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TAIL TRIVIALITY FOR SUMS OF STATIONARY RANDOM VARIABLES

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We study tail σ-fields and loss of memory associated with sums of stationary integer-valued random variables. An application concerns convergence in distribution of interarrival times in zero–one sequences.

1. Statement of results. Let $X_1, X_2, \dots$ be a *strictly stationary* sequence of *integer-valued* random variables and let $S_1, S_2, \dots$ be the sums

$$S_n = X_1 + \cdots + X_n, \qquad n \geq 1.$$

If the X_i are independent, then according to the Hewitt–Savage zero–one law [Breiman (1968)] the tail σ-field

$$\bigcap_{N>0} \sigma(S_n \colon n \geq N)$$

is trivial. Of course, without the independence this need no longer be true. The main question that will be addressed in this paper is what can be said about the tail behavior of the sums in this more general setting. An early reference is Blackwell and Freedman (1964), where the Hewitt–Savage zero–one law is generalized to Markov sequences. A later reference is Georgii (1976, 1979) for Gibbs states with finite state space.

It is natural to extend $(X_n)_{n\geq 1}$ to a double-sided process $X := (X_n)_{n\in\mathbb{Z}}$ and to extend also the sums to a double-sided process $(S_n)_{n\in\mathbb{Z}}$ by requiring that

$$S_0 = 0, \qquad S_n - S_{n-1} = X_n, \qquad n \in \mathbb{Z}. \tag{1.1}$$

The process X is defined on the probability space $(\Omega, \mathscr{F}, P)$ with $\Omega = \mathbb{Z}^{\mathbb{Z}}$, X the identity on Ω, $\mathscr{F}$ the product σ-field generated by the discrete topology on $\mathbb{Z}$ and P a T-invariant probability measure with T the shift defined by $(TX)_n = X_{n+1}$, $n \in \mathbb{Z}$. We shall be interested in the following tail σ-fields associated with (S_n):

$$\mathscr{G}_\infty := \bigcap_{M,\, N\geq 0} \sigma((S_m, S_n) \colon m \leq -M,\ n \geq N),$$

$$\mathscr{G}_\infty^{\text{inv}} := \bigcap_{M,\, N\geq 0} \sigma(S_n - S_m \colon m \leq -M,\ n \geq N).$$

Note that $\mathscr{G}_\infty^{\text{inv}} = \bigcap_N T^{-N}\mathscr{G}_\infty$, so $\mathscr{G}_\infty^{\text{inv}}$ is a T-invariant σ-field included in the

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double tail σ-field $\mathscr{G}_\infty$. The following example shows that $\mathscr{G}_\infty^{\text{inv}}$ and $\mathscr{G}_\infty$ may be different.

EXAMPLE 1.2. Let (Y_n) be i.i.d. with $P(Y_n = -1) = P(Y_n = 1) = \frac{1}{2}$ and let $X_n = Y_n - Y_{n-1}$, $n \in \mathbb{Z}$. Then $S_n - S_m = X_{m+1} + \cdots + X_n = Y_n - Y_m$ and so $\mathscr{G}_\infty^{\text{inv}}$ is trivial by the Kolmogorov zero–one law [Breiman (1968)]. On the other hand, $S_n = X_1 + \cdots + X_n = Y_n - Y_0$ and hence $\lim_{N\to\infty} N^{-1}\Sigma_{n=1}^N S_n = -Y_0$ a.s. Thus Y_0 is $\mathscr{G}_\infty$-measurable, so $\mathscr{G}_\infty$ is not trivial. In fact, $\mathscr{G}_\infty = \sigma(Y_0)$ a.s.

Let

$$\mathscr{F}_\infty := \bigcap_{M,\, N \ge 0} \sigma(X_n\colon m \le -M,\, n > N)$$

be the double tail σ-field associated with X. Since

$$\sigma(S_n - S_m\colon m \le -M,\, n \ge N) = \sigma(S_N - S_{-M}) \vee \sigma(X_n\colon m \le -M,\, n > N)$$

for all $M, N \ge 0$, it is clear that $\mathscr{G}_\infty^{\text{inv}} \supset \mathscr{F}_\infty$. In Section 2 we prove

THEOREM 1.3. *Let (X_n) be stationary integer-valued and let (S_n) be given by* (1.1). *If X_0 has finite entropy, then $\mathscr{G}_\infty^{\text{inv}} = \mathscr{F}_\infty$ a.s.*

COROLLARY 1.4. *If X_0 has finite entropy, then $\mathscr{G}_\infty^{\text{inv}}$ is trivial iff $\mathscr{F}_\infty$ is trivial.*

As Example 1.2 shows, to obtain triviality of the larger tail σ-field $\mathscr{G}_\infty$ we need to assume more than triviality of $\mathscr{F}_\infty$. It will not be enough to impose stronger mixing conditions on X. In Section 3 we prove the following zero–two theorem which will be seen to be the key to our study of $\mathscr{G}_\infty$.

THEOREM 1.5. *Let (X_n) be stationary, ergodic and real-valued and let (S_n) be given by* (1.1). *For every real h, either*

$$\|P(S_N \in \cdot|\mathscr{F}_N^*) - P(S_N + h \in \cdot|\mathscr{F}_N^*)\| = 2 \quad \text{for all } N \ge 1 \text{ a.s.}$$

or else

$$\lim_{N\to\infty} \|P(S_N \in \cdot|\mathscr{F}_N^*) - P(S_N + h \in \cdot|\mathscr{F}_N^*)\| = 0 \quad \text{a.s.}$$

Here $\mathscr{F}_N^* := \sigma(X_n\colon n \notin \{1,\dots,N\})$ and $\|\cdot\|$ denotes total variation.

For i.i.d. sequences the corresponding zero–two theorem was proved by Stam (1966/67) and by Ornstein (1969), while Berbee (1979) gave a proof for mixing sequences. Most zero–two theorems in the literature relate to Markov processes.

Theorem 1.5 leads us in a natural way to associate a group $H \subset \mathbb{R}$ with (S_n) as follows:

$$H = \begin{array}{l}\{h \in \mathbb{R}\colon \text{there exists } N \geq 1 \text{ such that with positive} \\ \text{probability } P(S_N \in \cdot|\mathscr{F}_N^*) \text{ and } P(S_N + h \in \cdot|\mathscr{F}_N^*) \\ \text{have mass in common}\}.\end{array}$$

By Theorem 1.5 the set H is a group. For (X_n) integer-valued we shall say that (S_n) is *strongly aperiodic* if $H = \mathbb{Z}$.

Theorem 1.6 below gives sufficient conditions for triviality of $\mathscr{G}_\infty$ as well as of the right tail σ-field

$$\mathscr{G}_\infty^+ := \bigcap_{N \geq 0} \sigma(S_n\colon n \geq N).$$

Let

$$\mathscr{F}_\infty^+ := \bigcap_{N \geq 0} \sigma(X_n\colon n > N)$$

and note that $\mathscr{G}_\infty^+ \supset \mathscr{F}_\infty^+$. In Section 4 we prove

THEOREM 1.6. *Let (X_n) be stationary integer-valued and let (S_n) be given by* (1.1). *Assume that (S_n) is strongly aperiodic. If $\mathscr{F}_\infty^+$ is trivial, then $\mathscr{G}_\infty^+$ is trivial. If $\mathscr{F}_\infty$ is trivial, then $\mathscr{G}_\infty$ is trivial.*

Note that in Example 1.2 triviality of $\mathscr{F}_\infty$ holds because (X_n) is one-dependent but strong aperiodicity fails because $S_N = Y_N - Y_0$ while Y_0 and Y_N are $\mathscr{F}_N^*$-measurable, so $H = \{0\}$.

As an application of Theorem 1.6 we consider a stationary ergodic zero–one sequence (X_n). Let $T_1, T_2, \ldots$ be the random positive times at which X_n assumes the value 1,

$$\begin{aligned} T_1 &= \inf\{n \geq 1\colon X_n = 1\}, \\ T_k &= \inf\{n > T_{k-1}\colon X_n = 1\}, \qquad k > 1. \end{aligned} \tag{1.7}$$

By stationarity and ergodicity, if $P(X_1 = 1) > 0$ then $T_k < \infty$ a.s. for all $k \geq 1$. We call $T_{k+1} - T_k$ the kth *interarrival time*. In Section 5 we prove

COROLLARY 1.8. *Let (X_n) be a stationary zero–one sequence with $P(X_1 = 1) > 0$ and let (S_n) be given by* (1.1) *and (T_k) by* (1.7). *If $\mathscr{G}_\infty^+$ is trivial, then for any integer $t \geq 0$,*

$$\lim_{k \to \infty} P(T_{k+1} - T_k > t) = P(T_2 - T_1 > t | X_1 = 1).$$

Corollary 1.8, together with Theorem 1.6, extends a result by Janson (1984) for m-dependent sequences. Janson (1984) contains an example of a one-dependent sequence with nonconverging interarrival times. Corollary 1.4 has recently been applied by den Hollander (1988) in a paper on mixing properties for random walk in random scenery.

EXAMPLE 1.9. All the conditions of Theorems 1.3 and 1.6 hold for stationary extremal Gibbs states. For Gibbs states with finite state space Theorem 1.3 was proved by Georgii (1976, 1979).

2. Proof of Theorem 1.3. We start by recalling a few properties of entropy [see, e.g., Smorodinsky (1971) or Parry (1981)]. Let Z be a discrete random variable on our probability space $(\Omega, \mathscr{F}, P)$ taking values in a countable set. The *entropy* of Z is defined as

$$H(Z) := -\sum_z P(Z=z)\log P(Z=z)$$

($0\log 0 = 0$). For a sub-σ-field $\mathscr{A}\subset\mathscr{F}$ the *conditional entropy* of Z given $\mathscr{A}$ is defined as

$$H(Z|\mathscr{A}) := -E\sum_z P(Z=z|\mathscr{A})\log P(Z=z|\mathscr{A}).$$

For two sub-σ-fields $\mathscr{A}, \mathscr{B}\subset\mathscr{F}$ we denote by $\mathscr{A}\vee\mathscr{B}$ the smallest σ-field containing $\mathscr{A}$ and $\mathscr{B}$. The following properties will be used below:

$$H(Z|\mathscr{A}) \le H(Z|\mathscr{B}) \quad \text{if } \mathscr{B}\subset\mathscr{A}, \tag{2.1}$$

$$H(Z|\mathscr{A}) \le H(Z), \tag{2.2}$$

$$\begin{aligned} H(Z_1, Z_2|\mathscr{A}) &= H(Z_1|\mathscr{A}) + H(Z_2|\mathscr{A}\vee\sigma(Z_1)) \\ &= H(Z_2|\mathscr{A}) + H(Z_1|\mathscr{A}\vee\sigma(Z_2)), \end{aligned} \tag{2.3}$$

$$H(Z_1+Z_2) \le H(Z_1) + H(Z_2), \tag{2.4}$$

$$H(Z) \le \log|\{z: P(Z=z)>0\}|. \tag{2.5}$$

PROOF OF THEOREM 1.3. We introduce the notation

$$\begin{aligned} S_{m,n} &:= S_n - S_m = X_{m+1} + \cdots + X_n, \qquad m, n\in\mathbb{Z},\ m<n, \\ \mathscr{F}_N &:= \sigma(X_n: n\le -N, n>N), \\ \mathscr{G}_N &:= \sigma(S_{m,n}: m\le -N, n\ge N) = \sigma(S_{-N,N})\vee\mathscr{F}_N. \end{aligned}$$

Note that $\mathscr{F}_\infty = \bigcap_{N\ge 0}\mathscr{F}_N$ and $\mathscr{G}_\infty^{\text{inv}} = \bigcap_{N\ge 0}\mathscr{G}_N$.

By the martingale convergence theorem [Smorodinsky (1971)]

$$\begin{aligned} H(X_0|\mathscr{F}_\infty) &= \lim_{N\to\infty} H(X_0|\mathscr{F}_N), \\ H(X_0|\mathscr{G}_\infty^{\text{inv}}) &= \lim_{N\to\infty} H(X_0|\mathscr{G}_N). \end{aligned}$$

Here the monotonicity of $\mathscr{F}_N$ and $\mathscr{G}_N$ is used together with (2.2) and the assumption $H(X_0)<\infty$. As a first step towards proving $\mathscr{G}_\infty^{\text{inv}} = \mathscr{F}_\infty$ a.s. we claim that

$$H(X_0|\mathscr{F}_\infty) = H(X_0|\mathscr{G}_\infty^{\text{inv}}). \tag{2.6}$$

To get (2.6), we note that

$$H(X_m|\mathscr{G}_N) \le H(X_0|\mathscr{G}_{N+|m|}) \quad \text{for all } m \text{ and } N. \tag{2.7}$$

This follows from stationarity, (2.1) and the inclusion $T^{-m}\mathscr{G}_{N+|m|} \subset \mathscr{G}_N$. Now fix two positive integers K, L and consider the quantity

$$H\big((X_{jK})_{|j|\le L-1}, S_{-LK,LK}|\mathscr{F}_{LK}\big),$$

that is, the conditional entropy given $\mathscr{F}_{LK}$ of the sum $S_{-LK,LK}$ and of the X_n with $|n| \le (L-1)K$ but *sampled over gaps* of size K. The idea is to write this out in two different ways, using (2.3),

$$\text{I} \quad H\big((X_{jK})_{|j|\le L-1}|\mathscr{F}_{LK}\big) + H\Big(S_{-LK,LK}|\mathscr{F}_{LK} \vee \sigma\big((X_{jK})_{|j|\le L-1}\big)\Big),$$

$$\text{II} \quad H(S_{-LK,LK}|\mathscr{F}_{LK}) + H\big((X_{jK})_{|j|\le L-1}|\mathscr{F}_{LK} \vee \sigma(S_{-LK,LK})\big),$$

to derive a lower bound for I and an upper bound for II, and thereby to get an inequality between these two bounds which can then be exploited to prove (2.6).

Indeed, if we iterate the first term of I using (2.3), then we get $2L-1$ terms each of which is bounded below by $H(X_0|\mathscr{F}_K)$ because of stationarity and (2.1). Hence, ignoring the second term of I we get

$$I \ge (2L-1)H(X_0|\mathscr{F}_K).$$

In II, on the other hand, the first term is bounded above by $H(S_{-LK,LK})$ because of (2.2), while the second term equals $H((X_{jK})_{|j|\le L-1}|\mathscr{G}_{LK})$ and is bounded above by $\sum_{|j|\le L-1}H(X_{jK}|\mathscr{G}_{LK})$ via (2.1) and (2.3). Together with (2.7) this gives

$$\text{II} \le H(S_{-LK,LK}) + (2L-1)H(X_0|\mathscr{G}_{2LK}).$$

Since I = II we thus have

$$H(X_0|\mathscr{F}_K) \le (2L-1)^{-1}H(S_{-LK,LK}) + H(X_0|\mathscr{G}_{2LK}).$$

Next let $L \to \infty$ while keeping K fixed, and use the following lemma which we shall prove below.

LEMMA 2.8. *If* $H(X_0) < \infty$, *then* $H(S_{-N,N}) = o(N)$.

Finally let $K \to \infty$ and we end up with $H(X_0|\mathscr{F}_\infty) \le H(X_0|\mathscr{G}_\infty^{\text{inv}})$. But this implies (2.6) since the reverse inequality holds by (2.1) and $\mathscr{F}_\infty \subset \mathscr{G}_\infty^{\text{inv}}$.

Now, (2.6) is only the first step in proving $\mathscr{G}_\infty^{\text{inv}} = \mathscr{F}_\infty$ a.s. The next step is to prove that for any positive integer M,

$$(2.9) \qquad H\big((X_n)_{|n|\le M}|\mathscr{F}_\infty\big) = H\big((X_n)_{|n|\le M}|\mathscr{G}_\infty^{\text{inv}}\big).$$

This can be done in exactly the same way as above. Instead of sampling $2L-1$ single variables X_n over gaps of size K, we must now sample $2L-1$ blocks of variables each of fixed length $2M+1$ separated by gaps of size K and then again take the limit $L \to \infty$ followed by $K \to \infty$. The key inequality is the obvious modification of (2.7). These steps are left for the reader to verify.

The assertion now follows from (2.9) and the following lemma which we shall prove below.

LEMMA 2.10. *Assume $H(X_0) < \infty$. Let $\mathscr{A}, \mathscr{B} \subset \mathscr{F}$ be two sub-σ-fields such that $\mathscr{A} \subset \mathscr{B}$. If*

$$H((X_n)_{|n| \le m} | \mathscr{A}) = H((X_n)_{|n| \le m} | \mathscr{B}) \quad \text{for all } m \ge 0,$$

then $\mathscr{A} = \mathscr{B}$ a.s.

This completes the proof of Theorem 1.3. □

PROOF OF LEMMA 2.8. We follow a standard truncation argument. Fix a positive integer M and define

$$Y_n = \begin{cases} 0 & \text{if } |X_n| \le M, \\ X_n & \text{if } |X_n| > M, \end{cases} \qquad n \in \mathbb{Z}.$$

Write $S_{-N,N} = \sum_{n=-N+1}^{N} Y_n + \sum_{n=-N+1}^{N} (X_n - Y_n)$. By stationarity and (2.4),

$$H(S_{-N,N}) \le 2NH(Y_0) + H\left(\sum_{n=-N+1}^{N} (X_n - Y_n) \right).$$

Since $X_n - Y_n$ can at most take $2M + 1$ distinct values, the last term is at most $\log(2N(2M + 1))$ by (2.5). Letting $N \to \infty$ while keeping M fixed, we get

$$\limsup_{N \to \infty} (2N)^{-1} H(S_{-N,N}) \le H(Y_0).$$

But $H(Y_0)$ can be made arbitrarily small by letting $M \to \infty$, because $H(X_0) < \infty$. □

PROOF OF LEMMA 2.10. Let $E \in \mathscr{B}$. Select $E_m \in \sigma((X_n)_{|n| \le m})$ such that

$$\lim_{m \to \infty} P(E_m \Delta E) = 0. \tag{2.11}$$

From (2.3), for all $m \ge 0$,

$$H(1_{E_m} | \mathscr{A}) + H((X_n)_{|n| \le m} | \mathscr{A} \vee \sigma(1_{E_m})) = H((X_n)_{|n| \le m} | \mathscr{A})$$

and by assumption this equals

$$H(1_{E_m} | \mathscr{B}) + H((X_n)_{|n| \le m} | \mathscr{B} \vee \sigma(1_{E_m})) = H((X_n)_{|n| \le m} | \mathscr{B}).$$

Because $\mathscr{A} \subset \mathscr{B}$ there must be term by term equality, so $H(1_{E_m} | \mathscr{A}) = H(1_{E_m} | \mathscr{B})$. But (2.11) implies $\lim_{m \to \infty} H(1_{E_m} | \mathscr{B}) = H(1_E | \mathscr{B}) = 0$, hence

$$\lim_{m \to \infty} H(1_{E_m} | \mathscr{A}) = H(1_E | \mathscr{A}) = 0.$$

This in turn implies $P(E | \mathscr{A}) = 0$ or 1 a.s., and so $E \in \mathscr{A}$ a.s. Thus $\mathscr{B} \subset \mathscr{A}$. □

3. Proof of Theorem 1.5. We follow the proof of the contraction lemma in Berbee (1986), which goes back to an idea of Bradley (1983). Berbee (1979) contains a proof for mixing sequences using coupling techniques. The ergodic theoretic proof below is considerably shorter but also less transparent.

Let us write $P(X \in \cdot|Z)$ for the conditional distribution of X given $\sigma(Z)$. The following facts will be used below:

$$(3.1)\qquad E\|P(X \in \cdot|Z) - P(Y \in \cdot|Z)\| = \|P((X,Z) \in \cdot) - P((Y,Z) \in \cdot)\|,$$

$$(3.2)\qquad C|P(f(X) \in \cdot) - P(f(Y) \in \cdot)| \le \|P(X \in \cdot) - P(Y \in \cdot)\|,$$

where X and Y are random variables taking values in the same measurable space, Z is any other random variable and f is any measurable function.

PROOF OF THEOREM 1.5. Let P_1 and P_2 be the two probability measures on $\mathscr{F}$ defined by

$$P_1 = P, \qquad P_2 = P \circ U_h$$

where $U_h\colon \Omega \to \Omega$ is the map

$$(U_h\omega)_n = \begin{cases} \omega_n, & n \ne 1, \\ \omega_n + h, & n = 1, \end{cases} \qquad \omega \in \Omega = \mathbb{R}^{\mathbf{Z}}.$$

We compare the restrictions of P_1 and P_2 to the nested sequence of σ-fields

$$\mathscr{G}_{M,N} := \sigma(S_{m,n}\colon m \le -M,\ n \ge N), \qquad M, N \ge 0.$$

Since $\mathscr{G}_\infty^{\text{inv}} = \bigcap_{M,N\ge 0}\mathscr{G}_{M,N}$, it follows via the martingale convergence theorem that

$$(3.3)\qquad \lim_{M,N\to\infty} \|P_1 - P_2\|_{\mathscr{G}_{M,N}} = \|P_1 - P_2\|_{\mathscr{G}_\infty^{\text{inv}}}.$$

Because $\mathscr{G}_\infty^{\text{inv}}$ is T-invariant and P is T-invariant and ergodic on $\mathscr{F}$, it is immediate that P_1 is T-invariant and ergodic on $\mathscr{G}_\infty^{\text{inv}}$. However, the same is true for P_2. This follows from the commutation rule $U_hT^{-1} = T^{-1}U_h$ which holds on $\mathscr{G}_{M,N}$ for any $M, N \ge 1$. Indeed, by this rule we have invariance because

$$P_2(T^{-1}A) = P(U_hT^{-1}A) = P(T^{-1}U_hA) = P(U_hA) = P_2(A),$$

while ergodicity follows because if A is T-invariant so is U_hA, again by this rule. By an elementary application of the ergodic theorem we get from these ergodic properties that P_1 and P_2 are either identical or mutually singular on $\mathscr{G}_\infty^{\text{inv}}$ and thus the right-hand side of (3.3) equals 0 or 2.

The left-hand side of (3.3) depends only on $N + M$ because of stationarity. If we take $M = 0$, note that $\mathscr{G}_{0,N} = \mathscr{F}_N^* \vee \sigma(S_N)$ and use (3.1), then (3.3) becomes equivalent to $\lim_{N\to\infty} EZ_N = 0$ or 2 with

$$Z_N = \|P(S_N \in \cdot|\mathscr{F}_N^*) - P(S_N + h \in \cdot|\mathscr{F}_N^*)\|.$$

Because S_{N+1} is (S_N, X_{N+1})-measurable, we get from (3.1) and (3.2) that Z_N is a reverse submartingale, that is, $Z_{N+1} \le E(Z_N|\mathscr{F}_{N+1}^*)$. Since $0 \le Z_N \le 2$, it follows that Z_N converges a.s. with limit either 0 or 2. In the latter case we have $Z_N = 2$ a.s. for all $N \ge 1$ because EZ_N is decreasing in N. □

4. Proof of Theorem 1.6. We need the following technical corollary of Theorem 1.5. Once this has been proved Theorem 1.6 will follow easily.

COROLLARY 4.1. *Let (X_n) be stationary, ergodic and integer-valued and let (S_n) be given by* (1.1). *If (S_n) is strongly aperiodic and if U and V are integer-valued (S_n)-measurable random variables, then for every $K \geq 0$,*

$$\lim_{N\to\infty} \Big\| P\big((S_n + U, S_{-n} + V)_{n\geq N} \in \cdot | (S_n)_{|n|\leq K}\big) - P\big((S_n, S_{-n})_{n\geq N} \in \cdot | (S_n)_{|n|\leq K}\big)\Big\| = 0 \quad a.s.$$

PROOF. Because U is integer-valued we can approximate it by random variables U_m which are $(S_n)_{|n|\leq m}$-measurable such that $\lim_{m\to\infty} P(U_m \neq U) = 0$. Hence it is enough to prove the assertion for U measurable with respect to $(S_n)_{|n|\leq m}$ for every fixed $m \geq 0$, and the same for V.

By Theorem 1.5 and property (3.1) we have that for any integer h,

$$\lim_{N\to\infty} \| P((S_n + h)_{n\geq N} \in \cdot | (S_n)_{n\leq 0}) - P((S_n)_{n\geq N} \in \cdot | (S_n)_{n\leq 0}) \| = 0 \quad \text{a.s.}$$

By stationarity this is the same as

$$\lim_{N\to\infty} \Big\| P\big((S_{n+m} - S_m + h)_{n\geq N} \in \cdot | (S_{n+m} - S_m)_{n\leq 0}\big) - P\big((S_{n+m} - S_m)_{n\geq N} \in \cdot | (S_{n+m} - S_m)_{n\leq 0}\big)\Big\| = 0 \quad \text{a.s. for every } m.$$

Now, $(S_{n+m} - S_m)_{n\leq 0}$ and $(S_n)_{n\leq m}$ determine each other when $m \geq 0$ (because $S_0 = 0$). Hence we may replace h by any $(S_n)_{n\leq m}$-measurable random variable. Moreover, since for all $N \geq -m$ also $(S_{-n} + V)_{n\geq N}$ is $(S_n)_{n\leq m}$-measurable we obtain

$$\lim_{N\to\infty} \Big\| P\big((S_n + U, S_{-n} + V)_{n\geq N} \in \cdot | (S_n)_{n\leq m}\big) - P\big((S_n, S_{-n} + V)_{n\geq N} \cap \cdot \,| (S_n)_{n\leq m}\big)\Big\| = 0 \quad \text{a.s.}$$

Via (3.1) and (3.2) it follows that for all $K \leq m$,

$$\lim_{N\to\infty} \Big\| P\big((S_n + U, S_{-n} + V)_{n\geq N} \in \cdot | (S_n)_{|n|\leq K}\big) - P\big((S_n, S_{-n} + V)_{n\geq N} \in \cdot | (S_n)_{|n|\leq K}\big)\Big\| = 0 \quad \text{a.s.}$$

The same type of argument shows that V may be replaced by 0 and that the role of U and V may be interchanged. The assertion now follows by using the triangle inequality. □

In the proof of Theorem 1.6 below we use the notation $\beta(X, Y)$ to denote the *dependence coefficient* of random variables X and Y defined as

$$\beta(X, Y) := E\| P(Y \in \cdot | X) - P(Y \in \cdot) \|.$$

We shall need the inequality

$$\beta(X, Y) \leq 2E\| P(Y \in \cdot | X) - P(Z \in \cdot) \|, \tag{4.2}$$

which holds for any random variable Z taking values in the same measurable space as Y. This follows from the inequality

$$\|P(Y \in \cdot) - P(Z \in \cdot)\| \le E\|P(Y \in \cdot|X) - P(Z \in \cdot)\|,$$

which is easily deduced from (3.1) and (3.2).

PROOF OF THEOREM 1.6. Let $K \le m$. Apply Corollary 4.1 with $U = -S_{-m}$ and $V = -S_{-m}$, and combine with the inequality

$$\begin{aligned} E\|P\big((S_n - S_m, S_{-n} - S_{-m})_{n \ge N} \in \cdot|(S_n)_{|n| \le K}\big) \\ - P\big((S_n - S_m, S_{-n} - S_{-m})_{n \ge N} \in \cdot\big)\| \\ =: \varepsilon_{m,K}, \end{aligned}$$

where we again use (3.1) and (3.2). This yields

$$\begin{aligned} \limsup_{N \to \infty} E\Big\|P\big((S_n - S_m, S_{-n} - S_{-m})_{n \ge N} \in \cdot\big) \\ -P\big((S_n, S_{-n})_{n \ge N} \in \cdot|(S_n)_{|n| \le K}\big)\Big\| \le \varepsilon_{m,K}. \end{aligned} \tag{4.3}$$

Now use (4.2) and (4.3) to obtain

$$\limsup_{N \to \infty} \beta\big((S_n)_{|n| \le K}, (S_n)_{|n| \ge N}\big) \le 2\varepsilon_{m,K}. \tag{4.4}$$

But if $\mathscr{F}_\infty$ is trivial, then $\lim_{m \to \infty} \varepsilon_{m,K} = 0$ for every fixed K (by a standard martingale argument) and so

$$\lim_{N \to \infty} \beta\big((S_n)_{|n| \le K}, (S_n)_{|n| \ge N}\big) = 0 \quad \text{for all } K \ge 0.$$

By the same token, the latter is equivalent to triviality of $\mathscr{G}_\infty$. Thus the second assertion is proved. The proof of triviality of $\mathscr{G}_\infty^+$ from triviality of $\mathscr{F}_\infty^+$ is an easy adaptation of the above proof. □

5. Proof of Corollary 1.8. The proof of Corollary 1.8 uses coupling. Let $(Z_n^1)_{n \ge 1}$ and $(Z_n^2)_{n \ge 1}$ be two arbitrary processes defined on probability spaces $(\Omega^1, \mathscr{F}^1, P^1)$ and $(\Omega^2, \mathscr{F}^2, P^2)$, respectively. A *coupling* is a construction of these processes on a common probability space $(\Omega^1 \times \Omega^2, \mathscr{F}^1 \times \mathscr{F}^2, P^{1,2})$ such that $P^{1,2}|_{\mathscr{F}^1} = P^1$ and $P^{1,2}|_{\mathscr{F}^2} = P^2$. Given two coupled processes taking values in the same measurable space $(\Omega^1, \mathscr{F}^1) = (\Omega^2, \mathscr{F}^2)$, we say that the coupling is *successful* if $P^{1,2}(Z_n^1 = Z_n^2$ for all n sufficiently large$) = 1$.

The key to Corollary 1.8 is the following maximal coupling theorem of Goldstein (1979).

MAXIMAL COUPLING THEOREM. *Let $(Z_n^1)_{n \ge 1}$ and $(Z_n^2)_{n \ge 1}$ be two arbitrary processes taking values in the same Borel space. There exists a successful coupling $P^{1,2}$ iff P^1 and P^2 agree on the tail σ-field.*

The idea is to apply this theorem to two copies (S_n^1) and (S_n^2) of our sum process (S_n) obtained by conditioning on $\{X_1 = 0\}$ and $\{X_1 = 1\}$, respectively.

That is,

$$P^1\big((S_n^1) \in \cdot\big) = P\big((S_n) \in \cdot|X_1 = 0\big),$$
$$P^2\big((S_n^2) \in \cdot\big) = P\big((S_n) \in \cdot|X_1 = 1\big).$$

If $\mathscr{G}_\infty^+$ is trivial, then obviously

$$P^1\big((S_n^1)_{n\geq 1} \in \cdot\big) = P^2\big((S_n^2)_{n\geq 1} \in \cdot\big) \quad \text{on } \mathscr{G}_\infty^+$$

and so by Goldstein's theorem there exists a successful coupling such that

$$P^{1,2}\big(S_n^1 = S_n^2 \text{ for all } n \text{ sufficiently large}\big) = 1.$$

Now let $(T_k^1)_{k\geq 1}$ and $(T_k^2)_{k\geq 1}$ denote the corresponding sequences of random positive times at which X_n^1 and X_n^2 assume the value 1. Then we have for any $t \geq 0$,

$$\begin{aligned} &|P(T_{k+1} - T_k > t|X_1 = 0) - P(T_{k+1} - T_k > t|X_1 = 1)| \\ &= \big|P^1\big(T_{k+1}^1 - T_k^1 > t\big) - P^2\big(T_{k+1}^2 - T_k^2 > t\big)\big| \\ &\leq P^{1,2}\big(T_k^1 < \tau^{1,2} \text{ or } T_k^2 < \tau^{1,2}\big), \end{aligned} \tag{5.1}$$

where

$$\tau^{1,2} = \inf\big\{m \geq 1 \colon S_n^1 = S_n^2 \text{ for all } n \geq m\big\}$$

is the coupling time. Since $P(\lim_{k\to\infty} T_k = \infty) = 1$, we have $P^{1,2}(\lim_{k\to\infty} T_k^1 = \lim_{k\to\infty} T_k^2 = \infty) = 1$ and so

$$\lim_{k\to\infty} P^{1,2}\big(T_k^1 < \tau^{1,2} \text{ or } T_k^2 < \tau^{1,2}\big) = P^{1,2}\big(\tau^{1,2} = \infty\big) = 0. \tag{5.2}$$

Conditioning on $\{X_1 = 1\}$ is natural because the process $(T_{k+1} - T_k)_{k\geq 1}$ is stationary given $\{X_1 = 1\}$, that is,

$$P(T_{k+1} - T_k > t|X_1 = 1) \text{ is independent of } k. \tag{5.3}$$

For this well known fact and its implications we refer to Kakutani (1943) and Kac (1947). If we combine (5.1) to (5.3), then the assertion follows. □

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