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Asymptotic Expansions for q-Gamma, q-Exponential, and q-Bessel Functions

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New asymptotic expansions are given for the q-gamma function, the q-exponential functions, and for the Hahn–Exton q-Bessel function. For the theta functions, four expansions are given. And for the Hahn–Exton q-Bessel difference equation, a new solution is given, which forms with the Hahn–Exton q-Bessel function a numerically satisfactory pair of solutions.

1. INTRODUCTION

In this paper we give new asymptotic expansions for the q-gamma function $\Gamma_q(x)$, for the q-exponential functions $e_q(x)$ and $E_q(x)$, and for the Hahn–Exton q-Bessel function $J_\nu(x, q)$. These functions are q-analogues of the classical functions and have been studied in many places in the literature. For a recent introduction to q-hypergeometric functions see Gasper and Rahman [3]. However, the asymptotic properties for these functions, as $|x| \to \infty$, have not been studied extensively. This may be due to the fact that these functions do not have nice integral representations, and are not solutions of linear differential equations.

For obtaining the asymptotic expansions of $\Gamma_q(x)$, in Section 2, and of $e_q(x)$ and $E_q(x)$, in Section 3, for $x \to \infty$, we use the *Abel–Plana formula.*

THEOREM 1.1. *Let $f(t)$ be holomorphic in $n \leqslant \operatorname{Re} t \leqslant m$, $n, m \in \mathbb{Z}$, and suppose that $f(t) = o(e^{2\pi|\operatorname{Im} t|})$ as $\operatorname{Im} t \to \pm\infty$, uniformly with respect to $\operatorname{Re} t \in [n, m]$, then*

$$\sum_{j=n}^{m} f(j) = \int_n^m f(t)\, dt + \frac{1}{2} f(n) + \frac{1}{2} f(m) + i \int_0^\infty \frac{f(n+iy) - f(m+iy) - f(n-iy) + f(m-iy)}{e^{2\pi y} - 1}\, dy, \qquad (1.1)$$

where $\mathbb{Z}$ is the set of integers.

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For a proof of this theorem see Olver [11, p. 290]. At the end of Section 3 we use the expansions of $e_q(x)$ and $E_q(x)$ for obtaining certain expansions for the theta functions ϑ_1, ϑ_2, ϑ_3, and ϑ_4.

In Section 4 the asymptotic expansion of $J_\nu(x, q)$, for $x \to \infty$, is obtained with a symmetry relation for the q-hypergeometric function ${}_1\Phi_1$. We show how this expansion can be obtained from the Hahn–Exton q-Bessel difference equation. For this second-order linear difference equation we find another solution, which forms with $J_\nu(x, q)$ a numerically satisfactory pair of solutions.

2. Asymptotic Expansions for the q-Gamma Function

2.1. *The q-gamma function.* Jackson [5] defined a q-analogue of the gamma function by writing

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad |q|<1, \tag{2.1}$$

where the product $(a;q)_\infty$ is defined by $(a;q)_\infty = \prod_{n=0}^\infty (1-aq^n)$, $a \in \mathbb{C}$, and $(a;q)_\nu = (a;q)_\infty/(aq^\nu;q)_\infty$, $\nu \in \mathbb{C}$. Thus for n a positive integer we have $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and

$$\Gamma_q(n+1) = \prod_{m=1}^{n}\left(\frac{1-q^m}{1-q}\right). \tag{2.2}$$

The factor $(1-q^a)/(1-q)$ is called *the basic number* of a, and q is called *the base*. It follows from $\lim_{q\to 1}(1-q^a)/(1-q) = a$ that $\Gamma_q(n+1)$ is a generalization of $\Gamma(n+1)$. In [3] a simple proof is given for $\lim_{q\uparrow 1}\Gamma_q(x) = \Gamma(x)$. For a rigorous justification of that formal proof, see Koornwinder [6]. And Askey [1] investigated thoroughly the q-gamma function as a function of both x and q.

2.2. *A simple asymptotic expansion for fixed q.* The q-binomial theorem (see [3]) reads

$$\frac{(ay;q)_\infty}{(y;q)_\infty} = \sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n} y^n, \qquad |y|<1. \tag{2.3}$$

By taking $a=0$ and $y=q^z$, in [1] the expansion

$$\Gamma_q(z) = (q;q)_\infty (1-q)^{1-z} \sum_{n=0}^\infty \frac{q^{nz}}{(q;q)_n}, \tag{2.4}$$

is obtained, which converges very fast for large positive z and $0<q<1$ fixed. But this simple expansion does not converge to Stirling's formula as $q \uparrow 1$.

2.3. *An asymptotic expansion which is valid for $q \uparrow 1$.* In this section we take $\operatorname{Re} z > 0$. The asymptotic expansion for $\ln \Gamma(z)$ reads

$$\ln \Gamma(z) \sim \left(z-\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} z^{1-2n}, \tag{2.5}$$

$|z| \to \infty$, $|\arg z| \leqslant \pi - \delta$, where B_m are the Bernoulli numbers. To derive an asymptotic expansion of the q-gamma function, we use the Abel–Plana formula with $f(t) = \ln(1-q^{z+t})$, $n=0$, $m=\infty$. Then from (2.1) we derive

$$\begin{aligned}\ln \Gamma_q(z) = \int_1^z \ln\left(\frac{1-q^w}{1-q}\right) dw - \frac{1}{2}\ln\left(\frac{1-q^z}{1-q}\right) + i\int_0^\infty \ln\left(\frac{1-q^{1+iy}}{1-q^{1-iy}}\right)\frac{dy}{e^{2\pi y}-1}\\ - i\int_0^\infty \ln\left(\frac{1-q^{z+iy}}{1-q^{z-iy}}\right)\frac{dy}{e^{2\pi y}-1}.\end{aligned} \tag{2.6}$$

Expression (2.6) holds for $\operatorname{Re} z > 0$, and it is a q-analogue of Binet's second expression for $\ln \Gamma(z)$ (see Whittaker and Watson [13, p. 251]). The function

$$\operatorname{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}\, dt = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots, \qquad |x| \leqslant 1, \tag{2.7}$$

is the dilogarithm function (see Lewin [8]). Using this, the first integral can be expressed as

$$\begin{aligned}\int_1^z \ln\left(\frac{1-q^w}{1-q}\right) dw &= z\ln\left(\frac{1-q^z}{1-q}\right) + \frac{\operatorname{Li}_2(1-q^z) - \operatorname{Li}_2(1-q)}{\ln q}\\ &= z\ln\left(\frac{1-q^z}{1-q}\right) + \sum_{k=1}^{\infty} \frac{(1-q^z)^k - (1-q)^k}{k^2 \ln q}.\end{aligned} \tag{2.8}$$

Thus

$$\lim_{q\uparrow 1} \int_1^z \ln\left(\frac{1-q^w}{1-q}\right) dw = z\ln(z) - z + 1. \tag{2.9}$$

The second integral in (2.6) does not depend on z, and it converges to

$$i\int_0^\infty \ln\left(\frac{1+iy}{1-iy}\right)\frac{dy}{e^{2\pi y}-1} = \frac{1}{2}\ln(2\pi) - 1, \tag{2.10}$$

as $q \uparrow 1$. In (2.10) we used Gradshteyn and Ryzhik [4, formula 4.552]. The final term in (2.6) can be expanded in an analogue of the infinite series in (2.5):

$$-i \int_0^\infty \ln\left(\frac{1-q^{z+iy}}{1-q^{z-iy}}\right) \frac{dy}{e^{2\pi y}-1}$$

$$= -i \int_0^\infty \ln\left[\left(\frac{i(q^{-z}-\cos(y \ln q))}{\sin(y \ln q)}+1\right) \Big/ \left(\frac{i(q^{-z}-\cos(y \ln q))}{\sin(y \ln q)}-1\right)\right]$$

$$\times \frac{dy}{e^{2\pi y}-1}$$

$$=^* \sum_{m=1}^{\infty} \frac{2(-1)^m}{2m-1} q^{(2m-1)z} \int_0^\infty \left(\frac{\sin(y \ln q)}{1-q^z \cos(y \ln q)}\right)^{2m-1} \frac{dy}{e^{2\pi y}-1}, \tag{2.11}$$

where $*$ holds for $|(q^{-z}-\cos(y \ln q))/\sin(y \ln q)| > 1$, thus for $\operatorname{Re} z > -\frac{1}{2} \ln 2/\ln q$. This series converges termwise to the final term of (2.5), as $q \uparrow 1$. Hence, we claim that the q-analogue of (2.5) is the convergent expansion

$$\ln \Gamma_q(z) = \left(z-\frac{1}{2}\right) \ln\left(\frac{1-q^z}{1-q}\right) + \frac{\mathrm{Li}_2(1-q^z)-\mathrm{Li}_2(1-q)}{\ln q}$$

$$+ i \int_0^\infty \ln\left(\frac{1-q^{1+iy}}{1-q^{1-iy}}\right) \frac{dy}{e^{2\pi y}-1}$$

$$+ \sum_{m=1}^{\infty} \frac{2(-1)^m}{2m-1} q^{(2m-1)z} \int_0^\infty \left(\frac{\sin(y \ln q)}{1-q^z \cos(y \ln q)}\right)^{2m-1} \frac{dy}{e^{2\pi y}-1}. \tag{2.12}$$

Expansion (2.12) holds for $\operatorname{Re} z > -\frac{1}{2} \ln 2/\ln q$, and it is an asymptotic expansion for $|\arg z| \leqslant \pi/2-\delta$. The integrals in the infinite series of (2.12) can be viewed as analogues of the Bernoulli numbers in (2.5). It seems it is not possible to express these integrals in terms of known special functions. For obtaining an expansion of these integrals we write

$$F_m(w, q) = \int_0^\infty \left(\frac{\sin(y \ln q)}{1-w \cos(y \ln q)}\right)^{2m-1} \frac{dy}{e^{2\pi y}-1}. \tag{2.13}$$

It is not difficult to prove that this function satisfies

$$2m(2m-1)\, F_{m+1}(w, q) = \left[(w^2-1)\frac{d^2}{dw^2} + 4mw\frac{d}{dw} + 2m(2m-1)\right] F_m(w, q). \tag{2.14}$$

The substitution of the Taylor expansion $F_m(w, q) = \sum_{n=0}^{\infty} c(n, m) w^n$ into (2.14) leads to

$$\int_0^{\infty} \left(\frac{\sin(y \ln q)}{1 - q^z \cos(y \ln q)} \right)^{2m-1} \frac{dy}{e^{2\pi y} - 1} = \sum_{n=0}^{\infty} c(n, m) q^{nz}, \tag{2.15}$$

where the coefficients have the recursion relation

$$c(n, m+1) = \frac{(n+2m-1)(n+2m)}{2m(2m-1)} c(n, m) - \frac{(n+1)(n+2)}{2m(2m-1)} c(n+2, m), \tag{2.16}$$

$m = 1, 2, \ldots$ The initial values are

$$\begin{aligned} c(n, 1) &= \int_0^{\infty} \frac{\sin(y \ln q)(\cos(y \ln q))^n}{e^{2\pi y} - 1} dy \\ &=^* \frac{-1}{2^{n+2}} \sum_{k=0}^{[n/2]} \left(\binom{n}{k} - \binom{n}{k-1} \right) \left[\frac{1 + q^{n+1-2k}}{1 - q^{n+1-2k}} + \frac{2}{(n+1-2k) \ln q} \right]. \end{aligned} \tag{2.17}$$

In $*$ we first expanded $\sin(x) \cos^n(x)$ as a sum of $\sin(kx)$, and then we used [4, formula 3.911.2]. And we have chosen $\binom{n}{-1} = 0$.

Remark 1. Observe that the asymptotic expansion (2.12) converges termwise to the well-known expansion (2.5) as $q \uparrow 1$. The asymptotic expansions derived in the following sections, for the q-exponential functions and for the Hahn–Exton q-Bessel functions, do not share this property.

Remark 2. With the Euler–Maclaurin formula (see [11, p. 285]) the following q-analogue of (2.5) is obtained in Moak [10]:

$$\begin{aligned} \ln \Gamma_q(z) \sim &\left(z - \frac{1}{2} \right) \ln \left(\frac{1 - q^z}{1 - q} \right) + \frac{\mathrm{Li}_2(1 - q^z)}{\ln q} + \frac{1}{2} \ln(1 - q) + \ln(q; q)_{\infty} \\ &- \frac{\pi^2}{6 \ln q} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \left(\frac{\ln q}{q^z - 1} \right)^{2n-1} q^z P_{2n-3}(q^z), \end{aligned} \tag{2.18}$$

as $z \to \infty$, where P_n is a polynomial of degree n satisfying

$$P_n(z) = (z - z^2) P'_{n-1}(z) + (nz + 1) P_{n-1}(z), \qquad P_0 = 1, \quad n \geqslant 1. \tag{2.19}$$

In [10] it is proven that the coefficients of $P_n(z)$ are all positive, and $P_n(1) = (n+1)!$. Although the series converges termwise to (2.5) as $q \uparrow 1$, it seems that the expansion does not have an asymptotic property as $z \to \infty$, because

$$\frac{B_{2n}}{(2n)!}\left(\frac{\ln q}{q^z-1}\right)^{2n-1} q^z P_{2n-3}(q^z) \Big/ \frac{B_{2n+2}}{(2n+2)!}\left(\frac{\ln q}{q^z-1}\right)^{2n+1} q^z P_{2n-1}(q^z)$$
$$\sim \frac{B_{2n}}{B_{2n+2}} \frac{(2n+1)(2n+2)}{\ln^2 q}, \tag{2.20}$$

as $z \to \infty$. Hence, successive terms of the series in (2.18) are not of lower order, as $z \to \infty$, and q and n are fixed.

3. Asymptotic Expansions for the q-Exponential Functions

3.1. *The q-exponential functions.* In this section we take $0 < q < 1$. The q-analogue $e_q(z)$ of the exponential function is defined by

$$e_q(z) = \frac{1}{(z; q)_\infty} =^* \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} z^n, \tag{3.1}$$

where $*$ follows from the q-binomial theorem (2.3), with $a = 0$; it holds for $|z| < 1$. Another q-analogue of the exponential function is defined by

$$E_q(z) = (-z; q)_\infty =^* \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} z^n, \tag{3.2}$$

and, again, $*$ follows from the q-binomial theorem (2.3), with $y = -z/a$ and letting $a \to \infty$; it holds for all $z \in \mathbb{C}$. Now the $q \uparrow 1$ limits read

$$\lim_{q \uparrow 1} e_q((1-q)z) = e^z, \qquad \lim_{q \uparrow 1} E_q((1-q)z) = e^z. \tag{3.3}$$

Note that $e_q(z)\, E_q(-z) = 1$.

In the next subsections we derive asymptotic representations of $e_q(z)$ and $E_q(z)$ as $z \to \infty$. As becomes clear these expansions do not converge to known representations of $e_q(z)$ and $E_q(z)$ as $q \uparrow 1$. This is due to the fact that the $q \uparrow 1$ limits require arguments $(1-q)z$, as shown in (3.3).

3.2. *An asymptotic expansion for $E_q(z)$.* In the following analysis we suppose that $z > 1$. For obtaining an asymptotic expansion for $\ln E_q(z)$, we use the Abel–Plana formula (1.1), with $n = 0$, $m = \infty$, and $f(t) = \ln(1 + zq^t)$. This function has singularities (branch points) at $t = -\ln z/\ln q - (2p+1)\pi i/\ln q$, $p \in \mathbb{Z}$, and these singularities give an extra term in the Abel–Plana formula. We obtain

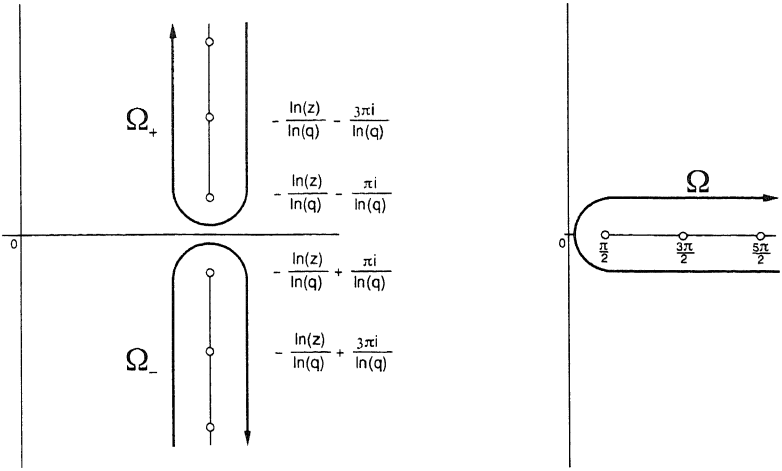

FIG. 3.1. The contours of integration of (3.5) and (3.10).

$$\ln E_q(z) = \sum_{n=0}^{\infty} \ln(1+zq^n)$$

$$= \int_0^\infty \ln(1+zq^t)\,dt + \frac{1}{2}\ln(1+z)$$

$$+ i\int_0^\infty \ln\left(\frac{1+zq^{iy}}{1+zq^{-iy}}\right)\frac{dy}{e^{2\pi y}-1} + F(z,q), \tag{3.4}$$

with

$$F(z,q) = \int_{\Omega_+} \frac{\ln(1+zq^t)}{1-e^{-2\pi i t}}\,dt + \int_{\Omega_-} \frac{\ln(1+zq^t)}{e^{2\pi i t}-1}\,dt, \tag{3.5}$$

where the contours of integration look like those in Fig. 3.1.

The first integral in (3.4) can be expanded as

$$\int_0^\infty \ln(1+zq^t)\,dt = \frac{1}{\ln q}\operatorname{Li}_2(-z)$$

$$=^* -\frac{1}{\ln q}\left[\frac{1}{6}\pi^2 + \frac{1}{2}\ln^2(z) + \sum_{k=1}^{\infty}\frac{(-z)^{-k}}{k^2}\right], \tag{3.6}$$

where $\mathrm{Li}_2(z)$ is defined in (2.7), and $*$ is proved in [8, p. 4]. The second integral can be expanded as follows:

$$
\begin{aligned}
& i\int_0^\infty \ln\left(\frac{1+zq^{iy}}{1+zq^{-iy}}\right)\frac{dy}{e^{2\pi y}-1} \\
&\quad = i\int_0^\infty\left[\ln\left(1+\frac{q^{-iy}}{z}\right)-\ln\left(1+\frac{q^{iy}}{z}\right)+2iy\ln q\right]\frac{dy}{e^{2\pi y}-1} \\
&\quad =^{*} -\frac{1}{12}\ln q-2\sum_{k=1}^{\infty}\frac{(-z)^{-k}}{k}\int_0^\infty\frac{\sin(ky\ln q)}{e^{2\pi y}-1}\,dy \\
&\quad =^{**} -\frac{1}{12}\ln q+\sum_{k=1}^{\infty}(-z)^{-k}\left[\frac{1}{2k}\frac{1+q^k}{1-q^k}+\frac{1}{k^2\ln q}\right] \\
&\quad =^{***} -\frac{1}{12}\ln q+\frac{1}{\ln q}\sum_{k=1}^{\infty}\frac{(-z)^{-k}}{k^2}-\frac{1}{2}\ln\left(1+\frac{1}{z}\right) \\
&\qquad -\ln\left(\prod_{m=1}^{\infty}\left(1+\frac{q^m}{z}\right)\right).
\end{aligned}
\tag{3.7}
$$

In $*$ we used $\int_0^\infty (y\,dy/(e^{2\pi y}-1))=\frac{1}{4}B_2=\frac{1}{24}$, in $**$ we used [4, formula 3.911.2], and in $***$ we used

$$
\begin{aligned}
\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-z)^{-k}}{k}\frac{1+q^k}{1-q^k} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-z)^{-k}}{k}+\sum_{k=1}^{\infty}\frac{(-z)^{-k}}{k}\frac{q^k}{1-q^k} \\
&= -\frac{1}{2}\ln\left(1+\frac{1}{z}\right)+\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-z)^{-k}q^{mk}}{k} \\
&= -\frac{1}{2}\ln\left(1+\frac{1}{z}\right)-\sum_{m=1}^{\infty}\ln\left(1+\frac{q^m}{z}\right) \\
&= -\frac{1}{2}\ln\left(1+\frac{1}{z}\right)-\ln\left(-\frac{q}{z};q\right)_\infty.
\end{aligned}
\tag{3.8}
$$

The infinite series in (3.6) and (3.7) are defined for $z>1$. It is not difficult to show that the function $F(z, q)$ satisfies

$$F(zq, q)=F(z, q). \tag{3.9}$$

We call functions with this property *q-periodic* functions. The function $F(z, q)$ has the Fourier series

$$
\begin{aligned}
F(z,q) &=^{*} \frac{-4}{\ln q}\operatorname{Im}\left[\int_{\Omega} \frac{\ln(1+e^{-2ix})}{\exp((-2\pi/\ln q)(2x-i\ln z))-1}\,dx\right] \\
&= \frac{-4}{\ln q}\sum_{k=1}^{\infty}\operatorname{Im}\left[\exp\left(-2\pi i k\frac{\ln z}{\ln q}\right)\int_{\Omega}\ln(\cos x)\exp\left(\frac{4\pi k}{\ln q}x\right)dx\right] \\
&=^{**} -2\sum_{k=1}^{\infty}\operatorname{Im}\left[\frac{\exp(-2\pi i k(\ln z/\ln q))}{2\pi k}\int_{\Omega}\frac{\sin x}{\cos x}\exp\left(\frac{4\pi k}{\ln q}x\right)dx\right] \\
&= -2\sum_{k=1}^{\infty}\operatorname{Im}\left[i\frac{\exp(-2\pi i k(\ln z/\ln q))}{k}\sum_{m=0}^{\infty}\exp\left(\frac{2\pi^2 k(2m+1)}{\ln q}\right)\right] \\
&= \sum_{k=1}^{\infty}\frac{\cos(2\pi k(\ln z/\ln q))}{k\sinh(2\pi^2 k/\ln q)},
\end{aligned}
\tag{3.10}
$$

where the contour of integration Ω looks like the contour Ω in Fig. 3.1. In $*$ we used (3.5) with the substitution $t=(\mp 2ix-\ln z)/\ln q$, and in $**$ we integrated by parts. Thus for $z>1$ and $0<q<1$ we have obtained the convergent expansion

$$
\begin{aligned}
E_q(z) &= (-z;q)_\infty \\
&= \frac{1}{(-q/z;q)_\infty}\exp\left[\frac{1}{2}\ln z-\frac{1}{\ln q}\left(\frac{1}{6}\pi^2+\frac{1}{2}\ln^2 z\right)-\frac{1}{12}\ln q\right. \\
&\qquad \left.+\sum_{k=1}^{\infty}\frac{\cos(2\pi k(\ln z/\ln q))}{k\sinh(2\pi^2 k/\ln q)}\right].
\end{aligned}
\tag{3.11}
$$

It is not difficult to prove that (3.11) holds for $|\arg z|\leqslant\pi$, and that the $\arg z\uparrow\pi$ limit of the right-hand side of (3.11) is the same as the $\arg z\downarrow -\pi$ limit. For large $|z|$ the dominant part of (3.11) is $\exp(-\frac{1}{2}\ln^2 z/\ln q)$. The factor $1/(-q/z;q)_\infty$ has the asymptotic expansion $\sum_{n=0}^{\infty}(-q/z)^n/(q;q)_n$, $|z|>q$, see (3.1). The infinite Fourier series, (3.10), is a fast converging series for $|\arg z|<\pi-\delta$, with $\delta>0$ fixed. As remarked earlier, (3.11) does not converge as $q\uparrow 1$.

3.3. *An asymptotic expansion for $e_q(z)$.* Taking $z>0$ and replacing z in (3.11) by $e^{i\pi}z$ or by $e^{-i\pi}z$, and using the Fourier series

$$
\left.\begin{aligned}
\sum_{k=1}^{\infty}\frac{\sin(ky)}{k} &= \frac{1}{2}(\pi-y), \\
\sum_{k=1}^{\infty}\frac{\cos(ky)}{k} &= -\frac{1}{2}\ln(2(1-\cos y)),
\end{aligned}\right\}\quad 0<y<2\pi,
\tag{3.12}
$$

we obtain for $z>0$

$$\begin{aligned}\frac{1}{e_q(z)}&=(z;q)_\infty\\&=2\,\frac{\sin(\pi(\ln z/\ln q))}{(q/z;q)_\infty}\exp\left[\frac{1}{2}\ln z-\frac{1}{\ln q}\left(-\frac{1}{3}\pi^2+\frac{1}{2}\ln^2 z\right)-\frac{1}{12}\ln q\right.\\&\quad\left.+\sum_{k=1}^{\infty}\left(\cos\left(2\pi k\frac{\ln z}{\ln q}\right)\exp\left(\frac{2\pi^2 k}{\ln q}\right)\Big/k\sinh\left(\frac{2\pi^2 k}{\ln q}\right)\right)\right].\end{aligned}\tag{3.13}$$

It is not difficult to prove that (3.13) holds for $|\arg z|\leqslant 2\pi$, and that the $\arg z\uparrow 2\pi$ limit of the right-hand side of (3.13) is the same as the $\arg z\downarrow -2\pi$ limit. Again, for large $|z|$ the dominant part of (3.13) is $\exp(-\frac{1}{2}\ln^2 z/\ln q)$. The factor $1/(q/z;q)_\infty$ has the asymptotic expansion $\sum_{n=0}^{\infty}(q/z)^n/(q;q)_n$, $|z|>q$. The infinite Fourier series is a fast converging series for $|\arg z|<2\pi-\delta$, with $\delta>0$ fixed. Again, (3.13) does not converge as $q\uparrow 1$.

3.4. *Certain expansions for theta functions.* In the previous subsections we have obtained expansions for the infinite products $(z;q)_\infty\,(q/z;q)_\infty$ and $(-z;q)_\infty\,(-q/z;q)_\infty$. In this subsection we use these expansions for obtaining certain expansions for the theta functions. The theta functions have the well-known representation

$$\begin{aligned}\vartheta_1(-i\ln z,q)&=i\sum_{n=-\infty}^{\infty}(-1)^n q^{((2n-1)/2)^2}z^{2n-1}\\&\overset{*}{=}\frac{i}{z}q^{1/4}(q^2;q^2)_\infty\,(z^2;q^2)_\infty\left(\frac{q^2}{z^2};q^2\right)_\infty,\\\vartheta_2(-i\ln z,q)&=\sum_{n=-\infty}^{\infty}q^{((2n-1)/2)^2}z^{2n-1}\\&\overset{*}{=}\frac{1}{z}q^{1/4}(q^2;q^2)_\infty\,(-z^2;q^2)_\infty\left(-\frac{q^2}{z^2};q^2\right)_\infty,\\\vartheta_3(-i\ln z,q)&=\sum_{n=-\infty}^{\infty}q^{n^2}z^{2n}\overset{*}{=}(q^2;q^2)_\infty\,(-z^2q;q^2)_\infty\left(-\frac{q}{z^2};q^2\right)_\infty,\\\vartheta_4(-i\ln z,q)&=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}z^{2n}\overset{*}{=}(q^2;q^2)_\infty\,(z^2q;q^2)_\infty\left(\frac{q}{z^2};q^2\right)_\infty,\end{aligned}\tag{3.14}$$

where $*$ follows from Jacobi's triple product identity, see [3]. Using (3.11) and (3.13), we obtain for $0<q<1$ the expansions

$$\vartheta_1(-i\ln z, q) = i(q^2; q^2)_\infty\, 2\sin\left(\pi\frac{\ln z}{\ln q}\right)\exp\left[-\frac{1}{\ln q}\left(-\frac{1}{6}\pi^2+\ln^2 z\right)\right.$$
$$+\frac{1}{12}\ln q+\sum_{k=1}^{\infty}\left(\cos\left(2\pi k\frac{\ln z}{\ln q}\right)\right.$$
$$\left.\left.\times\exp\left(\frac{\pi^2 k}{\ln q}\right)\Big/k\sinh\left(\frac{\pi^2 k}{\ln q}\right)\right)\right], \tag{3.15.a}$$

$$\vartheta_2(-i\ln z, q) = (q^2; q^2)_\infty \exp\left[-\frac{1}{\ln q}\left(\frac{1}{12}\pi^2+\ln^2 z\right)\right.$$
$$\left.+\frac{1}{12}\ln q+\sum_{k=1}^{\infty}\frac{\cos(2\pi k(\ln z/\ln q))}{k\sinh(\pi^2 k/\ln q)}\right], \tag{3.15.b}$$

$$\vartheta_3(-i\ln z, q) = (q^2; q^2)_\infty \exp\left[-\frac{1}{\ln q}\left(\frac{1}{12}\pi^2+\ln^2 z\right)\right.$$
$$\left.+\frac{1}{12}\ln q+\sum_{k=1}^{\infty}(-1)^k\frac{\cos(2\pi k(\ln z/\ln q))}{k\sinh(\pi^2 k/\ln q)}\right], \tag{3.15.c}$$

$$\vartheta_4(-i\ln z, q) = (q^2; q^2)_\infty\, 2\cos\left(\pi\frac{\ln z}{\ln q}\right)\exp\left[-\frac{1}{\ln q}\left(-\frac{1}{6}\pi^2+\ln^2 z\right)\right.$$
$$+\frac{1}{12}\ln q+\sum_{k=1}^{\infty}(-1)^k\left(\cos\left(2\pi k\frac{\ln z}{\ln q}\right)\right.$$
$$\left.\left.\times\exp\left(\frac{\pi^2 k}{\ln q}\right)\Big/k\sinh\left(\frac{\pi^2 k}{\ln q}\right)\right)\right], \tag{3.15.d}$$

Expansions (3.15.a) and (3.15.d) hold for $|\arg z| \leqslant \pi$, and the expansions (3.15.b) and (3.15.c) hold for $|\arg z| \leqslant \pi/2$. These four expansions can be used for numerical computations of the theta functions. A direct proof for these expansions uses Jacobi's transformation for theta functions. For $\vartheta_1(z, q)$ Jacobi's transformation reads

$$\vartheta_1(-i\ln z, q) = i\sqrt{\frac{\pi}{-\ln q}}\exp\left(-\frac{\ln^2 z}{\ln q}\right)\vartheta_1\left(\pi\frac{\ln z}{\ln q}, \exp\left(\frac{\pi^2}{\ln q}\right)\right). \tag{3.16}$$

Now use Jacobi's triple product identity for the theta function on the right-hand side of (3.16), and some simple manipulations lead to the first expansion of (3.15).

Remark. The results in this section overlap with the results in Littlewood [9]. Our result (3.11) is similar to [9, (2), p. 395], but the proof of (3.11) is new, and it can be used for other infinite products. In [9] a complete asymptotic expansion for $(-z;q)_\infty$ is given, as $|z|\to\infty$, which is valid for $0<|q|<1$. For a correct version of this complete asymptotic expansion see Chen *et al.* [2], where a result similar to (3.11) is proven also. Our result (3.15.b) is similar to [9, (5), p. 400].

4. Asymptotic Expansions for the Hahn–Exton q-Bessel Function

4.1. *The Hahn–Exton q-Bessel function.* This function has been thoroughly investigated by Swarttouw [12]. In this section we take $0<|q|<1$. The definition of this function is in terms of the q-hypergeometric ${}_1\Phi_1$ function. Let

$${}_1\Phi_1\left(\begin{matrix}0\\ w\end{matrix}\,\middle|\, q, z\right)=\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{(w;q)_k\,(q;q)_k}(-z)^k, \qquad z\in\mathbb{C}, \tag{4.1}$$

then the definition of the Hahn–Exton q-Bessel function $J_\nu(x,q^2)$ reads

$$\begin{aligned}J_\nu(x,q^2)&=x^\nu\frac{(q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty}\,{}_1\Phi_1\left(\begin{matrix}0\\ q^{2\nu+2}\end{matrix}\,\middle|\, q^2, x^2q^2\right)\\ &=x^\nu\frac{(q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^{2\binom{k}{2}}}{(q^{2\nu+2};q^2)_k\,(q^2;q^2)_k}(-x^2q^2)^k, \qquad x\in\mathbb{C}.\end{aligned} \tag{4.2}$$

This function has many nice properties, which are analogues of properties of the Bessel function $J_\nu(x)$. We have taken q^2 as the base, to make the notations easier. In [12] it is shown that a q-extension of Bessel's equation is the *Hahn–Exton q-difference equation*

$$f(x)+q^{-\nu}(x^2q^2-1-q^{2\nu})\,f(xq)+f(xq^2)=0, \tag{4.3}$$

of which $f(x)=J_\nu(x,q^2)$ is a solution. A second solution is $J_{-\nu}(xq^{-\nu},q^2)$. The Hahn–Exton q-Bessel function has the following $q\uparrow 1$ limit:

$$\lim_{q\uparrow 1} J_\nu(x(1-q),q^2)=J_\nu(x). \tag{4.4}$$

4.2. *An asymptotic expansion for* $J_\nu(x,q^2)$. In Koornwinder and Swarttouw [7] it is proven that the symmetry relation

$$(w;q)_\infty\,{}_1\Phi_1\left(\begin{matrix}0\\ w\end{matrix}\,\middle|\, q, z\right)=(z;q)_\infty\,{}_1\Phi_1\left(\begin{matrix}0\\ z\end{matrix}\,\middle|\, q, w\right), \qquad w,z\in\mathbb{C}, \tag{4.5}$$

holds. Thus with (4.2) we obtain

$$J_\nu(x, q^2) = x^\nu \frac{(x^2q^2; q^2)_\infty}{(q^2; q^2)_\infty} \, {}_1\Phi_1\left(\begin{matrix} 0 \\ x^2q^2 \end{matrix} \middle| q^2, q^{2\nu+2}\right)$$
$$= x^\nu \frac{(x^2q^2; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{2\binom{k}{2}}}{(x^2q^2; q^2)_k \, (q^2; q^2)_k} (-q^{2\nu+2})^k, \qquad x \in \mathbb{C}. \tag{4.6}$$

The factor $(x^2q^2; q^2)_\infty$ is the function $1/e_{q^2}(x^2q^2)$, for which we have derived an expansion in (3.13). The infinite series in (4.6) converges fast for large x. And from $(x^2q^2; q^2)_k/(x^2q^2; q^2)_{k+1} = (1 - x^2q^{2k+2})^{-1}$ it follows that this series has an asymptotic property for $x \to \infty$. The poles of $1/(x^2q^2; q^2)_k$ at $x = q^{-m}$, $m \in \{1, 2, . .., k\}$, are removed by the zeros of $(x^2q^2; q^2)_\infty$. Note that (4.6) is also an asymptotic expansion for $\nu \to \infty$.

4.3. *A derivation of the asymptotic expansion from the Hahn–Exton q-difference equation.* The following method for obtaining asymptotic expansions for solutions of the difference equation (4.3) is inspired by the formal solutions substitution method (see [11, Chap. 7]) for obtaining asymptotic expansions for solutions of differential equations with irregular singularities. For obtaining a q-exponential part of a formal solution of (4.3), we first solve the equation $h(x) = (q^{-\lambda} - x^2q^{2-\nu})\, h(xq)$, $\lambda \in \mathbb{C}$, which we call a reduced equation of (4.3). The simplest non-trivial solution of this reduced equation is $x^\lambda(x^2q^{2-\nu+\lambda}; q^2)_\infty$. This observation leads to the following representation of a solution of (4.3):

$$f_\lambda(x) = x^\lambda(x^2q^{2-\nu+\lambda}; q^2)_\infty \, g_\lambda(x), \qquad \lambda \in \mathbb{C}. \tag{4.7}$$

Substituting (4.7) in (4.3), we obtain for $g_\lambda(x)$ the following q-difference equation:

$$g_\lambda(x) - g_\lambda(xq) + \frac{1 - q^\lambda(q^{-\nu} + q^\nu)}{1 - x^2q^{2-\nu+\lambda}} g_\lambda(xq)$$
$$+ \frac{q^{2\lambda}}{(1 - x^2q^{2-\nu+\lambda})(1 - x^2q^{4-\nu+\lambda})} g_\lambda(xq^2) = 0. \tag{4.8}$$

Substituting the formal expansion

$$g_\lambda(x) = \sum_{k=0}^{\infty} \frac{a_k}{(x^2q^{2-\nu+\lambda}; q^2)_k} \tag{4.9}$$

in (4.8) and using

$$\frac{1}{(x^2q^{4-\nu+\lambda}; q^2)_k} = \frac{1 - q^{-2k}}{(x^2q^{2-\nu+\lambda}; q^2)_{k+1}} + \frac{q^{-2k}}{(x^2q^{2-\nu+\lambda}; q^2)_k}, \tag{4.10}$$

we obtain

$$
\begin{aligned}
a_1 &= \frac{q^2}{1-q^2}\,(1-q^\lambda(q^{-\nu}+q^\nu))\,a_0, \\
a_k &= \frac{q^{2k}}{1-q^{2k}}\,[(q^{2-2k}-q^\lambda(q^{-\nu}+q^\nu))\,a_{k-1}+q^{2\lambda}a_{k-2}], \qquad k=2,3,\ldots
\end{aligned}
\tag{4.11}
$$

We write

$$
a_k = \frac{q^{2k}}{(q^2;q^2)_k}\,\alpha_k, \tag{4.12}
$$

and with (4.11) we obtain that $\alpha_1=(1-q^\lambda(q^{-\nu}+q^\nu))\alpha_0$ and

$$
\alpha_{k+1}=\alpha_k-q^{2k}[q^\lambda(q^{-\nu}+q^\nu)\alpha_k+q^{2\lambda-2}(q^{2k}-1)\alpha_{k-1}], \qquad k=1,2,\ldots \tag{4.13}
$$

By induction with respect to k it is easy to prove that

$$
|\alpha_k| \leqslant (-M;|q|^2)_{k+1}\,|a_0| \leqslant (-M;|q|^2)_\infty\,|a_0|, \tag{4.14}
$$

where $M=|q|^{\Re\lambda-4}(|q|^{-\Re\nu}+|q|^{\Re\nu})+2\,|q|^{2\Re\lambda-4}$. With (4.13) it follows that $\{\alpha_k\}_{k=0,1,2,\ldots}$ is a bounded Cauchy sequence. Thus there is a constant $C_\lambda\in\mathbb{C}$ such that a_kq^{-2k} converges to C_λ as $k\to\infty$. It follows that for all $\lambda\in\mathbb{C}$ the expansion (4.9) converges.

Substituting

$$
g_\lambda^N(x)=\sum_{k=0}^{N}\frac{a_k}{(x^2q^{2-\nu+\lambda};q^2)_k} \tag{4.15}
$$

into the left-hand side of the q-difference equation (4.8), we obtain

$$
\begin{aligned}
&g_\lambda^N(x)-g_\lambda^N(xq)+\frac{1-q^\lambda(q^{-\nu}+q^\nu)}{1-x^2q^{2-\nu+\lambda}}\,g_\lambda^N(xq) \\
&\qquad+\frac{q^{2\lambda}}{(1-x^2q^{2-\nu+\lambda})(1-x^2q^{4-\nu+\lambda})}\,g_\lambda^N(xq^2) \\
&\quad=\frac{(q^{-2(N+1)}-1)\,a_{N+1}}{(x^2q^{2-\nu+\lambda};q^2)_{N+1}}+\frac{q^{2\lambda}a_N}{(x^2q^{2-\nu+\lambda};q^2)_{N+2}}.
\end{aligned}
\tag{4.16}
$$

Thus the right-hand side of (4.16) converges to zero only when $C_\lambda=0$. Hence, only for those λ, with $C_\lambda=0$, the expansion (4.9) is a solution of the q-difference equation (4.8). The case that $\lambda=\pm\nu$ is one of the simplest cases with $C_\lambda=0$. Another example with $C_\lambda=0$ is $\lambda=\pm\nu-2$.

We obtain the two solutions $f_{\pm\nu}(x)$ of (4.3), and it is easy to show that with the appropriate a_0, we have $f_\nu(x)=J_\nu(x,q^2)$ and $f_{-\nu}(x)=J_{-\nu}(xq^{-\nu},q^2)$.

4.4. *Other solutions of the Hahn–Exton q-difference equation.* In this subsection we start with the reduced equation $h(xq)=(q^{-\lambda}-x^2q^{-\nu})\,h(x)$, $\lambda\in\mathbb{C}$. The simplest non-trivial solution of this reduced equation is the product $x^{-\lambda}/(x^2q^{\lambda-\nu};q^2)_\infty$. Our second formal solution is written in the form

$$\tilde{f}_\lambda(x)=\frac{x^{-\lambda}}{(x^2q^{\lambda-\nu};q^2)_\infty}\,\tilde{g}_\lambda(x). \tag{4.17}$$

With the method of the previous subsection, we obtain two new solutions of (4.3). The first new solution is

$$\begin{aligned} J_\nu^{(2)}(x,q^2)&=x^{-\nu}\frac{(q^2;q^2)_\infty}{(x^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^{2k\nu}}{(x^2q^{-2};q^{-2})_k\,(q^2;q^2)_k}\\ &=x^{-\nu}\frac{(q^2;q^2)_\infty}{(x^2;q^2)_\infty}\,{}_1\Phi_1\left(\begin{matrix}0\\x^{-2}q^2\end{matrix}\,\middle|\,q^2,x^{-2}q^{2\nu+2}\right),\qquad x\in\mathbb{C}, \end{aligned} \tag{4.18}$$

and the second new solution is $J_{-\nu}^{(2)}(xq^{-\nu},q^2)$.

The solution $J_\nu(x,q^2)$ of (4.3) has the asymptotic behaviour $J_\nu(x,q^2)\sim x^\nu(x^2q^2;q^2)_\infty/(q^2;q^2)_\infty$ as $|x|\to\infty$, and the solution $J_\nu^{(2)}(x,q^2)$ has the asymptotic behaviour $J_\nu^{(2)}(x,q^2)\sim x^{-\nu}(q^2;q^2)_\infty/(x^2;q^2)_\infty$ as $|x|\to\infty$. With this behaviour for large $|x|$, and with the theory of the next subsections, it follows that $J_\nu(x,q^2)$ and $J_\nu^{(2)}(x,q^2)$ form a *numerically satisfactory pair of solutions* (see [11, §5.7] for an extensive treatment of this concept) of (4.3). Again, note that (4.18) is also an asymptotic expansion for $\nu\to\infty$.

4.5. *Second-order linear q-difference equations.* A general second-order linear q-difference equation is of the form

$$a(x,q)\,f(x)+b(x,q)\,f(xq)+c(x,q)\,f(xq^2)=0. \tag{4.19}$$

We take as the q-Wronskian

$$W_q(f_1(x),f_2(x))=f_1(xq)\,f_2(x)-f_1(x)\,f_2(xq), \tag{4.20}$$

which is a slightly different q-Wronskian than the q-Wronskian introduced in [12]. It is not difficult to show that $W_q/x(1-q)$ tends to the ordinary Wronskian if $q\uparrow 1$. Let f_1 and f_2 be solutions of (4.19), such that

$W_q(f_1(x), f_2(x))$ is not identically zero. Then it follows that any solution $f(x)$ of (4.19) can be written as $f(x)=p_1(x)\, f_1(x)+p_2(x)\, f_2(x)$, with

$$p_1(x)=\frac{W_q(f(x), f_2(x))}{W_q(f_1(x), f_2(x))} \qquad \text{and} \qquad p_2(x)=\frac{W_q(f_1(x), f(x))}{W_q(f_1(x), f_2(x))},$$

which are q-periodic functions, that is, $p_j(xq)=p_j(x)$.

Note that for two solutions f_1, f_2 of (4.19) we have $W_q(f_1(x), f_2(x))= c(x, q)/a(x, q)\, W_q(f_1(xq), f_2(xq))$. Thus $x \mapsto W_q(f_1(x), f_2(x))$ is q-periodic, if and only if $a(x, q)=c(x, q)$.

4.6. *The Wronskians of the solutions of the Hahn–Exton q-difference equation.* With the last remark of the previous subsection it follows that for two solutions f_1, f_2 of the Hahn–Exton q-difference equation (4.3), the Wronskian $w(x)=W_q(f_1(x), f_2(x))$ is q-periodic. Hence, $w(x)= \lim_{k\to\infty} w(xq^{\pm k})$, and we can compute the Wronskian from the expansions of $f_1(x)$ and $f_2(x)$ at $x=0$ or at $x=\infty$. In [12] the Wronskian

$$W_q(J_\nu(x, q^2), J_{-\nu}(xq^{-\nu}, q^2))=-\frac{q^{\nu(\nu-1)}(1-q^2)}{\Gamma_{q^2}(\nu)\, \Gamma_{q^2}(1-\nu)} \tag{4.21}$$

is obtained. Thus for ν an integer, the Wronskian is identically zero. In [12] a q-analogue of the Bessel function $Y_\nu(x)$ is introduced, which is a solution of (4.3). It is defined by

$$\begin{aligned} Y_\nu(x, q^2)=&\frac{\Gamma_{q^2}(\nu)\, \Gamma_{q^2}(1-\nu)\, q^{\nu(\nu+1)}}{\Gamma_{q^2}(1/2-\nu)\, \Gamma_{q^2}(1/2+\nu)} J_\nu(x, q^2) \\ &-\frac{\Gamma_{q^2}(\nu)\, \Gamma_{q^2}(1-\nu)}{\Gamma_{q^2}(1/2)\, \Gamma_{q^2}(1/2)} J_{-\nu}(xq^{-\nu}, q^2), \end{aligned} \tag{4.22}$$

and it has the limit relation

$$\lim_{q\uparrow 1} Y_\nu(x(1-q), q^2)=Y_\nu(x). \tag{4.23}$$

The Wronskians

$$\begin{aligned} W_q(J_\nu(x, q^2), Y_\nu(x, q^2))&=\frac{q^{\nu(\nu-1)}(1-q^2)}{\Gamma_{q^2}(1/2)\, \Gamma_{q^2}(1/2)}, \\ W_q(J_\nu(x, q^2), J_\nu^{(2)}(x, q^2))&=-q^{-\nu}, \\ W_q(J_{-\nu}(xq^{-\nu}, q^2), J_{-\nu}^{(2)}(xq^{-\nu}, q^2))&=-q^\nu, \end{aligned} \tag{4.24}$$

are not identically zero, for all $\nu \in \mathbb{C}$. But

$$W_q(J_\nu^{(2)}(x, q^2), J_{-\nu}^{(2)}(xq^{-\nu}, q^2))=0, \tag{4.25}$$

which follows from

$$\frac{J_\nu^{(2)}(x, q^2)}{J_{-\nu}^{(2)}(xq^{-\nu}, q^2)} = q^{\nu^2} x^{-2\nu} \frac{(x^2 q^{-2\nu}; q^2)_\infty \, (x^{-2} q^{2\nu+2}; q^2)_\infty}{(x^2; q^2)_\infty \, (x^{-2} q^2; q^2)_\infty}, \tag{4.26}$$

which is q-periodic, and which can be obtained by using (4.5) in (4.18).

4.7. *A representation for $J_\nu^{(2)}(x, q^2)$ in terms of the functions $J_\nu(x, q^2)$ and $Y_\nu(x, q^2)$.* In this subsection we find the unique q-periodic functions $a(x, q)$ and $b(x, q)$ such that

$$J_\nu^{(2)}(x, q^2) = a(x, q)\, J_\nu(x, q^2) + b(x, q)\, Y_\nu(x, q^2). \tag{4.27}$$

We substitute (4.27) in the second equation of (4.24), and we obtain

$$b(x, q) = \frac{-q^{-\nu}}{W_q(J_\nu(x, q^2), Y_\nu(x, q^2))} = \frac{-q^{-\nu^2}}{1-q^2} \Gamma_{q^2}^2\left(\frac{1}{2}\right). \tag{4.28}$$

To obtain a representation for $a(x, q)$ we substitute (4.27) and representation (4.22) in $a(x, q) = W_q(J_\nu^{(2)}(x, q^2), Y_\nu(x, q^2))/W_q(J_\nu(x, q^2), Y_\nu(x, q^2))$ and we use (4.26). We obtain

$$\begin{aligned} a(x, q) = {} & \frac{q^\nu}{1-q^2} \frac{\Gamma_{q^2}(\nu)\, \Gamma_{q^2}(1-\nu)\, \Gamma_{q^2}^2(1/2)}{\Gamma_{q^2}(1/2-\nu)\, \Gamma_{q^2}(1/2+\nu)} \\ & - \frac{q^{2\nu} x^{-2\nu}}{1-q^2} \frac{(x^2 q^{-2\nu}; q^2)_\infty \, (x^{-2} q^{2\nu+2}; q^2)_\infty}{(x^2; q^2)_\infty \, (x^{-2} q^2; q^2)_\infty} \Gamma_{q^2}(\nu)\, \Gamma_{q^2}(1-\nu). \end{aligned} \tag{4.29}$$

With representation (4.27) and some hard analysis it can be proven that in contrast to (4.4) and (4.8) the function $J_\nu^{(2)}(x(1-q), q^2)$ does not converge as $q \uparrow 1$.

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