Mathematical Morphology: a Geometrical Approach in Image Processing

Henk J.A.M. Heijmans
CWI, P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Mathematical morphology is a theory of image operators and image functionals which is based on set-theoretical, geometrical and topological concepts. The methodology is particularly useful for the analysis of the geometrical structure in an image. The main goal of this paper is to give an impression of the underlying philosophy and the mathematical theories which are relevant to this field. We have tried to achieve this goal by discussing a number of theoretical problems we have been dealing with in the past five years.

1. Introduction

In the early sixties two researchers at the Paris School of Mines in Fontainebleau, Georges Matheron and Jean Serra, worked on a number of problems in mineralogy and petrography. Their main goal was to characterize physical properties of certain materials, e.g. the permeability of a porous medium, by examining their geometrical structure. Their investigations ultimately led to a new quantitative approach in image analysis, nowadays known as mathematical morphology.

Thirty years later, mathematical morphology has achieved a status as a powerful method for image processing which, besides having been applied successfully in various disciplines such as mineralogy, medical diagnostics and histology, has also become a solid mathematical theory leaning on concepts from algebra, topology, integral and stochastic geometry. To a large extent the current status is due to its founders MATHERON and SERRA [27, 39, 40]

The central idea of mathematical morphology is to examine the geometrical structure of an image by matching it with small patterns at various locations in the image. By varying the size and the shape of the matching patterns, called *structuring elements*, one can obtain useful information about the shape of the different parts of the image and their interrelations. In general the procedure results in nonlinear image operators which are well-suited for the analysis of the geometrical and topological structure of an image.

Originally, mathematical morphology has been developed for binary images which can be represented mathematically as sets. The corresponding morphological operators essentially use only four ingredients from set theory, namely set intersection, union, complementation and translation. But from the very beginning there was a need for a more general theory powerful enough to deal with object spaces such as the closed subsets of a topological space, the convex sets of a (topological) vector space, and grey-scale images. It has been observed first by SERRA [39] that a theory of morphology essentially requires the underlying image space to be a complete lattice.

This paper intends to give the reader a flavour of mathematical morphology. As such we do not aim for completeness. In fact the paper is rather fragmentary and restricts to those subjects we consider interesting to a mathematical readership. In the final section we point out a few other subjects which are not discussed in this paper but which may also be of interest. For a general account on mathematical morphology we refer to the two books by SERRA [39, 40] and to a monograph by MATHERON [27] which contains a comprehensive discussion on random sets and integral geometry. Actually, it is this probabilistic branch which has made morphology into such a powerful methodology, and it is somewhat unfortunate that this aspect has been given so little attention in the recent literature. Furthermore we refer the interested reader to a forthcoming monograph by the author [15] dealing with various mathematical aspects of morphology. Some other elementary references are [6, 7, 9].

Besides this introduction this paper comprises six sections, all dealing with different topics. In Section 2 we introduce the reader to morphology and discuss a number of classical, binary morphological operators which are invariant under translations. In Section 3 we discuss the extension to the framework of complete lattices. Such an abstract theory also enables the construction of operators which are invariant under other transformation groups than translations. In Section 4 we introduce the reader into the world of morphological filters; these are defined as operators which preserve the partial order structure and which are idempotent. We indicate how one can construct filters by iteration of operators which are not idempotent but do have certain continuity properties. Geometrical aspects of morphology are discussed in Section 5. Despite the patchy contents of that section we hope it gives the reader an intuition for the kind of problems which occur. Then, in Section 6 we discuss some extensions of the binary theory to grey-scale images, and finally in Section 7 we mention some problems which have not been given a place in this paper.

2. WHAT IS MATHEMATICAL MORPHOLOGY?

A convenient way to model binary (=black and white) images, both continuous and discrete, is by means of sets. Unless stated otherwise we assume that $E = \mathbb{R}^d$ or $\mathbb{Z}^d$. By $\mathcal{P}(E)$ we denote the power set of E. Let $X \subseteq E$ be a binary image. The key principle underlying mathematical morphology is to gain geometric information about X by probing it with another small set, called the *structuring element*, at every position $h \in E$. By 'probing' we mean testing whether the set A_h hits X (that is $A_h \cap X \neq \emptyset$) misses X (that is, $A_h \cap X = \emptyset$), or lies entirely inside X (that is, $A_h \subseteq X$). Here A_h denotes the translate of A along the vector h, $A_h = \{a + h \mid a \in A\}$. The *hit-or-miss operator* is a mapping on the space of binary images $\mathcal{P}(E)$ which is based on this intuitive idea. Let $A, B \subseteq E$ be two structuring elements such that $A \cap B = \emptyset$ and define

$$X \circledast (A, B) = \{h \in E \mid A_h \subseteq X \text{ and } B_h \subseteq X^c\}. \tag{2.1}$$

Here X^c denotes the complement of X, or, in image processing terminology, the background of the image X. See Figure 1 for an example. It is obvious that A and B must have an empty intersection because otherwise the resulting set would be empty.

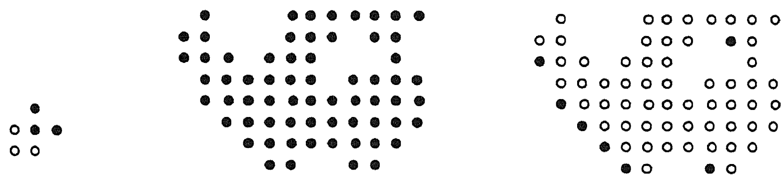

FIGURE 1. Hit-or-miss operator for a discrete image. The structuring element (left) is such that the operator detects the lower-left corner points. The black dots in the right image form the transformed image $X \circledast (A, B)$ of the original image X (middle).

The hit-or-miss operator is an easy example of a *set operator* (i.e., a mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$) which is *translation invariant*, that is,

$$\psi(X_h) = [\psi(X)]_h, \quad \text{for} X \in \mathcal{P}(E) \text{ and } h \in E. \tag{2.2}$$

Moreover, one can show that every translation invariant set operator can be represented as a union of hit-or-miss operators.

PROPOSITION 2.1. *Let $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ be a translation invariant operator. There is a family of pairs of structuring elements $\{(A_i, B_i) \mid i \in I\}$ such that*

$$\psi(X) = \bigcup_{i \in I} X \circledast (A_i, B_i).$$

In fact, it is possible to give a characterization of the structuring elements required in this representation [1]. If we take $B = \emptyset$ in (2.1) we obtain the *Minkowski difference*

$$X \ominus A = \{h \in E \mid A_h \subseteq X\},$$

which we shall call henceforth the *erosion* of X by A. Instead of $X \ominus A$ we shall also write $\varepsilon_A(X)$. Note that

$$X \circledast (A, B) = (X \ominus A) \cap (X^c \ominus B). \tag{2.4}$$

Erosion is a translation invariant operator which is *increasing*, that is

$$X \subseteq Y \implies X \ominus A \subseteq Y \ominus A.$$

Instead of (2.3) we can also write

$$X \ominus A = \bigcap_{a \in A} X_{-a}. \tag{2.5}$$

Another important operator is *Minkowski sum* given by

$$X \oplus A = \bigcup_{a \in A} X_a, \tag{2.6}$$

from now on called the *dilation* of X by A, and denoted as $\delta_A(X)$. Note that $X \oplus A = \{x + a \mid x \in X,\ a \in A\}$. Dilation and erosion are depicted in Figure 2.

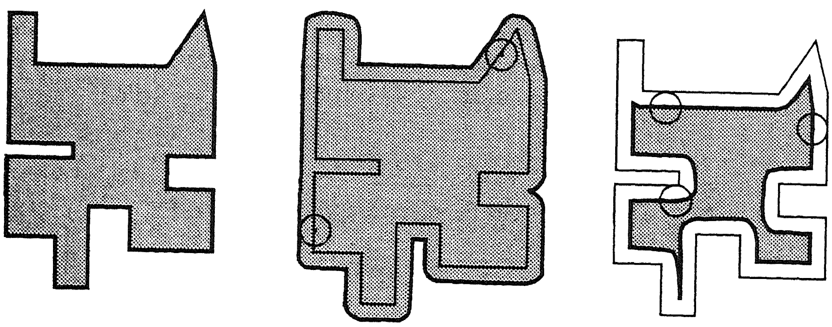

FIGURE 2. From left to right (in grey): the original set X and its dilation and erosion by a disk.

Defining the reflection of A with respect to the origin as

$$\check{A} = \{-a \mid a \in A\},$$

we can also write

$$X \oplus A = \{h \in \mathbb{R}^d \mid \check{A}_h \cap X \neq \emptyset\}.$$

It is clear that dilation defines an increasing translation invariant operator.

After these definitions we could give a long list of properties of erosion and dilation. However, we refrain from doing so and mention only those properties which we think are essential here. If $X_i \subseteq E$ for all i in some index set I, then

$$(\bigcup_{i \in I} X_i) \oplus A = \bigcup_{i \in I} (X_i \oplus A) \tag{2.7}$$

$$(\bigcap_{i \in I} X_i) \ominus A = \bigcap_{i \in I} (X_i \ominus A). \tag{2.8}$$

In the next section where we discuss morphological operators in the context of complete lattices we meet these properties again.

If $X, A, B \subseteq E$ then

$$(X \oplus A) \oplus B = X \oplus (A \oplus B) \tag{2.9}$$

$$(X \ominus A) \ominus B = X \ominus (A \oplus B). \tag{2.10}$$

These properties open the way to construct decompositions of erosion and dilation with large structuring elements in terms of erosions and dilations with smaller structuring elements. It is evident that 'clever' decomposition procedures yield a powerful method for fast implementations of dilations and erosions. For example one has

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \oplus \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \oplus \begin{pmatrix} \bullet & \cdot & \bullet \\ \cdot & \cdot & \cdot \\ \bullet & \cdot & \bullet \end{pmatrix},$$

where the $\bullet$'s represent the points in the structuring element. Note that in the last decomposition one of the 3×3-squares is replaced by its extreme points. This reduces the number of operations in expressions like (2.9) and (2.10). For larger structuring elements this reduction becomes even more significant. We refer to [3] where this property is used to obtain logarithmic decompositions.

There is a considerable amount of literature about decomposition of structuring elements using Minkowski addition [46, 47]. We do not give any details here but only point out that discrete convexity plays an important role in these theories. To understand this we make the following observation: a compact set $A \subseteq \mathbb{R}$ is convex if and only if

$$(\lambda + \mu)A = \lambda A \oplus \mu A, \tag{2.11}$$

for $\lambda, \mu > 0$. The relation (2.11) also holds in the 2-dimensional discrete case. In higher dimensions it is no longer true. We will meet property again in Section 5 when we discuss granulometries.

We can specialize Proposition 2.1 to increasing translation invariant operators. To be specific, we define the *kernel* $\mathcal{V}(\psi)$ of a translation invariant operator ψ as

$$\mathcal{V}(\psi) = \{A \subseteq E \mid 0 \in \psi(A)\}.$$

The following result is due to MATHERON [27].

THEOREM 2.2. *Let ψ be an increasing translation invariant operator on $\mathcal{P}(E)$. Then*

$$\psi(X) = \bigcup_{A \in \mathcal{V}(\psi)} X \ominus A = \bigcap_{A \in \mathcal{V}(\psi^*)} X \oplus \check{A}.$$

Here ψ^* denotes the complementary operator given by

$$\psi^*(X) = [\psi(X^c)]^c. \tag{2.12}$$

The kernel of an operator is too large to be of any practical use. Namely, it follows from the increasingness that $A \in \mathcal{V}(\psi)$ implies that also $A' \in \mathcal{V}(\psi)$ for every set $A' \supseteq A$. This has motivated MARAGOS [24] to introduce the notion of a *minimal kernel element* and to look for (continuity) conditions on ψ which guarantee that ψ can be represented as a union of erosions with these minimal kernel elements.

Dilation and erosion are also complementary operators in the following sense:

$$(X \oplus A)^c = X^c \ominus \check{A}. \tag{2.13}$$

In image processing terms, dilation of the image has the same effect as erosion of the background (with the reflected structuring element). But dilation and erosion are dual in yet another sense, namely

$$Y \oplus A \subseteq X \iff Y \subseteq X \ominus A, \tag{2.14}$$

for every pair of sets $X, Y \subseteq E$. This so-called *adjunction relation* forms the basis of the extension of morphological operators to complete lattices discussed in the following section. Since the left-hand-side is satisfied if we take $X = Y \oplus A$ we derive by substitution of this expression at the right-hand-side:

$$X \subseteq (X \oplus A) \ominus A =: X \bullet A. \tag{2.15}$$

In a similar way one can show that

$$X \supseteq (X \ominus A) \oplus A =: X \circ A. \tag{2.16}$$

We call $X \bullet A$ and $X \circ A$ respectively the *closing* and *opening* of X by A. From (2.15) and (2.16) we derive that

$$X \oplus A \subseteq ((X \oplus A) \ominus A) \oplus A = (X \oplus A) \circ A \subseteq X \oplus A,$$

yielding that

$$\big((X \oplus A) \ominus A\big) \oplus A = X \oplus A. \tag{2.17}$$

Similar arguments yield that

$$((X \ominus A) \oplus A) \ominus A = X \ominus A. \tag{2.18}$$

As a direct consequence we achieve the following identities:

$$(X \circ A) \circ A = X \circ A$$
$$(X \bullet A) \bullet A = X \bullet A,$$

saying that opening and closing are idempotent operators. We can derive the following alternative characterization of the opening and the closing:

$$X \circ A = \bigcup\{A_h \mid h \in E \text{ and } A_h \subseteq X\}, \tag{2.19}$$

that is, $X \circ A$ is built up of all translates of the structuring element A which are contained in X. For the closing $X \bullet A$ we derive

$$X \bullet A = \{k \in E \mid k \in \check{A}_h \implies \check{A}_h \cap X \neq \emptyset\}.$$

Figure 3 illustrates the opening and the closing.

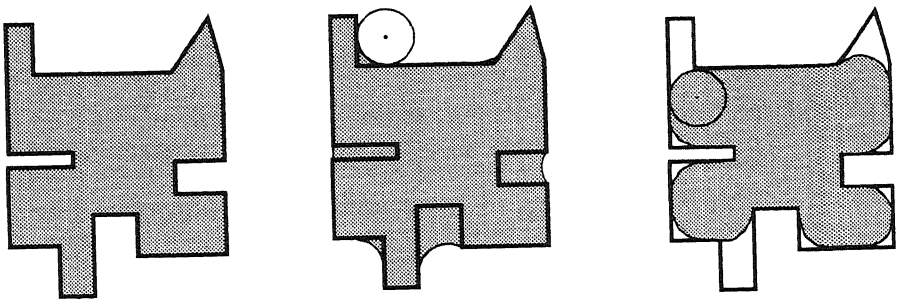

FIGURE 3. Closing and opening of a set $X \subseteq \mathbb{R}^2$ by a disk.

Opening and closing are complementary operators in the following sense:

$$(X \circ A)^c = X^c \bullet \check{A}. \tag{2.20}$$

We now give the following general definition.

DEFINITION 2.3. An operator $\alpha : \mathcal{P}(E) \to \mathcal{P}(E)$ is called an *opening* if α is increasing, $\alpha^2 = \alpha$ (idempotence) and $\alpha(X) \subseteq X$ for $X \subseteq E$ (anti-extensivity). An operator $\beta : \mathcal{P}(E) \to \mathcal{P}(E)$ is called a *closing* if β is increasing, $\beta^2 = \beta$ (idempotence) and $X \subseteq \beta(X)$ for $X \subseteq E$ (extensivity).

To distinguish the opening given in (2.16) from the general concept above we call it a *structural opening* to emphasize that it uses only one structuring element. There are many openings which are not of the structural type. In fact, if α_i is an opening for every i in some arbitrary index set I, then the operator given by $\alpha(X) = \bigcup_{i \in I} \alpha_i(X)$ is also an opening. Analogously, an arbitrary intersection of closings defines again a closing. In the literature one can find a number of alternative ways to construct openings. We refer in particular to [39,35,34]. In Section 5 we discuss an iterative method to construct openings and closings. The theorem below shows that the structural openings and closings form a basis for all openings and closings. We define the invariance domain $\mathsf{Inv}(\psi)$ of an operator ψ as the collection of all fixpoints of ψ, that is $\mathsf{Inv}(\psi) = \{X \subseteq E \mid \psi(X) = X\}$. It is easy to prove that the invariance domain of an opening is closed under unions. Moreover, if the opening is translation invariant then its invariance domain is closed under translations as well.

2.4. THEOREM.

(a) *Let α be a translation invariant opening on $\mathcal{P}(E)$. Then*

$$\alpha(X) = \bigcup_{A \in \mathsf{Inv}(\alpha)} X \circ A, \tag{2.21}$$

for every $X \subseteq E$.

(b) *Let β be a translation invariant closing on $\mathcal{P}(\mathbb{R})$. Then*

$$\beta(X) = \bigcap_{A \in \mathsf{Inv}(\beta)} X \bullet A, \tag{2.22}$$

for every $X \subseteq E$.

Openings and closings play a rather prominent role in mathematical morphology. We shall see one particular application in Section 5 where we discuss granulometries. A second application is the cleaning of images. If an image is corrupted by noise, one can try to recover it by applying openings and/or closings. This is the key idea underlying the so-called *alternating sequential filters.* We point out here that it is a good custom in morphology to preserve the name 'filter' for those operators which are increasing and idempotent. Consider a family of structuring elements A_n, $n \geq 1$ such that A_n is A_{n-1}-open, that is $A_n \circ A_{n-1} = A_n$. Let α_n resp. β_n be the opening resp. closing by A_n. Then

$$\alpha_n \alpha_{n-1} = \alpha_n \quad \text{and} \quad \beta_n \beta_{n-1} = \beta_n.$$

We define the alternating sequential filter of order n as

$$\nu_n = (\beta_n \alpha_n)(\beta_{n-1}\alpha_{n-1}) \cdots (\beta_1 \alpha_1).$$

It has been shown [40] that ν_n is idempotent, whence the name filter. Furthermore, the filters ν_n satisfy the semigroup property

$$\nu_n \nu_m \leq \nu_m = \nu_m \nu_n, \quad \text{if } m \geq n.$$

Here $\varphi \leq \psi$ means that $\varphi(X) \subseteq \psi(X)$ for $X \subseteq E$.

We conclude this section with a brief discussion of the relation between morphological operators and Boolean functions. A thorough exposition can be found in [15]; see also [3]. For a smooth introduction to Boolean functions we refer to [10]. We restrict ourselves to the case $E = \mathbb{Z}^d$. If one wants to perform a dilation with the 3×3-square

$$A = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix},$$

one has actually to repeat for every pixel h the following procedure: look at the values of $X(\cdot)$ (the characteristic function of X) at h and its eight neighbours: if at least one of these values is 1 then h lies in the dilated set. One can easily generalize this to arbitrary Boolean functions. Assume that A is a structuring element containing n points $a_1, a_2, \cdots, a_n$, and that b is a Boolean function of n variables. We define the operator ψ_b as

$$\psi_b(X) = \{h \in \mathbb{Z}^d \mid b(X(a_1 + h), X(a_2 + h), \cdots, X(a_n + h)) = 1\}. \quad (2.23)$$

Note that ψ_b also depends on A. We say that the translation invariant operator $\psi : \mathcal{P}(\mathbb{Z}^d) \to \mathcal{P}(\mathbb{Z}^d)$ is *finite* if there exists a finite window $A \subseteq \mathbb{Z}^d$ such that

$$h \in \psi(X) \iff h \in \psi(X \cap A'_h),$$

for every $h \in \mathbb{Z}^d$, $X \subseteq \mathbb{Z}^d$ and $A' \supseteq A$. It is not difficult to show that every finite translation invariant operator is of the form (2.23). The dilation can be represented through a Boolean function of the form $b(x_1, \ldots, x_n) = x_1 + \cdots + x_n$, where '+' denotes the logic 'OR'. If b is a positive (=increasing) Boolean function then ψ_b is increasing. If b is additive (as in the example above) resp. multiplicative then ψ_b is a dilation resp. an erosion.

As a first example consider the hit-or-miss operator $X \to X \circledast (B, C)$ where $B \cap C = \emptyset$. This operator can be represented in terms of a Boolean function as follows. Let $B = \{a_1, a_2, \cdots, a_m\}$ and $C = \{a_{m+1}, a_{m+2}, \cdots, a_n\}$, define $A = B \cup C$ and

$$b(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_m \cdot \overline{x_{m+1}} \cdot \overline{x_{m+2}} \cdot \cdots \overline{x_n}.$$

Here $\bar{x}$ means 'NOT x'. Using (2.23) we find that $h \in \psi_b(X)$ if and only if $a_1 + h, a_2 + h, \cdots, a_m + h \in X$ and $a_{m+1} + h, a_{m+2} + h, \cdots, a_n + h \notin X$, that is $B_h \subseteq X$ and $C_h \subseteq X^c$. This yields that $\psi_b(X) = X \circledast (B, C)$.

As a second example we discuss the *rank operators*, sometimes called *order statistics*. Let r_k, $k \leq n$ be the positive Boolean function of n variables which takes the value 1 if at least k variables are equal to 1 and 0 otherwise. It is clear that $r_1(x_1, \cdots, x_n) = x_1 + \cdots + x_n$ and $r_n(x_1, \cdots, x_n) = x_1 \cdot \cdots \cdot x_n$. If A is a structuring element with n points and if $k \leq n$ then we define the rank operator $\rho_{A,k}$ as the operator given by (2.23) with $b = r_k$. It is evident that $\rho_{A,1}(X) = X \oplus \check{A}$ and $\rho_{A,n}(X) = X \ominus A$. If n is odd and $k = (n+1)/2$ then $\rho_{A,k}$ is usually referred to as the *median operator*. It is easy to check that the median operator is *selfdual*. (An operator ψ is called selfdual if $\psi^* = \psi$.) In Figure 4 we have illustrated the rank operator for different values of k.

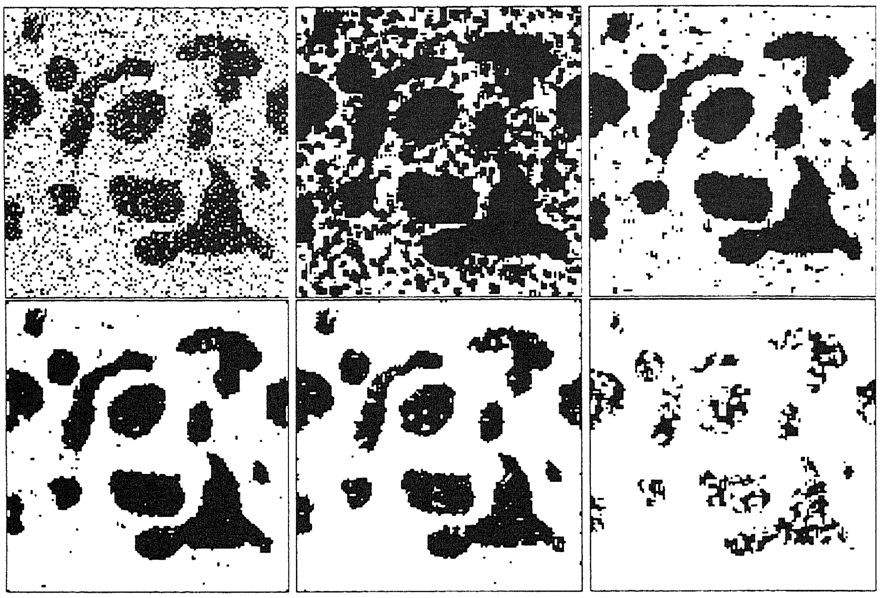

FIGURE 4. The rank operator applied to a noisy image X (black pixels) for different values of k. The structuring element is the 3×3-square consisting of 9 pixels. From left to right and top to bottom: the original image X and the transformed image $\rho_{A,k}(X)$ for respectively k =2,4,5 (median),6,8.

3. AN ALGEBRAIC APPROACH

From the previous section, mathematical morphology appears as a collection of image transforms based on set-theoretical operations such as union, intersection and complementation. Furthermore, these transforms are built in such a way that they are translation invariant. Although this paper deals exclusively with transforms mapping one image into another, the goal of mathematical morphology is much wider. Using techniques from integral geometry, stereology, and stochastic geometry the field is also concerned with image measurements, i.e., transforms mapping an image onto a number or a collection of numbers (e.g., the moments of a size distribution).

In the previous section we were only concerned with binary images, in which case the image space can be modeled appropriately by $\mathcal{P}(E)$, E being the Euclidean space $\mathbb{R}^d$ or the discrete space $\mathbb{Z}^d$. But depending on the kind of images one wants to consider, the sort of information one needs to extract, or the mathematical techniques one would like to use, other object spaces might be more appropriate. For instance, integral geometry relies heavily upon the theory of convex sets [8, 27], meaning that if one is about to use integral geometric tools, a restriction to convex sets may be necessary, or at least helpful. In stochastic geometry one has to supply the underlying space with a topology. Such is possible if one takes the space of closed sets or the space of compact sets as the image space [27]. Furthermore, it is important to extend mathematical morphology to grey-scale images which can be modeled mathematically as functions. Here several possibilities occur: one can choose continuous, discrete or finite grey-value sets, one can restrict to (semi-) continuous functions, convex functions, or functions with compact domains.

Besides this enormous variation in object spaces there is yet another generalization which is quite important. It is, namely, by no means obvious why morphological operators have to be translation invariant. In fact, one can think of a number of situations (e.g. radar imaging) where rotation invariance is to be considered more appropriate. Furthermore, one can easily think of situations where perspective transformations come in naturally; think, for instance, of the problem of monitoring the traffic on a highway with a camera at a fixed position. It is obvious that in such a configuration the detection algorithms should take the distance between the camera and the object (e.g. a car) into account.

These considerations are sufficient motivation to think about an abstraction of mathematical morphology which includes all mentioned object spaces and allows the generalization of translation invariance to other transformation groups. Such an abstraction was initiated by MATHERON and SERRA in [40]. They argued that the structure of a complete lattice is the appropriate framework for a general theory of mathematical morphology. Their work, however, did not include invariance under transformation groups. Such an inclusion was first made by HEIJMANS and RONSE in [17, 36] for Abelian transformation groups. Later this was generalized to non-Abelian transformation groups by ROERDINK [31, 32]. For a comprehensive discussion of the complete lattice framework of morphology we refer to our forthcoming book [15].

In this section we give a short overview of the main results in [17, 36] and illustrate them by a concrete example. In fact, Section 6, where we discuss morphology for grey-scale functions, comprises another application.

Although we expect that most of the readers will be familiar with the basic theory of complete lattices, we briefly recall some basic notions. A *complete lattice* is a set $\mathcal{L}$ with a partial order '$\leq$' such that every subset $\mathcal{H}$ of $\mathcal{L}$ has a least upper bound $\bigvee \mathcal{H}$, called the *supremum*, and a greatest lower bound $\bigwedge \mathcal{H}$ called the *infimum*. The least and greatest element of $\mathcal{L}$ are respectively denoted by O and I. We refer to [2] for a general account of the theory of complete lattices. From now on we denote by $\mathcal{L}$ an arbitrary complete lattice. A trivial but very useful fact is the observation that $\mathcal{L}$ endowed with the opposite ordering $X \leq' Y$ iff $Y \leq X$, is also a complete lattice, called the *opposite* or *dual* of $\mathcal{L}$, and denoted by $\mathcal{L}'$. To every definition, statement, property, etc. referring to $\mathcal{L}$ there corresponds a dual one referring to $\mathcal{L}'$, interchanging the role of $\leq$ and $\leq'$. This principle is known as the *duality principle*.

We mention some examples of complete lattices which are relevant in the context of mathematical morphology. We have already met $\mathcal{P}(E)$ in the previous section; this is a complete lattice if we take set inclusion as the partial ordering. If E is a topological space, then the closed sets $\mathcal{F}(E)$ ordered by inclusion form a complete lattice; in this case the supremum of a collection $X_i,\ i \in I$, of closed sets is given by $\overline{\bigcup_{i\in I} X_i}$, where the bar denotes closure. Analogously, the open sets form a complete lattice (this lattice is isomorphic to the opposite of $\mathcal{F}(E)$). If E is a real vector space, then $\mathcal{C}(E)$, the convex subsets of E ordered by inclusion define a complete lattice. The infimum is the ordinary set intersection and the supremum is given by

$$\bigvee_{i\in I} X_i = \mathsf{co}(\bigcup_{i\in I} X_i).$$

Here $\mathsf{co}(\cdot)$ denotes the convex hull. If E is a nonvoid space and $\mathcal{T}$ a complete lattice then we denote by $\mathsf{Fun}(E,\mathcal{T})$ the space of all functions mapping E into $\mathcal{T}$. With the pointwise ordering of $\mathcal{T}$ ($F \leq F'$ iff $F(x) \leq F'(x)$ for every $x \in E$) this becomes a complete lattice. This space will play an important role in Section 6, A choice for $\mathcal{T}$ which is particularly important is $\mathcal{T} = \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. As a special example of the function lattice we mention the space of Boolean functions of n variables.

The lattice $\mathcal{L}$ is called *distributive* if

$$\begin{aligned} X \wedge (Y \vee Z) &= (X \wedge Y) \vee (X \wedge Z) \\ X \vee (Y \wedge Z) &= (X \vee Y) \wedge (X \vee Z), \end{aligned}$$

for all $X, Y, Z \in \mathcal{L}$. $\mathcal{L}$ is called *modular* if it satisfies the identity

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge Z \quad \text{if } X \leq Z,$$

for all $X, Y, Z \in \mathcal{L}$. Any distributive lattice is modular but the converse is not true in general.

If $X, Y \in \mathcal{L}$ are such that $X \wedge Y = 0$ and $X \vee Y = I$ then Y is called the *complement* of X. We write $Y = X^c$. A *Boolean lattice* is a complete distributive lattice in which every element has a complement. An operator $\psi : \mathcal{L} \to \mathcal{L}$ is said to be *increasing* if it preserves the order structure, that is $X \leq Y$ implies that $\psi(X) \leq \psi(Y)$. It is called *decreasing* if it reverses the order structure. The increasing operators on $\mathcal{L}$ define a complete lattice under the ordering

$$\psi \leq \psi' \iff \psi(X) \leq \psi'(X) \quad \text{for all } X \in \mathcal{L}. \tag{3.1}$$

A mapping ψ on $\mathcal{L}$ is called a lattice automorphism if ψ is an increasing bijection. Then $\psi(\bigvee_{i\in I} X_i) = \bigvee_{i\in I} \psi(X_i)$ and $\psi(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \psi(X_i)$, for every collection X_i. A decreasing bijection is called a *dual automorphism.* A dual automorphism which satisfies

$$\psi^2 = \mathsf{id},$$

id denoting the identity operator, is called a *negation.* If ψ is a negation then $X^* = \psi(X)$ is called the *negative of* X; although such a nomenclature is in general ambiguous due to the fact that negations are not necessarily unique, it will not give rise to any confusion. Note that on a Boolean lattice the complement operator is a negation. On Fun $(E, \overline{\mathbb{R}})$ the mapping $F \to -F$ defines a negation.

If ψ is an operator mapping $\mathcal{L}$ into another complete lattice $\mathcal{M}$, and if both lattices have a negation, then we define the *complementary operator* $\psi^* : \mathcal{L} \longrightarrow \mathcal{M}$ as

$$\psi^*(X) = [\psi(X^*)]^*. \tag{3.2}$$

Note that ψ^* is increasing (decreasing) if and only if ψ is increasing (decreasing).

The basis for the definition of morphological operators on complete lattices is formed by the concept of adjunction.

DEFINITION 3.1. Let $\mathcal{L}, \mathcal{M}$ be complete lattices, let ε be an operator from $\mathcal{L}$ into $\mathcal{M}$ and δ an operator from $\mathcal{M}$ into $\mathcal{L}$. The pair (ε, δ) is called an *adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$ if

$$\delta(Y) \leq X \iff Y \leq \varepsilon(X), \tag{3.3}$$

for every $X \in \mathcal{L}$ and $Y \in \mathcal{M}$. Then δ is called a *dilation* and ε an *erosion.* We summarize some basic properties

PROPOSITION 3.2. *Let* (ε, δ) *be an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$*. Then*
(a) $\varepsilon(I) = I$ *and* $\delta(O) = O$.
(b) $\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i)$ *for every collection* $X_i \in \mathcal{L}$ $(i \in I)$.
(c) $\delta(\bigvee_{i\in I} Y_i) = \bigvee_{i\in I} \delta(Y_i)$ *for every collection* $Y_i \in \mathcal{M}$ $(i \in I)$.
(d) $\varepsilon\delta \geq \mathsf{id}_{\mathcal{M}}$.
(e) $\delta\varepsilon \leq \mathsf{id}_{\mathcal{L}}$.
(f) $\varepsilon\delta\varepsilon = \varepsilon$.
(g) $\delta\varepsilon\delta = \delta$.
(h) $\varepsilon(X) = \bigvee\{Y \in \mathcal{M} \mid \delta(Y) \leq X\}$.
(i) $\delta(Y) = \bigwedge\{X \in \mathcal{L} \mid Y \leq \varepsilon(X)\}$.

From (2.14) it follows that the pair $(\varepsilon_A, \delta_A)$ defines an adjunction on $\mathcal{P}(E)$.

PROPOSITION 3.3. *With every erosion* $\varepsilon : \mathcal{L} \longrightarrow \mathcal{M}$ *one can associate a unique* *dilation* $\delta : \mathcal{M} \to \mathcal{L}$ *such that* (ε, δ) *forms an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$. *Dually, with every dilation* $\delta : \mathcal{M} \to \mathcal{L}$ *one can associate a unique erosion* $\varepsilon : \mathcal{L} \to \mathcal{M}$ *such that* (ε, δ) *forms an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$.

PROPOSITION 3.4. *If* $(\varepsilon_i, \delta_i)$ *is an adjunction for every* $i \in I$, *then* $(\bigwedge_{i\in I} \varepsilon_i, \bigvee_{i\in I} \delta_i)$ *is an adjunction as well.*

PROPOSITION 3.5. *Let* $\mathcal{L}, \mathcal{M}$ *be complete lattices with a negation and let* (ε, δ) *be an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$. *Then* $(\delta^*, \varepsilon^*)$ *defines an adjunction between* $\mathcal{M}$ *and* $\mathcal{L}$.

Let T form an abelian group of automorphisms on $\mathcal{L}$. An operator ψ on $\mathcal{L}$ is called *T-invariant* if

$$\psi\tau = \tau\psi, \quad \tau \in T.$$

A T-invariant operator is called a T-operator. Similarly, a T-invariant dilation is called a T-dilation. If $\mathcal{L} = \mathcal{P}(\mathbb{R})$ and T is the group of translations, then 'T-invariant' means 'translation invariant'. If (ε, δ) is an adjunction on $\mathcal{L}$ and if ε or δ is T-invariant, then both operators are T-invariant. If $\tau \in T$ then (τ^{-1}, τ) is a T-adjunction. Together with Proposition 3.4 this yields that $(\bigwedge_{i\in I} \tau_i^{-1}, \bigvee_{i\in I} \tau_i)$ is a T-adjunction if $\tau_i \in T$ for $i \in I$. Below we will give assumptions under which every T-adjunction is of this form. It is not difficult to show that in the translation invariant case discussed in the previous section this is true indeed. Essentially, two properties of the space $\mathcal{P}(E)$ are used here: (1) every set is a union of points, and (2) the translation group acts simply transitive on the points. The latter means that for every two points there is a unique translation mapping the first point onto the second. The Basic Assumption below essentially generalizes these two properties.

A subset ℓ of $\mathcal{L}$ is called a *sup-generating family* if every element of $\mathcal{L}$ can be obtained as a supremum of elements of ℓ. As a special example we mention the set of atoms in an atomic lattice: see [2]. Note that $\mathcal{P}(E)$ is atomic, the atoms being the singletons.

3.6. BASIC ASSUMPTION

$\mathcal{L}$ is a complete lattice which possesses a sup-generating family ℓ and T is an abelian automorphism group on $\mathcal{L}$ such that (i) ℓ is invariant under T, (ii) T is simply transitive on ℓ.

From now on we assume that the Basic Assumption holds. We indicate how one can define Minkowski addition and subtraction on $\mathcal{L}$. First we fix an origin $o \in \mathcal{L}$. For every $h \in \ell$ there is a unique $\tau_h \in T$ which transforms o to h. We can now define a Minkowski addition and subtraction as follows:

$$\delta_A(X) := X \oplus A = \bigvee_{a\in\ell(A)} \tau_a(X)$$
$$\varepsilon_A(X) := X \ominus A = \bigwedge_{a\in\ell(A)} \tau_a^{-1}(X),$$

for $X, A \in \mathcal{L}$. Here $\ell(A) = \{a \in \ell \mid a \leq A\}$. Note that $\delta_A = \bigvee_{a \in \ell(A)} \tau_a$ and $\varepsilon_A = \bigwedge_{a \in \ell(A)} \tau_a^{-1}$.

3.7. PROPOSITION *For every $A \in \mathcal{L}$, the pair $(\varepsilon_A, \delta_A)$ forms a T-adjunction on $\mathcal{L}$. Moreover, any T-adjunction has this form.*

Now one can proceed by establishing properties similar to those stated in the previous section, including the representation theorem 2.2. We will not do so, however; the interested reader can find the details in [17] or [15].

Taking Definition 2.3 as a starting point, the concept of the opening and closing can easily be extended to the given abstract framework. As in the translation invariant case the concept of a structural opening plays an important role. A comprehensive discussion can be found in [36] and [15].

We conclude this section, mainly intended as a trend-setter, with the following example. Take $E = \mathbb{R}^2 \setminus \{0\}$ and $\mathcal{L} = \mathcal{P}(E)$. Furthermore, let T be the abelian group of rotations and scalar multiplications. It is obvious that the Basic Assumption is satisfied, so we can define Minkowski addition and subtraction in this case. Note that here the size of a translate of the structuring element depends on its distance from the origin; see Figure 5.

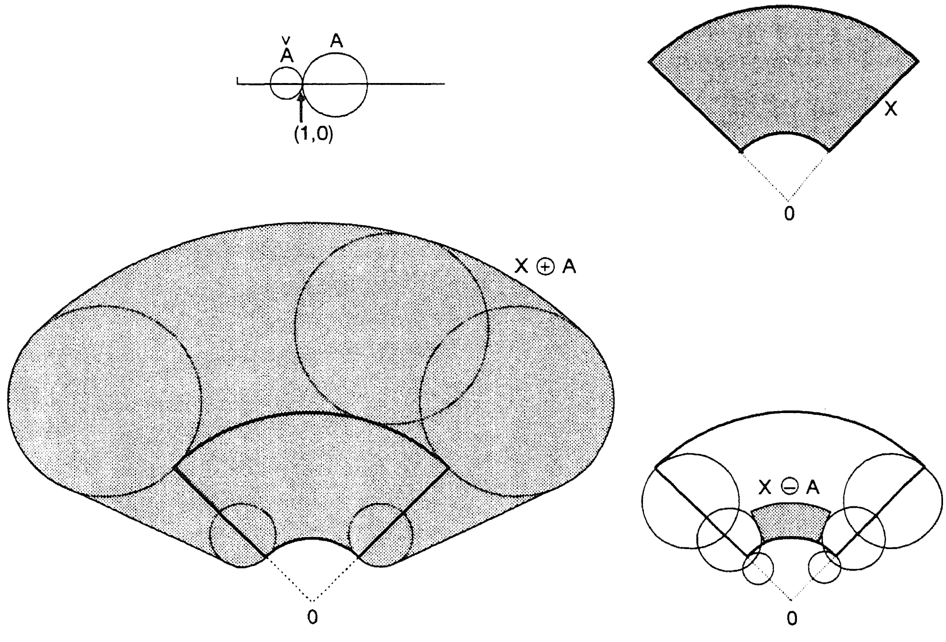

FIGURE 5. The effect of a dilation and erosion which are invariant under rotation and scalar multiplication. To compute the erosion we have used that $X \ominus A = (X^c \oplus \check{A})^c$.

If the assumption that T is abelian is dropped then we have to distinguish between two different cases, (i) the simply transitive case, and (ii) the multi-transitive case. Needless to say that this situation is considerably more complex than the Abelian case. We refer to [31, 32] and [15] for a detailed discussion. As an important application of the nonabelian, multiply transitive case we mention the rotation-translation group on $\mathcal{P}(\mathbb{R}^2)$.

4. Morphological filtering

In Section 2 we have already indicated the importance of openings and closings (and alternating sequential filters) with respect to image filtering, i.e., the pre-processing of an image with the intention to remove noise, to sharpen the edges, to enhance the contrast, etc. Suppose we apply an operator ψ to an image X in order to make it more suitable to analysis. If ψ is not idempotent then $\psi^2(X) \neq \psi(X)$ in general. If $\psi(X)$ contains less noise than X, then its seems reasonable to assume that $\psi^2(X)$ contains less noise than $\psi(X)$. This suggests that we should apply ψ until the result remains unchanged. Denoting by $\tilde{\psi}(X)$ the final result of such an iteration procedure (presuming that it converges eventually) it is clear that $\psi\tilde{\psi} = \tilde{\psi}$, and hence that $\tilde{\psi}$ is idempotent. Obviously, idempotence is a desirable property in any image filtering procedure. As we have seen, openings, closings, and alternating sequential filters satisfy this requirement, which makes them suited for the removal of noise.

The iteration procedure above raises a number of questions.

Question 1. What kind of convergence has to be considered?

Question 2. For which operators ψ does iteration yield an idempotent operator?

Before we deal with these problems we give some basic concepts from the theory of morphological filtering. Again we do not pursue completeness, but rather do we intend to give the reader a flavour of the mathematical concepts and tools which play a role. Throughout this section we assume that our image space $\mathcal{L}$ is an arbitrary complete lattice. Unlike in the previous section we do not assume our operators to be invariant under some transformation group.

The theory of morphological filtering was initiated by Matheron. His main ideas are contained in Chapter 6 of [40]; see also other chapters in this book. Most of the concepts introduced below can be found in this reference. See also [12, 18, 23, 36]. In Section 2 we have seen that an arbitrary union of openings is again an opening. This is not true for intersections; furthermore, the composition of two openings is not an opening in general. This shows in particular that the property of idempotence is not preserved under suprema, infima and compositions: if φ, ψ are increasing idempotent operators, then neither of the operators $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi\psi$ is idempotent in general. Though this is rather unfortunate, it does not mean that nothing can be said. In fact, supremum preserves one half of the idempotence property. To be specific, $(\varphi \vee \psi)^2 \geq \varphi^2 \vee \psi^2 = \varphi \vee \psi$ and we say that $\varphi \vee \psi$ is an overfilter.

DEFINITION 4.1. Let ψ be an increasing operator on $\mathcal{L}$. We say that ψ is
(a) a *(morphological) filter* if $\psi^2 = \psi$
(b) an *underfilter* if $\psi^2 \leq \psi$
(c) an *overfilter* if $\psi^2 \geq \psi$
(d) an *inf-overfilter* if $\psi = \psi(\mathsf{id} \wedge \psi)$
(e) a *sup-underfilter* if $\psi = \psi(\mathsf{id} \vee \psi)$
(f) a *strong filter* if ψ is both an inf-overfilter and a sup-underfilter
(g) an *opening* if ψ is increasing, idempotent and anti-extensive ($\psi \leq \mathsf{id}$)
(h) a *closing* if ψ is increasing, idempotent and extensive ($\psi \geq \mathsf{id}$).

The latter two definitions generalize Definition 2.3 to the complete lattice framework. From Proposition 3.2 (d)-(g) it follows immediately that $\delta\varepsilon$ is an opening on $\mathcal{L}$ and that $\varepsilon\delta$ is a closing on $\mathcal{M}$ if (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$.

It is clear that underfilters and overfilters are dual concepts in the sense of the duality principle. The same remark applies to inf-overfilters and sup-underfilters. For that reason we will restrict to (inf-) overfilters. We start with some basic properties.

- every inf-overfilter is an overfilter;
- every extensive operator is an inf-overfilter;
- ψ is a filter iff it is both an underfilter and an overfilter;
- both the set of overfilters and inf-overfilters is closed under suprema;
- the set of (inf-) overfilters is closed under self-composition (i.e., if ψ is an (inf-) overfilter then ψ^n is such as well for every integer $n \geq 1$);
- openings and closings are strong filters;
- if ψ is an inf-overfilter then $\mathsf{id} \wedge \psi$ is an opening.

This last property is very important because it provides a powerful technique for the construction of openings: see [34, 36] for more details.

We define $\check{\psi}$ as the largest opening $\leq \psi$, that is $\check{\psi} = \bigvee\{\alpha \mid \alpha \text{ is an opening and } \alpha \leq \psi\}$. Since every supremum of openings is again an opening this definition makes sense. It can be shown that

$$\mathsf{Inv}(\check{\psi}) = \mathsf{Inv}(\mathsf{id} \wedge \psi) = \{X \in \mathcal{L} \mid \psi(X) \geq X\},$$

and

$$\psi\check{\psi} = \check{\psi}.$$

Let $\mathcal{C}$ be the smallest collection of increasing operators on $\mathcal{L}$ which contains $\mathsf{id} \wedge \psi$ and which is closed under compositions and infima. It is easy to see that $\check{\psi} \leq \varphi \leq \mathsf{id} \wedge \psi$ for every $\varphi \in \mathcal{C}$. Let $\alpha := \inf \mathcal{C}$, then $\alpha \in \mathcal{C}$ because $\mathcal{C}$ is closed under infima. In particular $\alpha^2 \leq \alpha$ since $\alpha \leq \mathsf{id}$. On the other hand, $\alpha^2 \in \mathcal{C}$ and by the very definition of α this yields that $\alpha^2 \geq \alpha$. Therefore $\alpha^2 = \alpha$ and we conclude that α is an opening $\leq \psi$. Since $\check{\psi}$ is the largest such opening and $\alpha \geq \check{\psi}$ we may conclude that $\alpha = \check{\psi} = \inf \mathcal{C}$.

For the mathematical connoisseurs we point out that the mapping $\psi \to \check{\psi}$ defines an opening on the complete lattice of increasing operators on $\mathcal{L}$. Dually we define $\hat{\psi}$ as the smallest closing $\geq \psi$. The proof of the next result can be found in [40, Chapter 6].

PROPOSITION 4.2. *Let ψ be an increasing operator on $\mathcal{L}$.*
(a) $\check{\psi}\psi$ is an overfilter. It is the largest overfilter $\leq \psi$.
(b) If ψ is an underfilter then $\check{\psi}\psi$ is a filter. It is the largest filter $\leq \psi$.
(c) $\psi\check{\psi}$ is an inf-overfilter. It is the largest inf-overfilter $\leq \psi$.
(d) Assume that $\mathcal{L}$ is modular. If ψ is a sup-underfilter then $\psi\check{\psi}$ is a strong filter.

Note that (a) can be restated as follows: the mapping $\psi \to \check{\psi}\psi$ on the complete lattice of increasing operators on $\mathcal{L}$ is an opening with invariance domain the set of overfilters. We mention the following important consequence of this result.

COROLLARY 4.3. *The filters on $\mathcal{L}$ define a complete lattice.*

Here we have provided the set of filters with the order given in (3.1). To prove this result, take an arbitrary collection ψ_i of filters and define the underfilter λ and the overfilter μ resp. by

$$\lambda := \bigwedge_{i\in I} \psi_i \quad \text{and} \quad \mu := \bigvee_{i\in I} \psi_i.$$

Then, by Proposition 4.2(b), $\check{\lambda}\lambda$ is a filter, the largest filter $\leq \psi_i$, $i \in I$, and therefore the infimum of the ψ_i in the lattice of filters. Similarly, $\hat{\mu}\mu$ is the supremum of the ψ_i in this lattice. We can use similar arguments to show that the set of strong filters, the set of openings, and the set of closings form complete lattices.

If $\mathcal{L}$ has a negation, the mapping $\psi \to \psi^*$ given by (3.2) transforms an (inf-) overfilter into a (sup-) underfilter, an opening into a closing and a (strong) filter into a (strong) filter. A filter ψ is called *selfdual* if $\psi^* = \psi$. In image processing selfdual filters are rather popular since they treat fore- and background of an image in the same way. The example at the end of this section deals with the construction of a selfdual filter.

The above considerations make clear how important it is to develop tools for the computation of the so-called *lower* and *upper envelope* $\check{\psi}$ and $\hat{\psi}$. Below we explain that under certain assumptions on ψ, the lower envelope can be computed through iteration of the anti-extensive operator $\mathsf{id} \wedge \psi$. Dually, iteration of $\mathsf{id} \vee \psi$ yields the upper envelope $\hat{\psi}$.

Assume that ψ is anti-extensive (so that $\mathsf{id} \wedge \psi = \psi$). In that case

$$\cdots \leq \psi^{n+1} \leq \psi^{n} \leq \psi^{n-1} \leq \cdots \leq \psi \leq \mathsf{id}.$$

We define $\psi^{\infty} := \bigwedge_{n\geq 1} \psi^n$. If ψ^{∞} is idempotent, or equivalently if $\psi\psi^{\infty} = \psi^{\infty}$, then ψ^{∞} is an opening and we have $\check{\psi} = \psi^{\infty}$. Unfortunately, ψ^{∞} need not be idempotent. In [11] we have given the following counterexample. Let $\psi : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ be the increasing, anti-extensive translation invariant operator given by $\psi(X) = (X \oplus A) \cap X$, where $A = \{\ldots, -5, -3, -1, 2\}$. Take $X = \{0, 1, 3, 5, \ldots\}$, then, by a straightforward calculation, $\psi^n(X) = \{0, 2n+1, 2n+3, 2n+5, \ldots\}$ (see Figure 6), and so $\psi^{\infty}(X) = \{0\}$. However, $\psi\psi^{\infty}(X) = \psi(\{0\}) = \emptyset \neq \psi^{\infty}(X)$.

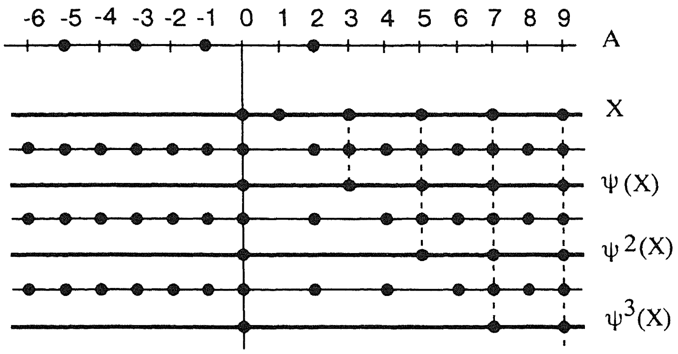

FIGURE 6. An example of an anti-extensive operator ψ for which $\psi^{\infty} \neq \psi^{\infty+1}$.

Let X_n be a sequence in $\mathcal{L}$, we say that $X_n \downarrow X$ if X_n is decreasing and $X = \bigwedge_{n\geq 1} X_n$. Similarly, we define $X_n \uparrow X$. An increasing operator ψ on $\mathcal{L}$ is called $\downarrow$-continuous if $X_n \downarrow X$ implies that $\psi(X_n) \downarrow \psi(X)$. If the anti-extensive operator ψ is $\downarrow$-continuous, then

$$\psi\psi^{\infty}(X) = \psi(\textstyle\bigwedge_{n\geq 1} \psi^n(X)) = \bigwedge_{n\geq 1} \psi^{n+1}(X) = \psi^{\infty}(X),$$

yielding that $\check{\psi} = \psi^{\infty}$; we refer to [36] for a number of examples.

We show how we can extend such arguments for operators which are not necessarily increasing. See [18] for a comprehensive theory. For a sequence X_n in $\mathcal{L}$ we define

$$\liminf X_n = \bigvee_{N\geq 1} \bigwedge_{n\geq N} X_n \qquad \limsup X_n = \bigwedge_{N\geq 1} \bigvee_{n\geq N} X_n.$$

It is obvious that $\liminf X_n \leq \limsup X_n$. We say that $X_n \to X$ if $\liminf X_n = \limsup X_n = X$. If $X_n \downarrow X$ or $X_n \uparrow X$ then $X_n \to X$. An operator ψ on $\mathcal{L}$ is said to be $\downarrow$-*continuous* if $X_n \to X$ implies that $\limsup \psi(X_n) \leq \psi(X)$, and

↑-continuous if $X_n \to X$ implies that $\liminf \psi(X_n) \geq \psi(X)$. The operator ψ is called *continuous* if it is both ↓-continuous and ↑-continuous. It is not difficult to show that for increasing operators this definition is consistent with the definition given above.

In [18] and [15] one can find a general account of ↓- and ↑-continuous operators. We quote some of the main results.

- Every erosion is ↓-continuous and every dilation is ↑-continuous.
- Every automorphism is continuous.
- Let $\mathcal{L}$ have a negation, then ψ is ↓-continuous iff ψ^* is ↑-continuous.
- The infimum of an arbitrary collection of ↓-continuous operators is ↓-continuous.
- If $\mathcal{L}$ is atomic then any finite supremum of ↓-continuous operators is ↓-continuous. (It is worthwile to remark that in [18] a more general result is stated which also includes the function lattice $\mathsf{Fun}(E), \overline{\mathbb{R}}$.)

REMARK 4.4. The complete lattice $\mathcal{F}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$ can be endowed with a topology which is based on the hit-or-miss operator of Section 2. This topology is called the hit-or-miss topology and has been investigated in great detail by Matheron [27]. In [18] we have pointed out the relation between the lattice convergence defined here and convergence in the sense of the hit-or-miss topology.

In Section 2 we have introduced the notion of a finite operator in combination with translation invariance. We can easily extend this definition for operators which are not translation invariant.

DEFINITION 4.5. Let E be an arbitrary set. The operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is *finite* if for every $h \in E$ there exists a finite window $A(h) \subseteq E$ such that

$$h \in \psi(X) \iff h \in \psi(X \cap A')$$

for $X \subseteq E$ and $A' \supseteq A(h)$.

For translation invariant operators one can take $A(h) = A_h$.

PROPOSITION 4.6. *Every finite operator on $\mathcal{P}(E)$ is continuous.*

One can show that finite unions, intersections and compositions of finite operators are finite. Furthermore, ψ is finite if and only if ψ^* is finite.

For an arbitrary operator ψ on $\mathcal{L}$ we define

$$\psi^\infty(X) = \limsup \psi^n(X)$$
$$\psi_\infty(X) = \liminf \psi^n(X).$$

If $\psi_\infty = \psi^\infty$ then we write $\psi^n \to \psi^\infty$.

If ψ is anti-extensive, or more generally if $\psi^2 \leq \psi$, then $\psi^{n+1} \leq \psi^n$ and in that case we have $\psi^\infty = \psi_\infty = \bigwedge_{n\geq 1} \psi^n$. In [18] we have established the following results.

THEOREM 4.7. *Let ψ be an arbitrary operator on $\mathcal{L}$.*
(a) *If ψ is anti-extensive and $\downarrow$-continuous then $\psi^\infty = \bigwedge_{n\geq 1} \psi^n$ is idempotent.*
(b) *If ψ is extensive and $\uparrow$-continuous then $\psi^\infty = \bigvee_{n\geq 1} \psi^n$ is idempotent.*
(c) *If $\psi^n \to \psi^\infty$ and ψ is continuous then ψ^∞ is idempotent.*

COROLLARY 4.8. *Let ψ be an increasing operator on $\mathcal{L}$.*
(a) *If ψ is $\downarrow$-continuous then $\check{\psi} = \bigwedge_{n\geq 1}(\mathsf{id} \wedge \psi)^n$.*
(b) *If ψ is $\uparrow$-continuous then $\hat{\psi} = \bigvee_{n\geq 1}(\mathsf{id} \vee \psi)^n$.*

In [36] we have discussed some examples of openings generated by iteration. Here we discuss the construction of a morphological filter which utilizes Theorem 4.7(c). For E we take the discrete hexagonal grid. Let H be the discrete hexagon with radius one centered at the origin. Then H contains seven points, the origin a_0 and its six neighbours, denoted by $a_1, \cdots, a_6$; see Figure 7(a). Let b be the Boolean threshold function (see [29] and [3])

$$b(x_0, x_1, \cdots, x_6) = \begin{cases} 1, & if 3x_0 + x_1 + x_2 + \cdots + x_6 \geq 5 \\ 0, & otherwise. \end{cases}$$

Let $\psi = \psi_b$ be given by (2.23):

$$\psi(X) = \{h \in E \mid 3X(h) + X(a_1 + h) + \cdots + X(a_6 + h) \geq 5\}.$$

A moment of reflection shows that this operator acts as follows: a point h in the set X lies in the transformed set $\psi(X)$ if and only if at least two of its six hexagonal neighbours also lie inside X. If, say, only one neighbour lies in X then $3X(h) + X(a_1 + h) + \cdots + X(a_6 + h) = 4$ and the threshold 5 is not reached. If h lies in X^c it will be contained in $\psi(X)$ if at least five of its hexagonal neighbours lie in X. In Figure 7(b) ψ_b changes the value of the central point whereas in (c) this value remains unchanged.

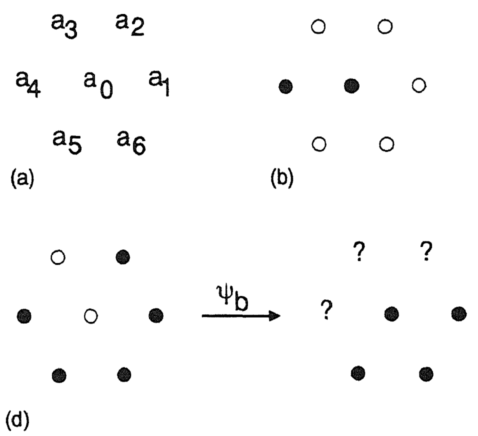

FIGURE 7. See text

In this procedure the background is treated analogously: a point in the background X^c with at least two neighbours in X^c stays in the background after transformation. In other words, the operator ψ is selfdual. In fact, the selfduality of ψ is a consequence of the selfduality of the generating Boolean function b. Since b is positive the operator ψ is increasing. Furthermore, it is evident that ψ is finite and from Proposition 4.6 we conclude that it is continuous. We show that the sequence ψ^n is convergent. This is an immediate consequence of the following nice property of ψ: for every X the sequence $X, \psi(X), \psi^2(X), \cdots$ is *pointwise monotone.* This means that for every point h the value $[\psi^n(X)](h)$ (which is 0 or 1) can change at most at one instance. So either it is of the form $0, 0, \cdots, 0, 1, 1, 1 \cdots$ or of the opposite form. To see why this property holds consider the configuration in Figure 7(d). Here the central point changes its value from 0 to 1; however, it's neighbours a_5, a_6, a_1 remain unchanged at this step. But the configuration formed by the points a_0, a_5, a_6, a_1 is stable under ψ, meaning that it's values remain constant at subsequent iteration steps. We may now conclude that $\psi^n \to \psi^\infty$, where $\psi^\infty(X)$ is the pointwise limit under the iteration procedure described above. From Theorem 4.7(c) we conclude that ψ^∞ is a filter. Since ψ is selfdual, the operator ψ^∞ is selfdual as well.

5. GEOMETRICAL ASPECTS

The theory discussed so far may have given the reader the impression that mathematical morphology is a highly algebraic theory. This impression is correct to a limited extent only. As we pointed out in the introduction, the primary goal of morphology is to extract geometric information from an image. For that matter topological and geometrical concepts are of paramount importance to the field. In this section we illustrate this fact by means of two examples. First we introduce the geodesic operators; these form a class of transformations which has proved very useful for many different applications, including image segmentation. And secondly, we discuss granulometries which form the basis for the computation of size distributions. For both examples the notion of a metric plays an important role.

Throughout this section we assume that $E = \mathbb{R}^d$.

DEFINITION 5.1. A *metric* on E is a function $d : E \times E \to \mathbb{R}_+$ such that for $x, y, z \in E$,
(D1) $d(x,y) = 0 \iff x = y$
(D2) $d(x,y) = d(y,x)$
(D3) $d(x,z) \le d(x,y) + d(y,z)$.
The latter property is called the *triangle inequality*. d is called a *translation invariant metric* or *T-metric* if it satisfies the additional property
(D4) $d(x+h, y+h) = d(x,y)$
for $x, y, h \in E$. If, moreover,
(D5) $d(\lambda x, \lambda y) = |\lambda| d(x,y)$,
for $\lambda \in \mathbb{R}$, then d is called a *linear metric*.

Let $B(\lambda)$ be the ball centered at 0 with radius $\lambda \ge 0$,

$$B(\lambda) = \{x \in E \mid d(x,0) \le \lambda\}.$$

A set $X \subseteq E$ is called *symmetric* if $X = \check{X}$, where $\check{X}$ has been defined as $\{-x \mid x \in X\}$.

PROPOSITION 5.2. *Let d be a T-metric, then the balls $B(\lambda)$ have the following properties*
(B1) $B(0) = \{0\}$
(B2) $B(\lambda)$ *is symmetric and* $0 \in B(\lambda)$
(B3) $B(\lambda) \oplus B(\mu) \subseteq B(\lambda + \mu)$
(B4) $B(\lambda) = \bigcap_{\lambda' > \lambda} B(\lambda')$.

PROOF. (B1) and (B4) are obvious. (B2) follows readily from the observation that

$$d(x,0) = d(x-x, 0-x) = d(0,-x) = d(-x,0).$$

To prove (B3) take $x \in B(\lambda)$ and $y \in B(\mu)$. We show that $x + y \in B(\lambda + \mu)$. Since $d(0,x) \le \lambda$ and $d(0,y) = d(0,-y) \le \mu$ we get that $d(x,-y) \le \lambda + \mu$. Now our claim follows from the observation that

$$d(x,-y) = d(x+y,0) = d(0,x+y). \qquad \square$$

On the other hand, the existence of a family of sets $B(\lambda)$ which satisfy the axioms (B1)-(B4) implies the existence of a T-metric d, namely

$$d(x,y) = \inf\{\lambda \geq 0 \mid x - y \in B(\lambda)\}.$$

If, moreover, d is a linear metric, then the function $p : E \to \mathbb{R}_+$ given by

$$p(x) = d(x,0) \tag{5.1}$$

defines a norm, and the unit ball with respect to this norm

$$B = \{x \in E \mid p(x) \leq 1\}$$

is convex. Furthermore, this set is compact with respect to the topology generated by this norm. Conversely, if B is compact, convex and symmetric, and if 0 is contained in the interior of B, then

$$p_B(x) = \inf\{\lambda > 0 \mid x \in \lambda B\} \tag{5.2}$$

defines a norm on E. The function p_B is called the *Minkowski functional* or *gauge function*; see [42]. In this case the balls of radius λ are given by $B(\lambda) = \lambda B$. It is easy to check that

$$\lambda B \oplus \mu B = (\lambda + \mu)B, \quad \lambda, \mu \geq 0, \tag{5.3}$$

if B is convex. In fact, the inclusion '$\supseteq$' holds for any set B. One may wonder which sets B "solve" equation (5.3). Is convexity required? Obviously, the set $B \subseteq \mathbb{R}^2$ consisting of the closed first and third quarter-plane satisfies the equation but is definitely not convex. A very general answer to the question has been given by MATHERON [27]. Let $\mathcal{K}'$ be the space of nonvoid compact subsets of $\mathbb{R}^d$ provided with the topology generated by the Hausdorff metric [8, 27].

THEOREM 5.3. *Let $B(\cdot) : \mathbb{R}_+ \to \mathcal{K}'$ be a continuous mapping. Then*

$$B(\lambda) \oplus B(\mu) = B(\lambda + \mu), \quad \lambda, \mu \geq 0, \tag{5.4}$$

if and only if $B(\lambda) = \lambda B$ for some compact, convex set B.

Let $A, X \subseteq E$. We say that X is *A-open* if $X = X \circ A$, where the opening '$\circ$' has been defined in (2.16) It is obvious that $B(\mu)$ is $B(\lambda)$-open for $\mu \geq \lambda$ if the semigroup property (5.4) holds. Furthermore, MATHERON [27] has proved the following result.

THEOREM 5.4. *Let $B \subseteq E$ be compact. Then μB is λB-open for $\mu \geq \lambda$ if and only if B is convex.*

As a corollary we get that μB is λB-open ($\mu \geq \lambda$) if and only if the semigroup property (5.3) holds (given that B is compact). This result cannot be extended to families $B(\lambda)$ which are not obtained by scaling. In fact, introducing the notion of Stieltjes-Minkowski integral in the space $\mathcal{K}'$, Matheron constructed a class of mappings $B : \mathbb{R}_+ \to \mathcal{K}'$ such that $B(\mu)$ is $B(\lambda)$-open for $\mu \geq \lambda$. As a special member of this class we mention

$$B(\lambda) = \begin{cases} B_1, & 0 \leq \lambda \leq \lambda_1, \\ B_1 \oplus B_2, & \lambda > \lambda_1. \end{cases}$$

Note that this example does not satisfy the semigroup property (5.4).

The reason for this discussion on metrics and its relation to convexity becomes clear below where we treat granulometries. But first we introduce a class of metrics which is neither translation invariant nor homogeneous, but which is of great practical value. This class comprises the so-called *geodesic metrics.* Let $M \subseteq \mathbb{R}^d$ be a fixed set called the *mask set* or *mask image.* Let x, y be two points inside M. If x, y lie in the same (arc-connected) component of M then there exist paths inside M connecting x and y; see Figure 8 for an illustration.

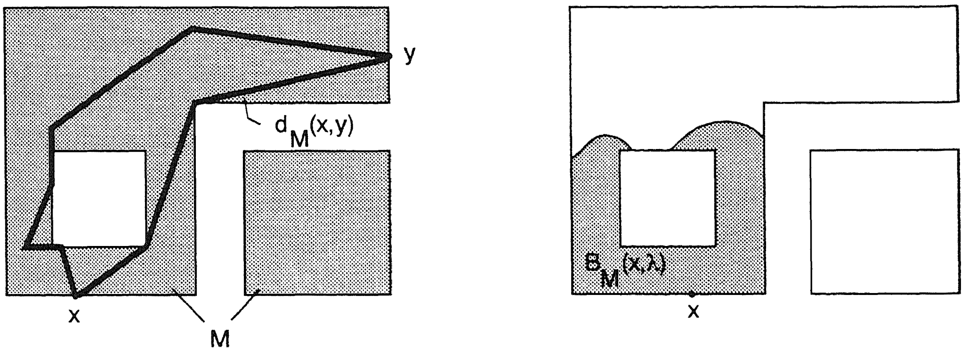

FIGURE 8. Left: a mask image M and two geodesic paths from x to y. The path at the right is the geodesic arc and its length is $d_M(x, y)$.
Right: the geodesic ball $B_M(x, \lambda)$ centered at x with radius λ. Note that $B_M(x, \lambda)$ has an empty intersection with the bottom-right component of M, no matter the magnitude of λ.

The shortest of these paths is called a *geodesic path* and its length is denoted by $d_M(x, y)$. We call this quantity the *geodesic distance between x and y.* If x and y lie in different components of M then we put $d_M(x, y) = \infty$. Since d_M may take the value ∞ it is not a metric in the classical sense of the word. However, it is not difficult to show that d_M satisfies the axioms (D1)–(D3). Geodesics have been studied in great detail by differential geometers. A nice treatment can be found in [4] The metric d_M is also called *intrinsic metric* in the literature [29].

Using the geodesic distance we can build a class of transformations to which we refer as *geodesic operators*. Let $B_M(x,\lambda)$ be the *geodesic ball* with centre x and radius λ,

$$B_M(x,\lambda) = \{y \in M \mid d_M(x,y) \leq \lambda\}; \tag{5.5}$$

see Figure 8.

On $\mathcal{P}(M)$ we define the geodesic dilations δ_M^λ and erosions ε_M^λ respectively as

$$\delta_M^\lambda(X) = \bigcup_{x \in X} B_M(x,\lambda), \tag{5.6}$$

$$\varepsilon_M^\lambda(X) = \{x \in X \mid B_M(x,\lambda) \subseteq X\}. \tag{5.7}$$

It is easy to check that the pair $(\varepsilon_M^\lambda, \delta_M^\lambda)$ forms an adjunction on $\mathcal{P}(M)$. Furthermore, these operators satisfy the semigroup relations

$$\delta_M^\lambda \circ \delta_M^\mu = \delta_M^{\lambda+\mu}, \quad \varepsilon_M^\lambda \circ \varepsilon_M^\mu = \varepsilon_M^{\lambda+\mu}. \tag{5.8}$$

As an application of the geodesic approach we mention the skeleton by influence zones, usually referred to as the SKIZ. Suppose that the set $X \subseteq M$ is made of n objects, $X = X_1 \cup X_2 \cup \cdots \cup X_n$, which have empty intersection. A point $h \in M$ is said to belong to the influence zone of the object X_k if $d_M(h, X_k) < d_M(h, X_l)$ for $l \neq k$. Here $d_M(h, X_k)$ is defined as the length of the shortest geodesic path from h to some point of X_k. The *SKIZ* is defined as the collection of boundaries which separate the influence zones. It is often used for segmentation problems. We refer to [22] and [39] for more information.

Geodesic reconstruction

A geodesic operator which turns out to be of great value in practice is the *geodesic reconstruction*, defined as

$$\rho_M(X) = \bigcup_{\lambda > 0} \delta_M^\lambda(X). \tag{5.9}$$

Since a union of dilations is again a dilation (Proposition 3.4) we get that ρ_M is a dilation. It extracts the connected components of M which have a non-empty intersection with X; see Figure 9.

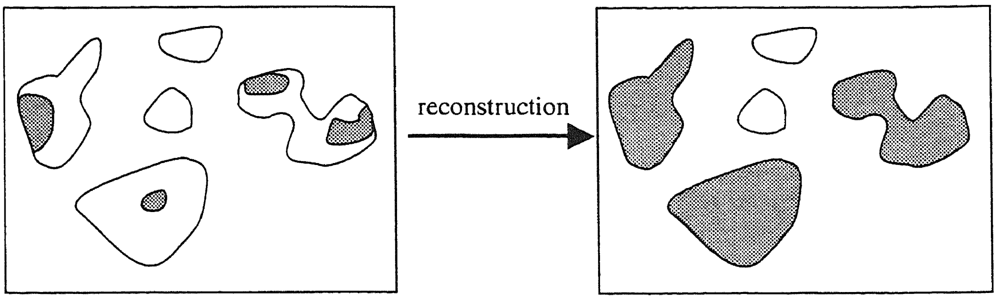

FIGURE 9. Geodesic reconstruction.

Here we mention one application of the reconstruction ρ_M. Suppose we apply some opening ψ to an image with the intention to remove small noise particles. This opening may also affect larger particles, in particular their contours. We can neutralize this effect by taking the reconstruction of $\psi(X)$ inside the mask set X. That is, we define

$$\psi'(X) = \rho_X(\psi(X)). \tag{5.10}$$

One can easily show that ψ' is again an opening. Furthermore, this procedure preserves symmetry properties of ψ. For instance, if ψ is translation invariant, then ψ' is such as well.

Granulometries

A *granulometry* on $\mathcal{P}(E)$ is a one-parameter family $\{\psi_\lambda \mid \lambda > 0\}$ of openings on $\mathcal{P}(E)$ such that

$$\psi_\lambda\psi_\mu = \psi_\mu\psi_\lambda = \psi_\mu, \quad \mu \geq \lambda, \tag{5.11}$$

or equivalently,

$$\psi_\mu \leq \psi_\lambda, \quad \mu \geq \lambda. \tag{5.12}$$

The proof of the equivalence of (5.11) and (5.12) is left as an excercise to the reader. Granulometries are used in practice to obtain size distributions. If m denotes Lebesgue measure then $m(\psi_\lambda(X))$ can be interpreted as the total volume of particles with size $\geq \lambda$.

If every opening ψ_λ is a structural opening (see Section 2) then we say that $\{\psi_\lambda\}$ is a *structural granulometry*. If every opening ψ_λ is translation invariant then we speak of a T-granulometry. The following observation is of great importance with respect to T-granulometries: if B is A-open then $X \circ B \subseteq X \circ A$ for every set $X \subseteq E$. If $B(\lambda)$ is a family of subsets of E which satisfy the semigroup property $B(\lambda)\oplus B(\mu) = B(\lambda+\mu)$, then the family ψ_λ given by $\psi_\lambda(X) = X\circ B(\lambda)$ defines a structural T-granulometry. Namely, if the semigroup relation holds then $B(\mu)$ is $B(\lambda)$-open if $\mu \geq \lambda$, yielding that $\psi_\mu \leq \psi_\lambda$.

If $\{\psi_\lambda\}$ is a T-granulometry which is scale-compatible in the sense that

$$\psi_\lambda(X) = \lambda\psi_1(\lambda^{-1}X), \text{ for every } \lambda > 0 \text{ and } X \subseteq E, \tag{5.13}$$

then $\{\psi_\lambda\}$ is called a *Euclidean granulometry*. It is apparent that condition (5.13) is equivalent to

$$\mathsf{Inv}(\psi_\lambda) = \lambda \mathsf{Inv}(\psi_1). \tag{5.14}$$

In fact, if $\{\psi_\lambda\}$ is a Euclidean granulometry then $\mathsf{Inv}(\psi_1)$ is closed under translation and multiplications with scalars $\lambda \geq 1$.

THEOREM 5.5. *Let $\{\psi_\lambda\}$ be a Euclidean granulometry. Then there is a family $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R}^d)$ such that*

$$\psi_\lambda(X) = \bigcup_{\mu\geq\lambda}\bigcup_{B\in\mathcal{B}} X\circ\mu B. \tag{5.15}$$

Conversely, if $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R}^d)$, then ψ_λ given by (5.15) defines a Euclidean granulometry.

Of particular interest are the structural Euclidean granulometries. From the considerations above it is easy to deduce that the openings in such a granulometry are of the form $\psi_\lambda(X) = X\circ\lambda B$. In order that μB is λB-open for $\mu \geq \lambda$ it is necessary and sufficient to assume that B is convex (that is, if we impose compactness on B); see Theorem 5.4.

It will be clear to the reader that a granulometry is a metric concept. In fact, we have seen that a structural Euclidean granulometry is characterized by a unique compact, convex set, and therefore by a linear metric (to make this precise one has to impose some extra conditions which guarantee that the convex set which defines the granulometry is compact and has nonempty interior, but we shall not deal with these technicalities here). If $\{\delta^\lambda \mid \lambda > 0\}$ is a family of dilations on $\mathcal{P}(E)$ which satisfy the semigroup property

$$\delta^\lambda\delta^\mu = \delta^{\lambda+\mu},$$

(see also (5.8)) and if ε^λ is the erosion related to δ^λ by adjunction, then the openings $\psi_\lambda = \delta^\lambda\varepsilon^\lambda$ define a granulometry. This observation holds for arbitrary complete lattices, and can e.g. be used as a basis for a theory of granulometries on discrete spaces where the notion of convexity is rather cumbersome. An important problem to be dealt with concerns a general construction of families of dilations which do have this semigroup property.

6. From sets to functions

We have seen that, though morphology has originally been developed for binary images, it can be extended to arbitrary complete lattices. In this section we consider lattices of functions which form representations of spaces of grey-scale images. There exist several, closely related, approaches to grey-scale morphology. A comprehensive discussion can be found in [15]. Other references include [13, 25, 39, 42].

Before we start our discussion we point out that in this paper we restrict ourselves to the case where the grey-value set is $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. However, we point out that many other choices are possible such as $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$ or the finite set $\{0, 1, \cdots, N\}$.

Again, let $E = \mathbb{R}^d$ or $\mathbb{Z}^d$ and denote by $\mathsf{Fun}(E)$ the space of functions mapping E into $\overline{\mathbb{R}}$. In Section 3 we have seen that $\mathsf{Fun}(E)$ is a complete lattice. Elements of $\mathsf{Fun}(E)$ are written F, G, etc. On $\mathsf{Fun}(E)$ we can define a *horizontal* or *spatial* translation along $h \in E$ as

$$F_h(x) = F(x - h),\ x \in E,$$

and a *vertical* or *grey-scale* translation along $v \in \mathbb{R}$ as

$$(F + v)(x) = F(x) + v,\ x \in E.$$

An operator ψ on $\mathsf{Fun}(E)$ is called an *H-operator* if it is invariant under horizontal translations,

$$\psi(F_h) = [\psi(F)]_h,\ F \in \mathsf{Fun}(E),\ h \in E,$$

and a *T-operator* if it is invariant under both type of translations,

$$\psi(F_h + v) = [\psi(F)]_h + v,\ F \in \mathsf{Fun}(E),\ h \in E,\ v \in \mathbb{R}.$$

The set $\mathsf{Fun}(E)$ is a complete lattice under the pointwise supremum (denoted '$\bigvee$') and infimum (denoted '$\bigwedge$'). The mapping $F \to -F$ defines a negation (see Section 3). The complementary operator of ψ, denoted as ψ^*, is defined by

$$\psi^*(F) = -\psi(-F). \tag{6.1}$$

For $F, G \in \mathsf{Fun}(E)$ we can define their Minkowski sum and difference respectively as

$$(F \oplus G)(x) = \bigvee_{h \in E} [F(x - h) + G(h)], \tag{6.2}$$

and

$$(F \ominus G)(x) = \bigwedge_{h \in E} [F(x + h) - G(h)]. \tag{6.3}$$

In fact, these two operations follow from the application of Proposition 3.7 to the complete lattice $\mathcal{L} = \mathsf{Fun}(E)$, with T the group of horizontal and vertical translations; for the missing details we refer to [35, 15]. In our terminology, $\Delta_G(F) = F \oplus G$ defines a T-dilation and $\mathcal{E}_G(F) = F \ominus G$ a T-erosion. From the abstract theory (Proposition 3.2) it follows that Δ_G commutes with suprema and that $\mathcal{E}_G$ commutes with infima. By composing (or alternatively, taking suprema and infima of) these two operators we can build openings, closings, alternating sequential filters, and many other grey-scale operators.

To visualize such operators the notion of an umbra turns out to be very useful. Here we only sketch the idea; a comprehensive discussion can be found in [6, 15]. Let us, however, caution the mathematical reader that the umbra approach has always given rise to a lot of confusion and, even worse, wrong statements.

A set $U \subseteq E \times \mathbb{R}$ is called an *umbra* if

$$(x,t) \in U \iff (x,s) \in U \text{ for } s < t. \tag{6.4}$$

If F is a function then $U(F) = \{(x,t) \mid F(x) \geq t\}$ defines an umbra. To transform a function, one can alternatively transform the corresponding umbra. The idea is illustrated in Figure 10 below. Here B is a ball in $E \times \mathbb{R}$.

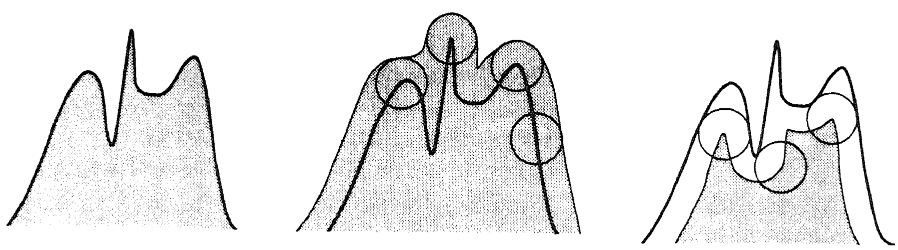

FIGURE 10. Visualization of $F \oplus B$ and $F \ominus B$, where B is a disk, by means of the umbra transform. Note that we may replace B by the smallest umbra containing B.

A useful transform in mathematical morphology is the so-called top-hat transform which can be used for the extraction of narrow peaks. The procedure is illustrated in Figure 11.

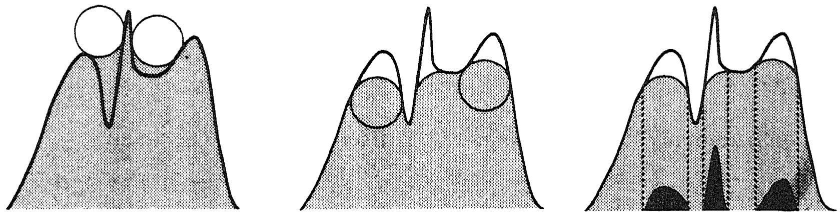

FIGURE 11. Rolling-ball closing and opening and the top-hat transform.

We start with a grey-scale image F, compute the opening $F \circ B = (F \ominus B) \oplus B$ where B is the ball depicted in Figure 11. This opening is ususally referred to as the *rolling-ball opening* [42]. Then $F \circ B \leq F$. The *top-hat transform* is the difference $F - (F \circ B)$; see Figure 11. Note that the peaks which are sufficiently wide and smooth are preserved.

Of particular interest in applications is the case where the structuring function G is *flat*, to be specific, $G(x) = 0$ if $x \in A$ and $-\infty$ elsewhere; here $A \subseteq E$ is the domain of G. We derive the following expressions:

$$F \oplus A = \bigvee_{h \in A} F_h, \quad F \ominus A = \bigwedge_{h \in A} F_{-h}. \tag{6.5}$$

In Figure 12 both operations are visualized by the umbra transform.

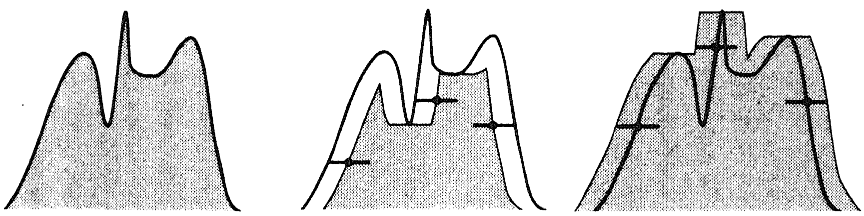

FIGURE 12. Flat erosion $\mathcal{E}_A$ and flat dilation Δ_A.

As an illustration of the use of flat structuring elements we mention the so-called *morphological gradient.* Recall that for a continuously differentiable function on $\mathbb{R}^d$ the gradient ∇F is defined as the d-vector $(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \cdots, \frac{\partial F}{\partial x_d})$. The morphological gradient is defined as

$$\mathsf{grad}(F) = \lim_{r \downarrow 0} \frac{1}{2r} \left[(F \oplus rD) - (F \ominus rD) \right],$$

where D is the d-dimensional ball with radius 1. It is easy to show that

$$\mathsf{grad}(F) = \|\nabla F\|,$$

if F is continuously differentiable. For a discrete image one defines

$$\mathsf{grad}(F) = \frac{1}{2} \left[(F \oplus D) - (F \ominus D) \right],$$

where D is the discrete analogon of the unit ball. In the 2-dimensional case one usually takes for D the square consisting of nine points. In Figure 13 we have computed the discrete gradient for a specific image.

The grey-scale operators given above have been introduced without reference to the binary case. It is obvious, however, that these operators are closely related. In fact, both cases are nothing but particular examples of the complete lattice framework described in Section 3. We now present an alternative way to construct grey-scale morphological operators. The basic idea is to threshold a function at any grey-value t (in other words, to take horizontal cross sections of the umbra), to apply a fixed increasing binary operator at every level (the resulting sets form again an umbra) and to compute the new function from the transformed umbra.

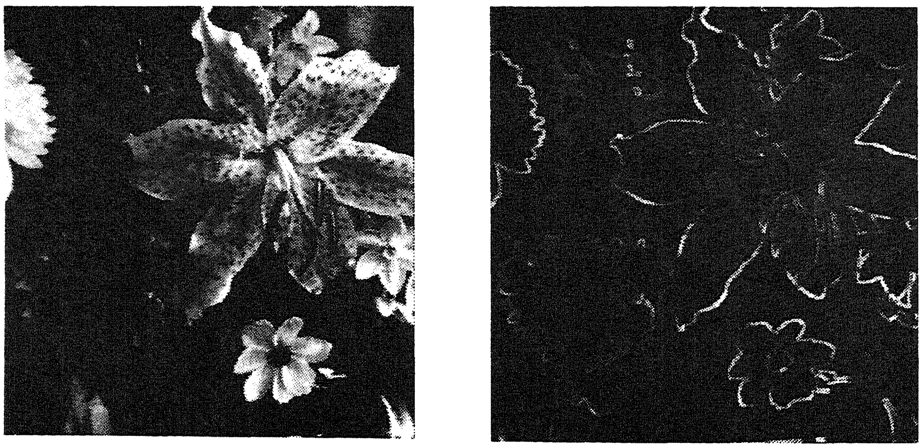

FIGURE 13. Morphological gradient.

This approach can be formalized if one returns to the lattice framework. This has the advantage that it applies to various grey-value sets, as well as to other function spaces, such as upper semi-continuous functions, a class of functions which has proved most relevant in the context of mathematical morphology. A comprehensive discussion can be found in [15].

Let $\mathcal{L}$ be a complete lattice. We define $\mathcal{L}\,\square\,\overline{\mathbb{R}}$ as the space of all mappings $\mathbf{X} : \overline{\mathbb{R}} \to \mathcal{L}$ which are decreasing, and $\mathcal{L}\triangle\,\overline{\mathbb{R}}$ as the subset of mappings which satisfy

$$\mathbf{X}(\bigvee_{i\in I} t_i) = \bigwedge_{i\in I} \mathbf{X}(t_i),$$

for every family $\{t_i \mid i \in I\}$ in $\overline{\mathbb{R}}$ (such mappings are sometimes called anti-dilations). Define the operator $\hat{\mathbf{I}} : \mathcal{L}\,\square\,\overline{\mathbb{R}}r \to \mathcal{L}\triangle\,\overline{\mathbb{R}}$ as

$$(\hat{\mathbf{I}}\mathbf{X})(t) = \bigwedge_{s<t} \mathbf{X}(s). \tag{6.6}$$

Then the mapping $\hat{\mathbf{I}}$ is a closing because it is increasing, idempotent, and satisfies $(\hat{\mathbf{I}}\mathbf{X})(t) \geq \mathbf{X}(t)$ for every $\mathbf{X} \in \mathcal{L}\,\square\,\overline{\mathbb{R}}$ and $t \in \overline{\mathbb{R}}$. The space $\mathcal{L}\triangle\overline{\mathbb{R}}$ is a complete lattice with the pointwise infimum of $\mathcal{L}$ and with supremum given by $\hat{\mathbf{I}}(\bigvee_{i\in I} \mathbf{X}_i)$, for any collection $\{\mathbf{X}_i \mid i \in I\}$ in $\mathcal{L}\triangle\overline{\mathbb{R}}$; here '$\bigvee$' denotes the supremum in $\mathcal{L}^{\overline{\mathbb{R}}}$.

For a function $F : E \to \overline{\mathbb{R}}$ we define its threshold set $\mathbf{X}(F, t)$ as

$$\mathbf{X}(F,t) = \{x \in E \mid F(x) \geq t\}. \tag{6.7}$$

This mapping defines an isomorphism between $\mathsf{Fun}(E)$ and $\mathcal{P}(E)\triangle\overline{\mathbb{R}}$ with inverse given by

$$\mathbf{F}(\mathbf{X})(x) = \bigvee\{t \in \overline{\mathbb{R}} \mid x \in \mathbf{X}(t)\}. \tag{6.8}$$

REMARK 6.1. If E is a topological space, e.g., $\mathbb{R}^d$, and $\mathcal{F}(E)$ is the complete lattice of closed subsets of E, then $\mathcal{F}(E)\triangle\overline{\mathbb{R}}$ is isomorphic with the space of upper semi-continuous (u.s.c.) functions on E. Recall that a function F is u.s.c. if for every $t \in \overline{\mathbb{R}}$ and $x \in E$ such that $t > F(x)$ there exists a neighbourhood V of x such that $t > F(y)$ for $y \in V$. One can easily show that a function F is u.s.c. if and only if every threshold set $\mathbf{X}(F, t)$ is closed. Using the representation of $\mathsf{Fun}(E)$ given above it is easy to extend an increasing operator ψ on $\mathcal{P}(E)$ to $\mathsf{Fun}(E)$. Namely, we can represent a function F by its threshold sets $\mathbf{X}(F, t)$. Applying ψ to any such set yields a family of sets $\psi(\mathbf{X}(F, t))$ which is decreasing with respect to t, but which does not necessarily lie inside $\mathcal{P}(E)\triangle\overline{\mathbb{R}}$. To achieve this, we apply $\hat{\mathbf{I}}$ to this family. This yields an element of $\mathcal{P}(E)\triangle\overline{\mathbb{R}}$ and hence an element $\psi(F)$ of $\mathsf{Fun}(E)$. It is obvious that the following relation holds:

$$\mathbf{X}(\psi(F), t) = \bigwedge_{s<t} \psi(\mathbf{X}(F, s)), \tag{6.9}$$

or alternatively

$$\psi(F)(x) = \bigvee\{t \in \overline{\mathbb{R}} \mid x \in \psi(\mathbf{X}(F,t))\}.$$

We call ψ the *flat function operator generated by* ψ. The reader can easily verify that this construction fails if ψ is not increasing. It can be shown that the given construction of flat operators is compatible with the formation of suprema, infima, compositions and negation. The latter means for example that if ψ is the generator of ψ, then ψ^* is the generator of ψ^*. Furthermore, if ψ is a dilation (erosion, closing, opening) then ψ is a dilation (erosion, closing, opening) as well. If we take for example ψ to be the Minkowski set addition $\psi(X) = X \oplus A$ as given by (2.6) then $\psi(F) = F \oplus A$ given by (6.5). We briefly discuss the class of flat function operators associated with the finite set operators on $\mathcal{P}(\mathbb{Z}^d)$. In Section 2 we have seen that every such operator is of the form

$$\psi_b(X) = \{h \in \mathbb{Z}^d \mid b(X(a_1 + h), X(a_2 + h), \cdots, X(a_n + h)) = 1\},$$

where $A = \{a_1, a_2, \cdots, a_n\}$ is a structuring element and b a Boolean function. The operator ψ_b is increasing if and only if b is positive. Let b be a Boolean function. We can extend b to a function $b^\sim : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ in the following way:

$$b^\sim(t_1, t_2, \cdots, t_n) = \sup\{t \in \overline{\mathbb{R}} \mid b([t_1 \geq t], [t_2 \geq t], \cdots, [t_n \geq t]) = 1\}, \qquad (6.11)$$

5 where $[t_1 \geq t]$ is a Boolean expression which equals 1 if $t_1 \geq t$ and 0 otherwise. It is easy to show that this procedure satisfies the following properties:

$$(b_1 \cdot b_2)^\sim = b_1^\sim \wedge b_2^\sim$$
$$(b_1 + b_2)^\sim = b_1^\sim \vee b_2^\sim,$$

where $\cdot$ and $+$ denote the logic AND and OR respectively. For example, if $b(x_1, x_2, x_3) = x_1 x_2 + x_3$ then $b^\sim(t_1, t_2, t_3) = (t_1 \wedge t_2) \vee t_3$. From now on we will denote a positive Boolean function and its extension to $\overline{\mathbb{R}}^n$ with the same symbol.

Now, if $A = \{a_1, a_2, \cdots, a_n\}$ and b a positive Boolean function of n variables, then we can define the operator Ψ_b on $\mathsf{Fun}(E)$ by

$$\psi_b(F)(x) = b(F(x + a_1), \cdots, F(x + a_n)).$$

We show that ψ_b coincides with the flat operator generated by ψ_b, which we denote by ψ for the moment. Combining (6.10) with the expression for ψ_b and (6.11) we get

$$\begin{aligned}
\psi(F)(x) &= \bigvee\{t \in \overline{\mathbb{R}} \mid x \in \psi_b(\mathbf{X}(F,t))\} \\
&= \bigvee\{t \in \overline{\mathbb{R}} \mid b([F(x + a_1) \geq t], \cdots, [F(x + a_n) \geq t]) = 1\} \\
&= \psi_b(F)(x)(F(x + a_1), \cdots, F(x + a_n)) \\
&= \psi_b(F)(x).
\end{aligned}$$

This proves our claim.

One can extend the geodesic operators introduced in Section 5 to grey-scale functions. Such operators appear to be quite useful in many applications. We refer to [45] for a number of examples. We conclude this section with a brief discussion on granulometries for grey-scale functions. A general account can be found in [21] The definition of grey-scale granulometries is identical to the binary case. A family of openings $\{\psi_\lambda \mid \lambda > 0\}$ on $\mathsf{Fun}(E)$ is called a granulometry if $\psi_\mu \leq \psi_\lambda$ for $\mu \geq \lambda$. It is apparent that the flat extension of a binary granulometry to $\mathsf{Fun}(E)$ defines again a granulometry, called *flat granulometry*. We argue below that this class of flat granulometries is quite important. If we want to extend the notion of Euclidean granulometry there are several possibilities. First, we can choose between H-openings and T-openings. And moreover, we can think of at least two different kind of scalings. The first one is the *umbral scaling* given by

$$(\lambda \cdot F)(x) = \lambda F(X/\lambda), \quad \lambda > 0,$$

which, as the name suggests, scales the entire umbra of the function. This scaling is also called a T-scaling, referring to the fact that it scales both the spatial and grey-scale variable. A second type of scaling is the *spatial scaling*

$$(\lambda \cdot F)(x) = F(X/\lambda), \quad \lambda > 0,$$

also called H-scaling. Both scalings are depicted in Figure 14.

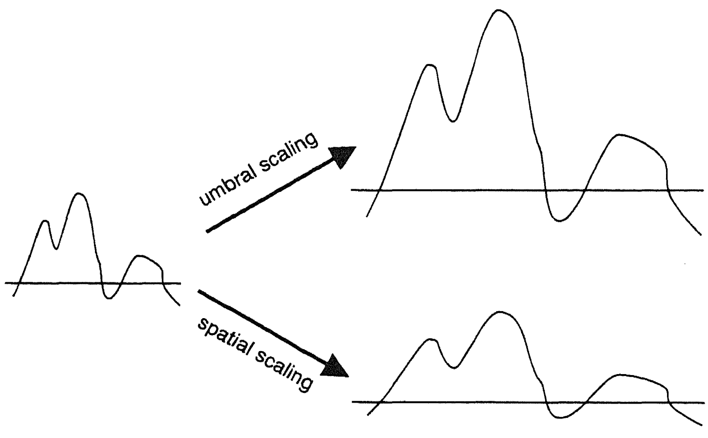

FIGURE 14. Umbral versus spatial scaling.

Therefore we can distinguish at least four types of Euclidean granulometries. All of them have been discussed in more or less detail in [21]. To give the reader an impression of the underlying mathematics we consider here the so-called (T,H)-Euclidean granulometry, where the first prefix 'T' indicates that we consider T-openings, and the second prefix 'H' that we consider H-scalings. A first observation to make is that the flat extension of a Euclidean granulometry on $\mathcal{P}(E)$ yields a (T,H)-Euclidean granulometry on $\mathsf{Fun}(E)$. We can formulate the following generalization of Theorem 5.5.

THEOREM 6.2. *Let $\{\psi_\lambda\}$ be a (T,H)-Euclidean granulometry. Then there is a family $\mathcal{G} \subseteq \mathsf{Fun}(E)$ such that*

$$\psi_\lambda(F) = \bigvee_{\mu \geq \lambda} \bigvee_{G \in \mathcal{G}} F \circ \mu \cdot G. \tag{6.12}$$

Conversely, if $\mathcal{G} \subseteq \mathsf{Fun}(E)$ then $\{\psi_\lambda\}$ given by 6.12 defines a (T,H)-Euclidean granulometry.

For practical applications, one requires elimination of the outer supremum in (6.12) so that we end up with $\Psi_\lambda(F) = \bigvee_{G \in \mathcal{G}} F \circ \lambda G$. As in the binary case this amounts to the following condition on the structuring function G:

$$\lambda G \circ G = \lambda G \text{ for } \lambda \geq 1. \tag{6.13}$$

In [21] we have proved the following theorem. Here the domain $D(G)$ of G is the set $\{x \in E \mid G(x) > -\infty\}$.

THEOREM 6.3. *Let $G \in \mathsf{Fun}(E)$ be u.s.c. and have compact domain. Then condition (6.13) holds if and only if $D(G)$ is convex and G is constant there.* In fact this result says that every structural (T,H)-Euclidean granulometry on $\mathsf{Fun}(E)$ ('structural' meaning that every opening involves only one structuring function) is the flat extension of a structural Euclidean granulometry on $\mathcal{P}(E)$. The proof of Theorem 6.3 employs Theorem 5.4, the Krein-Milman theorem and Zorn's Lemma; see [21].

7. AND SO FORTH...

In this paper we have only been able to touch upon a few aspects of mathematical morphology. Many other aspects have been kept unmentioned, and we will use this last section to devote some words to two or three of them.

We point out that our choice has largely been determined by our personal interest and knowledge. This is why nothing has been said about the implementation of morphological algorithms. It goes without saying that this is an extremely important subject, and, for that reason it has received a lot of attention in the literature. For those readers who are interested in the design of flexible data structures and fast algorithms we refer to the forthcoming book of SCHMITT and VINCENT [38].

Another subject which has been ignored here is the probabilistic approach. As we pointed out in the introduction, the strength of morphology lies in its intertwining with integral geometry and stochastic geometry. From the very beginning MATHERON and SERRA [39] have emphasized the importance of a joint development of a theory of mathematical morphology and random sets. A major ingredient for such a theory of random set, as initiated by Matheron, is the so-called *hit-or-miss topology* on the space of closed subsets of $\mathbb{R}^d$. As the name suggests, this topology is closely related to the hit-or-miss operator discussed in Section 2. During the last ten years, the theory of morphological operators (and more particularly, morphological filters) has started to lead its own life as a toolbox for image processing, including powerful algorithms for filtering, segmentation, skeletonization, etc. However, for a sound judgment of the merits of mathematical morphology as a methodology in image analysis it is necessary that one keeps in mind this stochastic component.

The hit-or-miss topology mentioned above has also been used to develop a theory of discretization; see [39, Chapter VII] and [14]. Such a theory is required to bridge the gap between the analytic approach using concepts from e.g. integral geometry and their digital implementations. The most common discretization of an image uses the regular discrete grid, either square or hexagonal. The hexagonal grid has the advantage that it possesses more isotropy (it allows rotations over multiples of 60°, whereas the square grid only allows 90°-rotations), but the visualization of an hexagonal image requires some more effort. Anyhow, discrete representations of images raise a number of problems which are quite familiar to people working in discrete geometry. Besides rotations, also such notions as distance, convexity, homotopy, etc. need reconsideration. But such problems concerning discrete topology and geometry cross the morphological borders and form a major challenge in digital image processing [20, 37].

For some applications, a regular grid is not the optimal discrete structure to model an image. For example, if X is an electron microscopic image of some cell tissue it seems plausible to model the cells in this population as the vertices of a neighbourhood graph. The edges of such a graph carry useful information about the spatial relationships between the individual cells. In his thesis [44] VINCENT has generalized many concepts, both algebraic and geometric, from classical morphology to the graph framework. His work has been extended in [16, 19].

We hope that we have succeeded in giving the reader a first impression of the underlying principles of mathematical morphology and in convincing him or her that this theory, besides taking benefit from the power of a mathematical framework, also contributes to mathematics by posing many challenging questions.

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