^p + C' \max_{0\leq t\leq t_n} \| \alpha_h(t) \| \tag{2.9}$$

with constants $C, C'>0$ independent of τ and h. So, in particular, C is not allowed to depend on L (which is $\sim h^{-2}$). It will turn out that this requirement may affect the order in time p.

3. Recursions for the global errors

Let $\epsilon_n = U_h(t_n)-u_n$ denote the global discretization errors. For the LOD method (1.4) it can be shown [4] that the global errors satisfy the recursion

$$\epsilon_{n+1} = R\,\epsilon_n + \delta_n \quad (n=0,1,2,...) \tag{3.1}$$

where $\epsilon_0 = 0$ and

$$R = r(\tau A_1)\, r(\tau A_2)\,, \tag{3.2}$$

$$\delta_n = (I-\tfrac{1}{2}\tau A_1)^{-1}(I-\tfrac{1}{2}\tau A_2)^{-1}\{\tfrac{1}{2}\tau^2(A_1F_2-A_2F_1)+\tau\alpha_h+O(\tau^3)\} \tag{3.3}$$

Here F_1, F_2 and α_h are evaluated in $t_n+\frac{1}{2}\tau$ and the $O(\tau^3)$ term is genuinely 3-th order (it can be bounded in norm by $C\tau^3$ with $C>0$ only depending on smoothness of the exact PDE solution U, so that no negative powers of h are hidden). As we see from (3.1) the global error at time t_{n+1} consists of two contributions: (i) the error ϵ_n at time t_n premultiplied by the companion (or stability-) matrix R, and (ii) an additional error δ_n introduced in the step from t_n to t_{n+1}. This δ_n can be considered as a local discretization error. (It should be noted that δ_n differs from the usual local discretization error which is obtained by substituting the exact solution in (1.4).)

For *stationary* solutions, where $0=\dot{U}_h = F_1+F_2+\alpha_h$, it follows that

$$\delta_n = \tfrac{1}{2}\tau[(I-\tfrac{1}{2}\tau A_1)^{-1}(I-\tfrac{1}{2}\tau A_2)^{-1}\tau A]F_2 + O(\tau)\alpha_h =$$

$$= \tfrac{1}{2}\tau[R-I]F_2 + O(\tau)\alpha_h\,. \tag{3.4}$$

The $O(\tau^3)$ term in (3.3) has cancelled due to the fact that the derivates of U_h all vanish. Stationary problems are considered here only as the most simple case to analyse the order in time of the methods. The LOD methods are not particularly suited if one only wants to find stationary solutions via time marching.

For the modified method (1.5) we obtain in the same way

$$\epsilon_{n+1} = R\epsilon_n + \delta_n,\quad \epsilon_{n+2} = R\epsilon_{n+1} + \delta'_{n+1} \quad (n=0,2,4,...) \tag{3.5}$$

where δ'_{n+1} differs from δ_{n+1}, as given by (3.3), only in that the indices 1 and 2 are interchanged (since A_1 and A_2 commute, R does not have to be replaced by an R'). Taking the two relations in (3.5) together, it follows that

$$\epsilon_{n+2} = R^2\epsilon_n + \Delta_n,\quad \Delta_n = R\delta_n + \delta'_{n+1} \quad (n=0,2,4,...)\,. \tag{3.6}$$

The vector Δ_n thus stands for the discretization error which is introduced in one single step of the process (1.5).

If we consider again a stationary solution $\dot{U}_h=0$, we see from (3.4) that

$$\delta'_{n+1}=\frac{1}{2}\tau[R-I]F_1+O(\tau)\alpha_h=-\frac{1}{2}\tau[R-I]F_2+O(\tau)\alpha_h\ ,$$

and therefore

$$\Delta_{n+1}=\frac{1}{2}\tau[R-I]^2F_2+O(\tau)\alpha_h\ . \tag{3.7}$$

4. Local error bounds

Order reduction for the LOD methods can already be observed for the simple case where we deal with a stationary solution. Omitting the space errors, the errors per step of the LOD methods (1.4), (1.5), respectively, are then given by

$$\delta_n=\frac{1}{2}\tau[R-I]F_2,\quad \Delta_n=\frac{1}{2}\tau[R-I]^2F_2\ .$$

Now, on a *fixed space grid* ($h=h_0>0$) we have $\| R-I\| = O(\tau)$, and thus

$$\|\delta_n\| = O(\tau^2),\ \|\Delta_n\| = O(\tau^3)\ .$$

These bounds are in agreement with the classical error bounds for nonstiff ODEs. They are not valid if both $\tau\downarrow 0$ and $h\downarrow 0$: we have

$$[R-I]F_2=\tau[(I-\frac{1}{2}\tau A_1)^{-1}(I-\frac{1}{2}\tau A_2)^{-1}]\,[AF_2]\ ,$$

$$\|(I-\frac{1}{2}\tau A_1)^{-1}(I-\frac{1}{2}\tau A_2)^{-1}\|\leq 1,\ \ \textit{but}\ \ \|AF_2\|\neq O(1)\ \ \textit{if}\ h\downarrow 0\ ,$$

due to the fact that F_2 is in general not zero near the boundaries.

It can be proved [4] that for arbitrary (time-dependent) problems of our model class there exist constants $C_0, C_0'>0$, independent of h, such that

$$\|\delta_n\| \leq C_0\tau^{5/4}+C_0'\tau\|\alpha_h(t_n+\frac{1}{2}\tau)\|\ , \tag{4.1}$$

$$\|\Delta_n\| \leq C_0\tau^{5/4}+C_0'\tau\,\|\alpha_h(t_n+\frac{1}{2}\tau)\|\ . \tag{4.2}$$

These bounds are also valid if $h\downarrow 0$, but instead of $O(\tau^2)$, $O(\tau^3)$ as on fixed space grids we only have an $O(\tau^{5/4})$ estimate for the temporal local errors. The bounds (4.1), (4.2) can be proved to be sharp, for example for stationary problems with $\tau=h$, see [4]. So, due to small meshwidths there is a large order reduction for the local errors.

Order reduction for local errors of the LOD method (1.4) was discussed already in [13] and [9] for the problems (2.1) with $G=0$ but with time varying boundary conditions. As we see here it may also occur from homogeneous boundary conditions.

5. Global error bounds

To estimate the global errors ϵ_n we use the error recursions (3.1), (3.6) and the fact that $\| R\| \leq 1$, which guarantees stability.

First we consider the standard approach to obtain bounds for the $\|\epsilon_n\|$. If

$$\epsilon_n = R\epsilon_{n-1} + \delta_{n-1},\ \| \delta_j \| \leq C_0 \tau^{p+1} \ \ (\textit{for all } j)\ ,$$

then

$$\| \epsilon_n \| \leq \| \epsilon_{n-1} \| + \| \delta_{n-1} \| \leq \cdots \leq \| \delta_0 \| + \| \delta_1 \| + \cdots + \| \delta_{n-1} \| \leq nC_0\tau^{p+1} \leq C_0 T\tau^p\ .$$

For both LOD methods (1.4) and (1.5) it follows in this way that there are C, $C'>0$, independent of h, such that

$$\| \epsilon_n \| \leq C\tau^{1/4} + C' \max_{0 \leq t \leq T} \| \alpha_h(t) \|\ . \tag{5.1}$$

The $\tau^{1/4}$ term is of course disappointing compared with the order on fixed space grids. Since the bounds (4.1), (4.2) are known to be sharp this seems at first sight to be the best possible.

For method (1.4) however this result can be improved by taking into account cancellation and damping effects. This is easy to show for stationary problems where we have, omitting the space errors,

$$\epsilon_n = R\epsilon_{n-1} + \frac{1}{2}\tau[R-I]F_2\ ,$$

$$\epsilon_n = \frac{1}{2}\tau[I+R+\cdots+R^{n-1}]\,[R-I]F_2 = \frac{1}{2}\tau[R^n-I]F_2\ , \tag{5.2}$$

and thus

$$\| \epsilon_n \| \leq \tau \| F_2 \| \leq C\tau\ .$$

This can also be proved for non-stationary problems [4]: the global errors of the LOD method (1.4) applied to a problem (2.1) can be bounded by

$$\| \epsilon_n \| \leq C\tau + C' \max_{0 \leq t \leq T} \| \alpha_h(t) \| \tag{5.3}$$

with C, $C'>0$ independent of h. Thus, although the local errors δ_n do suffer from an order reduction, we see that the contribution to the global error of time discretization is still of order 1, as on fixed space grids. In other words, the order reduction is annihilated in the transition from local to global errors.

For the modified method (1.5) the situation is different. When applied to a problem with stationary solution the error recursion for this method reads

$$\epsilon_n = R^2\epsilon_{n-2} + \frac{1}{2}\tau[R-I]^2F_2\ ,$$

where again space errors are omitted. Proceeding as above we can obtain

$$\epsilon_n = \frac{1}{2}\tau[I+R^2+\cdots+R^{n-2}]\,[R-I]^2F_2\ , \tag{5.4}$$

but now there is no cancellation of terms as in (5.2). In fact, it can be proved [4] that the global errors of method (1.5) for a smooth stationary solution satisfy

$$\| \epsilon_n \| \geq C\tau^{1/2} \ \ \textit{for } \tau = h,\ nh = t > 0 \tag{5.5}$$

with $C>0$ independent of τ and h. Hence, the error bound (5.1), which is too pessimistic for method (1.4), is nearly optimal for method (1.5). (The question whether the order is $\frac{1}{4}$ or $\frac{1}{2}$, or in between, is not so relevant since the convergence behaviour is very disappointing anyway.) For method (1.5), which is 2-d order on fixed space grids, there is a strong order reduction for the global errors.

As a numerical illustration we consider our model problem (2.1) with $T=2$ and stationary solution

$$U(x,y,t)=x(1-x)\,y(1-y)\,(16+y)\,.$$

The solution is chosen such that no space errors are present, and the source term G is adapted to the solution. Below we have listed the global errors $\|\epsilon_n\|$ at the endpoint $n\tau=T=2$ for a fixed grid $h=h_0$ and for $\tau/h=$const. These numerical results nicely illustrate the theory. On fixed space grids method (1.5) will become more accurate than method (1.4) for decreasing τ, but if both τ and h tend to 0 method (1.4) is the better one.

τ	$^1/_{10}$	$^1/_{20}$	$^1/_{40}$	$^1/_{80}$	$^1/_{160}$
(1.4)	.47 E-1	.23 E-1	.12 E-1	.59 E-2	.29 E-2
(1.5)	.69 E-1	.32 E-1	.10 E-1	.27 E-2	.68 E-3

TABLE 5.1. Global errors for the LOD methods (1.4), (1.5) on a fixed space grid $h={}^1/_5$.

τ	$^1/_{10}$	$^1/_{20}$	$^1/_{40}$	$^1/_{80}$	$^1/_{160}$
(1.4)	.47 E-1	.39 E-1	.20 E-1	.10 E-1	.53 E-2
(1.5)	.69 E-1	.63 E-1	.45 E-1	.31 E-1	.22 E-1

TABLE 5.2. Global errors for the LOD methods (1.4), (1.5) for $h=2\tau$.

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REFERENCES

[1] BRENNER, P., CROUZEIX, M., THOMÉE, V., *Single step methods for inhomogeneous linear differential equations in Banach space.* RAIRO Numer. Anal. 16 (1982), pp. 5-26.

[2] DEKKER, K., VERWER, J.G., *Stability of Runge-Kutta methods for stiff nonlinear differential equations.* North-Holland, Amsterdam - New York - Oxford, 1984.

[3] FRANK, R., SCHNEID, J., UEBERHUBER, C.W., *Order results for implicit Runge-Kutta methods applied to stiff systems.* SIAM J. Numer. Anal. 22 (1985), pp. 515-543.

[4] HUNDSDORFER, W.H., *Uniform convergence of some Crank-Nicholson LOD methods for initial-boundary value problems.* To appear.

[5] HUNDSDORFER, W.H., VERWER, J.G., *Stability and convergence of the Peaceman-Rachford ADI method for initial-boundary value problems.* To appear in Math. Comp. (1989).

[6] HOUWEN, P.J. VAN DER, VERWER, J.G., *One-step splitting methods for semi-discrete parabolic equations.* Computing 22 (1979), pp. 291-309.

[7] KRAAIJEVANGER, J.F.B.M., *B-convergence of the implicit midpoint rule and the trapezoidal rule.* BIT 25 (1985), pp. 652-666.

[8] MARCHUK, G.I., *Methods of numerical mathematics (2-d ed.).* Springer Verlag, New York - Heidelberg - Berlin, 1981.

[9] LE VEQUE, R.J., *Intermediate boundary conditions for LOD, ADI and approximate factorization methods.* ICASE report 85-21, NASA Langley Research Centre, 1985.

[10] SANZ-SERNA, J.M., VERWER, J.G., HUNDSDORFER, W.H., *Convergence and order reduction of Runge-Kutta schemes applied to evolutionary partial differential equations.* Numer. Math. 50 (1987), pp. 405-418.

[11] VERWER, J.G., *Contractivity of locally one-dimensional splitting methods.* Numer. Math. 44 (1984), pp. 247-259.

[12] VERWER, J.G., *Convergence and order reduction of diagonally implicit Runge-Kutta schemes in the method of lines.* Proc. Dundee Conf., (D.F. Griffiths & G.A Watson, eds.), Pitman Research Notes in Mathematics Series 140 (1986), pp. 220-237.

[13] YANENKO, N.N., *The method of fractional steps.* Springer Verlag, Berlin - Heidelberg - New York, 1971.