The Connectivity of Strongly Regular Graphs

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We prove that the connectivity of a connected strongly regular graph equals its valency.

A strongly regular graph (with parameters v, k, λ, μ) is a finite undirected graph without loops or multiple edges such that (i) has v vertices, (ii) it is regular of degree k, (iii) each edge is in λ triangles, (iv) any two-nonadjacent points are joined by μ paths of length 2. For basic properties see Cameron [1] or Seidel [3]. The connectivity of a graph is the minimum number of vertices one has to remove in order to make it disconnected (or empty). Our aim is to prove the following proposition.

PROPOSITION. *Let Γ be a connected strongly regular graph. Then its connectivity $\kappa(\Gamma)$ equals its valency k. Also, the only disconnecting sets of size k are the sets $\Gamma(x)$ of all neighbours of a vertex x.*

PROOF. Clearly $\kappa(\Gamma) \leq k$. Let S be a disconnecting set of vertices not containing all neighbours of some vertex. Let $\Gamma \backslash S = A + B$ be a separation of $\Gamma \backslash S$ (i.e. A and B are nonempty and there are no edges between A and B). Since the eigenvalues of A and B interlace those of Γ (which are k, r, s with $k > r \geq 0 > -2 \geq s$, where k has multiplicity 1) it follows that at least one of A and B, say B, has largest eigenvalue at most r. (cf. [2], theorem 0.10.) It follows that the average valency of B is at most r, and in particular that $r > 0$. (cf. [2], theorem 3.8.)

Now let $|S| \leq k$. Since B is not a singleton we can find two points x, y in B such that $|S \cap \Gamma(x)| \geq k - r$ and $|S \cap \Gamma(y)| \geq k - r$, so that these points have at least $k - 2r$ common neighbours in S.

Let us first consider the case where Γ has nonintegral eigenvalues. In this case we have for some t: $(v, k, \lambda, \mu) = (4t+1, 2t, t-1, t)$ and $r = (-1 + \sqrt{v})/2$. The inequality $\max(\lambda, \mu) \geq k - 2r$ gives here $2r \geq t$, i.e. $t \in \{1, 2\}$, but for $t = 2$ the eigenvalues are integral, so we have $t = 1$ and Γ is a pentagon. But clearly the proposition is true in this case.

Now assume r and s are integral. We have $r^2 + (\mu - \lambda)r + \mu - k = 0$ so $(r+1) \times (r - 1 + \mu - \lambda) = k - \lambda - 1$ and $(r+1)|(k - \lambda - 1)$. If $r + 1 = k - \lambda - 1$ then $s = -2$; otherwise $2(r+1) \leq k - \lambda - 1$ and $s \leq -3$.

If $s \leq -3$ then $\mu = k + rs \leq k - 3r$ and $\lambda \leq k - 2r - 3$ so that no two points can have $k - 2r$ common neighbours. Consequently $s = -2$, $\mu = k - 2r$ and $\lambda = k - r - 2$.

By Seidel's classification of strongly regular graphs with $s = -2$ [4], Γ is one of

(i)   the complement of the ladder graph $(r = 0)$, $(v, k, \lambda, \mu) = (2n, 2n - 2, 2n - 4, 2n - 2)$,

(ii)   a lattice graph, $(v, k, \lambda, \mu) = (n^2, 2(n-1), n-2, 2)$, $r = n - 2$,

(iii)   the Shrikhande graph, $(v, k, \lambda, \mu) = (16, 6, 2, 2)$, $\bar{\mu} = 6$,

(iv)   a triangular graph, $(v, k, \lambda, \mu) = (\binom{n}{2}, 2(n-2), n-2, 4)$, $r = n - 4$,

(v)   one of the three Chang graphs, $(v, k, \lambda, \mu) = (28, 12, 6, 4)$, $\bar{\mu} = 10$,

(vi)   the Petersen graph, $(v, k, \lambda, \mu) = (10, 3, 0, 1)$, $\bar{\mu} = 4$,

(vii)   the Clebsch graph, $(v, k, \lambda, \mu) = (16, 10, 6, 6)$

(viii) the Schläfli graph, $(v, k, \lambda, \mu) = (27, 16, 10, 8)$.

Since $r > 0$ (i) does not occur.

Since A contains an edge we have $|B| \leq \bar{\mu} = v - 2k + \lambda$; similarly $|A| \leq v - 2k + \lambda$, so $|B| \geq k - \lambda$ and in particular $v \geq 3k - 2\lambda$. This eliminates cases (vii) and (viii).

If B is a clique then $|B| \leq r+1 = k - \lambda - 1$, a contradiction. Let x, y be two nonadjacent points in B. The neighbours of x and y are in $(B \cup S) \setminus \{x, y\}$, thus $2k - \mu \leq |B| + |S| - 2$ and since $|S| \leq k$ we have $\bar{\mu} \geq |B| \geq k + 2 - \mu$.

In cases (iii), (v), (vi) we have $\bar{\mu} = k + 2 - \mu$ so $|B| = \bar{\mu}$ and $|S| = k$. Since $v < 2\bar{\mu} + k$ we have $|A| < \bar{\mu}$ and A must be a clique; but the Petersen graph does not contain 3-cliques and the Shrikhande graph does not contain 4-cliques; also, if A is a 6-clique in a Chang graph and a, b, $c \in A$ then $\Gamma(a) \cap S$, $\Gamma(b) \cap S$ and $\Gamma(c) \cap S$ are three 7-sets in the 12-set S such that any two meet in precisely two points—impossible. This eliminates cases (iii), (v) and (vi).

We are left with the two infinite families of the lattice graphs and the triangular graphs. In both cases it is very easy to see that if x, y are nonadjacent points then there exist k paths joining x and y and having no other vertices in common, where these paths are entirely contained within $\{x, y\} \cup \Gamma(x) \cup \Gamma(y)$. (It follows that $|S| \geq k$ and hence $|S| = k$.) Assume that S separates x and y, then $\Gamma(x) \cap \Gamma(y) \subset S$.

Also we have that $\Delta = \Gamma \setminus (\{x, y\} \cup \Gamma(x) \cup \Gamma(y))$ is connected (by inspection; this is certainly the case when $k - 2\mu > r$, which holds for $n > 4$ in lattice graphs and $n > 8$ in triangular graphs), and each vertex of the symmetric difference of $\Gamma(x)$ and $\Gamma(y)$ has a neighbour in Δ (again by insepction; $k > \lambda + \mu$ suffices) except in the case of the triangular graph with $v = \binom{5}{2}$. It follows that if $u \in \Gamma(x) \setminus S$ and $u' \in \Gamma(y) \setminus S$ then there is a path $x - u - \Delta - u' - y$ so that S cannot separate x and y. Finally in the case of the triangular graph with $v = \binom{5}{2}$, each point of $\Gamma(x) \setminus \Gamma(y)$ is joined to each point of $\Gamma(y) \setminus \Gamma(x)$ so that we find a path $x - u - u' - y$. This completes the proof of the proposition.

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