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PERIODICITY AND ABSOLUTE REGULARITY

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ABSTRACT
For a stationary ergodic process it is proved that the dependence coefficient associated with absolute regularity has a limit connected with a periodicity concept. Similar results can then be obtained for stronger dependence coefficients. The periodicity concept is studied separately and it is seen that the double tail σ-field can be trivial while the period is 2. The paper imbeds renewal theory in ergodic theory. The total variation metric is used.

1. Introduction

We study some "total variation" properties for a stationary sequence similar to 0–2 theorems for Markov chains.

Random variables are measurable mappings on a normalized measure space, the probability space. They induce a measure on their range, called their distribution. Let $\xi := (\xi_n)_{n\in\mathbf{Z}}$ be a sequence of random variables (*a process*) with values in a common measurable space. Write $T\xi$ for the process with

$$(T\xi)_n := \xi_{n+1}, \qquad n \in \mathbf{Z}.$$

In most of our results below we may assume that ξ is the coordinate process, i.e. the identity on sequence space, where T corresponds naturally to the shift transformation (see also the end of section 2). If $T\xi$ is distributed as ξ we say that ξ is *stationary*. Denote $\xi_+ := (\xi_n)_{n\geqq 1}$ and $\xi_- := (\xi_n)_{n\leqq 0}$. We say that tail $(\xi_+) := \bigcap_n \sigma((T^n\xi)_+)$ is trivial if it contains only sets with probability 0 or 1. We investigate here a periodicity concept for processes. Furthermore we discuss an asymptotic independence condition for processes, called *absolute regularity*, first studied by Volkonskii and Rozanov [24] who attributed it to Komogorov, and later introduced during the study of Bernoulli shifts under the name *weak*

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Bernoulli by Friedman and Ornstein [9]. The latter name is often used for countably valued processes. It can be defined as follows. The *total variation* $\|\nu\| = \|\nu\|_{\mathcal{F}}$ of a signed measure ν defined on a σ-field $\mathcal{F}$ is given by

$$\|\nu\| = \sup_{F \in \mathcal{F}} |\nu(F)| + |\nu(F^c)|.$$

Note that if $\mathcal{F}$ is replaced by a sub σ-field of $\mathcal{F}$ then the total variation decreases. This causes the monotonicity in total variation expressions below.

Let P_X denote the distribution of a random variable (vector) X. If X and Y are random variables on the same probability space, define their dependence

$$\beta(X, Y) := \tfrac{1}{2}\| P_{X,Y} - P_X \times P_Y \|.$$

It vanishes if X and Y are independent. Define as a measure of asymptotic independence of the past and the far future

$$\beta_n := \beta(\xi_-, (T^n\xi)_+), \qquad n \geqq 0.$$

We say ξ is absolutely regular if $\lim_{n\to\infty} \beta_n = 0$. For ergodic stationary processes ξ it will be shown that if $\beta_n < 1$ for some n, then

$$\beta_n \downarrow 1 - 1/p \qquad \text{as } n \to \infty \tag{1.1}$$

for an integer $p \geqq 1$ and we shall see that then ξ is in fact a "periodic" version of an absolutely regular process.

For a stationary ergodic process ξ the notion "periodicity" seems sufficiently nice to be studied also in isolation from absolute regularity. Note that the set of integers k for which

$$\| P_{\xi_-,(T^n\xi)_+} - P_{\xi_-,(T^{n+k}\xi)_+} \| \downarrow 0 \qquad \text{as } n \to \infty \tag{1.2}$$

has the form $p\,\mathbf{Z}$ or consists of $\{0\}$ only. We shall say that the *process* ξ has *period* p in the first case and has infinite period otherwise. If p is finite, then it will be seen that tail (ξ_+) is atomic but that its number r of atoms may be less than p. This phenomenon occurs for the well known skew product example (4.10). However in the absolutely regular situation (1.1) these numbers coincide again as is known in Markov chain theory where it is connected with the notion "cyclic moving subclass". For stationary ergodic sequences one has

$$\text{absolutely regular} \Rightarrow p = 1 \Rightarrow \text{tail } (\xi_+) \text{ trivial}.$$

For stationary Markov chains these notions coincide but by the examples at the end of section 4 this is not true in general.

In section 2 we discuss the "total variation" limit theorems. They are based on the simple fact that ergodic probability measures either coincide or are mutually disjoint. A result in Bradley [5] suggested the use we make of this property. In section 3 we study periodicity and indicate questions that arise when one formulates the notion periodicity for transformations instead of processes. This may even be more natural. Section 4 discusses examples. Section 5 considers absolute regularity for discrete time. At the end of the section we show how limit theorems for non-stationary processes could be obtained from them. Finally in section 6 we discuss a generalization to continuous time where no periodicity occurs.

2. Statement of the limit theorems

The result below shows for a process ξ with period p what happens if $k \notin p\mathbf{Z}$ in (1.2). Related earlier results in Berbee [2], p. 127, were only satisfying for countably valued mixing processes. However the "window-frame method" used there has some interest from a philosophical point of view.

THEOREM 2.1. *Suppose ξ is an ergodic stationary sequence. For any integer k*

$$\lim_{n\to\infty} \| P_{\xi_-,(T^n\xi)_+} - P_{\xi_-,(T^{n+k}\xi)_+} \| = 0 \quad \text{or} \quad 2. \tag{2.1}$$

So either the measures in (2.1) are mutually singular for all n or else they are asymptotically the same.

Ornstein and Sucheston [18] used the term 0–2 theorem in a study of Markov operators on a σ-finite measure space. There are clearly relations here (see also the application following the proof of Proposition 4.1), but in general the result above seems different.

In section 3 we study also the tail of ξ and for $p = 1$ we may conclude from these results that ξ is *mixing*, i.e.

$$\lim_{n\to\infty} P(\xi \in A,\ T^n\xi \in B) = P(\xi \in A)P(\xi \in B).$$

We assume here that the sets above are in the σ-field generated by all ξ_n-variables. The example below shows that from a certain point of view this generalizes renewal theory.

EXAMPLE 2.1. Suppose ξ is a stationary ergodic 0–1 valued process such that, given $\{\xi_0 = 1\}$, the set $\{n : \xi_n = 1\}$ has the form

$$\cdots < S_{-1} < S_0 = 0 < S_1 < \cdots$$

and we assume that (conditionally) the increments of (S_n) form an i.i.d. sequence with distribution F. If $F\{k\}>0$ one checks easily that the measures in (2.1) for $n=0$ are not mutually singular. Hence if g.c.d.$\{k : F\{k\}>0\}=1$ then ξ has period $p=1$, and because ξ is mixing we have the discrete renewal theorem

$$\lim_{n\to\infty} P(\xi_n = 1 \mid \xi_0 = 1) = P(\xi_0 = 1).$$

A stationary sequence as above can be constructed as in [21], ergodicity following from Kolmogorov's 0–1 law for i.i.d. sequences.

Let us now discuss absolute regularity. For ξ mixing Bradley [5] obtained the aperiodic version of the theorem below, strengthening a result in Volkonskii and Rozanov [24]. Ledrappier [15] gave a criterion for absolute regularity that is discussed in Note 5.1.

Define the double tail σ-field of ξ as $\mathcal{F}_\infty := \bigcap_n \sigma(\xi_i : |i| \geqq n)$.

THEOREM 2.2. *Suppose ξ is stationary ergodic. If $\beta_n < 1$ for some n, then ξ has finite period p and* (1.1) *holds. Moreover*

(i) *the double tail σ-field of ξ is partitioned by $\bigcup_{0\leqq i<p}\{T^i\xi \in E\}$ into atoms that are T^p-invariant,*

(ii) *the process ξ conditioned by the event $\{T^i\xi \in E\}$ is absolutely regular.*

NOTE. Given $\{T^i\xi \in E\}$ the process $\tilde{\xi}$ defined by

$$\tilde{\xi}_n := (\xi_{np+i})_{0\leqq i<p}, \qquad n \in \mathbf{Z},$$

is stationary. This need not be true for ξ.

It will be clear that the result above generalizes the notion "cyclic moving subclass" of Markov chain theory (see e.g. [6]), but as we mentioned already, this generalization does not carry over to the notion periodicity.

Bradley [4] remarks that the theorem above carries over easily to several stronger dependence coefficients by using his earlier results on these coefficients for mixing ξ in combination with the decomposition of our theorem (see also its proof). Following the notation of [12] we get that if ξ is ergodic stationary then unless for all n, $\phi_n = 1$ (or e.g. $I(n)=\infty$) we will have

$$\lim_{n\to\infty} \phi_n = 1 - \frac{1}{p} \left(\lim_{n\to\infty} I(n) = \log p \right)$$

where p is the period of ξ. However for the weaker dependence coefficient α_n it holds that $\lim_{n\to\infty} \alpha_n$ may be any value in $[0,\frac{1}{4}]$ by the example of theorem 6 of [3].

Before continuing we discuss some conventions. We study a stationary process

(ξ_n) with values in a measurable space $(\Gamma, \mathcal{T})$, so its distribution is defined on the product space $(\Gamma, \mathcal{T})^{\mathbf{Z}}$, and we can usually assume, without losing generality, that (ξ_n) is the *coordinate process* on this *sequence space*, given by

$$\xi_n(x) = x_n, \qquad x \in \Gamma^{\mathbf{Z}}.$$

We also write $x = (x_-, x_+)$ as above to denote the position of the first coordinate.

For measures μ' and μ'' on the same measurable space we define

$$\mu' \wedge \mu'' := \mu' - (\mu' - \mu'')^+ = \mu'' - (\mu'' - \mu')^+ \tag{2.4}$$

and if μ' and μ'' are probability measures they have *mass* $q := \|\mu' \wedge \mu''\|$ *in common*, such that

$$\tfrac{1}{2}\|\mu' - \mu''\| = 1 - q. \tag{2.5}$$

If f' (f'') denotes the density of μ' (μ'') with respect to e.g. $\mu = \frac{1}{2}(\mu' + \mu'')$ then we may also write

$$\mu' \wedge \mu'' = \min(f', f'')\mu.$$

3. Periodicity

We prove Theorem 2.1 but first show the following "contraction" lemma, a somewhat technical but simple consequence of the ergodic theorem.

LEMMA 3.1. *Let T be a transformation on a measurable space and suppose P and Q are probability measures on this space, not necessarily T-invariant. Assume $\mathcal{F}_n$, $n \geqq 1$, forms a decreasing sequence of σ-fields on this space, with a T-invariant intersection $\mathcal{F}_\infty$. If P and Q have mass in common on $\mathcal{F}_\infty$ and T is ergodic measure preserving for both P and Q on $\mathcal{F}_\infty$, then*

$$\lim_{n\to\infty} \|P - Q\|_{\mathcal{F}_n} = 0. \tag{3.1}$$

PROOF. Let $\mu := \frac{1}{2}(P + Q)$. Denote by f (and g) the density of P (and Q) with respect to μ. By the martingale convergence theorem

$$\|P - Q\|_{\mathcal{F}_n} = \int |E_\mu(f \mid \mathcal{F}_n) - E_\mu(g \mid \mathcal{F}_n)|\, d\mu$$

$$\to \int |E_\mu(f \mid \mathcal{F}_\infty) - E_\mu(g \mid \mathcal{F}_\infty)|\, d\mu = \|P - Q\|_{\mathcal{F}_\infty}.$$

So if P and Q coincide on $\mathcal{F}_\infty$ we have (3.1). Otherwise, by ergodicity, P and Q are mutually singular on $\mathcal{F}_\infty \subset \mathcal{F}_n$ and the terms in (3.1) all equal 2. □

PROOF OF THEOREM 2.1. We may assume ξ is the coordinate process. Define on the sequence space

$$P := P_{\xi_-,\xi_+} \quad \text{and} \quad Q := P_{\xi_-,(T^k\xi)_+}$$

and let $\mathcal{F}_n$ be generated by $(\xi_i, |i| \geqq n)$. Note that by stationarity (and monotonicity) the assertion of the lemma would imply the theorem. Only some care is needed in verifying the properties of Q in the lemma because Q may not be T-invariant. Define

$$Sx := (x_-, (Tx)_+) \qquad \text{for sequences } x.$$

Note also that

$$S^kTx = ((\dots, x_{-1}, x_0), (x_{k+1}, x_{k+2}, \dots)),$$

$$TS^kx = ((\dots, x_{-1}, x_k), (x_{k+1}, x_{k+2}, \dots))$$

coincide except possibly at the 0th coordinate. Hence for $A \in \mathcal{F}_\infty$ in the double tail σ-field

$$S^kTx \in A \qquad \text{iff } TS^kx \in A. \tag{3.2}$$

Because P is T-invariant, (3.2) implies that on $\mathcal{F}_\infty$ also $Q = PS^{-k}$ is T-invariant. Moreover if $A \in \mathcal{F}_\infty$ is T-invariant then also by this property $S^{-k}A$ is T-invariant, so ergodicity of T under P on $\mathcal{F}_\infty$ implies ergodicity under Q. Thus the lemma implies the theorem. ∎

THEOREM 3.2. *Let ξ be stationary ergodic with finite period p. The double tail σ-field of ξ is partitioned into at most p atoms of the form*

$$\{T^i\xi \in E\}, \qquad 0 \leqq i < r,$$

where r divides p. Moreover this tail field coincides with the T^p-invariant σ-field.

It follows that the double tail σ-field of ξ is trivial if $p = 1$.

PROOF. We use the notation of the proof above and let $E \in \mathcal{F}_\infty$ with positive probability. Because (1.2) holds with $k = p$ we have

$$P(A \cap E) = P(A \cap S^{-p}E) \tag{3.3}$$

for A ξ_--measurable, because also $E \in \mathcal{F}_\infty \subset \bigcap_n \sigma(\xi_-, (T^n\xi)_+)$. By stationarity A in (3.3) may also be any finite dimensional set (here we use (3.2) again). By stationarity we also have from (1.2)

$$\lim_{n\to-\infty} \| P_{(T^n\xi)_-,\xi_+} - P_{(T^{n+p}\xi)_-,\xi_+} \| = 0.$$

Writing

$$S_{-}x := ((Tx)_{-}, x_{+})$$

we get for A finite dimensional

$$P(A \cap E) = P(A \cap S_{-}^{-p}E).$$

Combining this with (3.3) and using that $S_{-}^{-p}S^{-p}E = T^{-p}E$ we obtain

$$P(A \cap E) = P(A \cap T^{-p}E).$$

Let $A = A_\varepsilon$ approximate E. We get $P(E) = P(E \cap T^{-p}E)$ so E is a.s. T^p-invariant. Hence

$$\bigcup_{0 \leqq i < p} T^{-i}E$$

is a.s. T-invariant and by ergodicity has probability 1. Therefore $P(E) \geqq 1/p$ and it follows that $\mathcal{F}_\infty$ is atomic under P.

Assume $E \in \mathcal{F}_\infty$ as above was chosen to be an atom. Let r be the smallest i with $E \cap T^{-i}E \neq \varnothing$ a.s. Necessarily because T is measure preserving and E is an atom, one even has $E = T^{-r}E$ a.s. So, also because T is measure preserving, the sequence $E, T^{-1}E, T^{-2}E, \dots$ repeats itself with period r and by the definition of r the sets $T^{-i}E$ and $T^{-j}E$ are a.s. disjoint iff $i - j$ does not divide r and these sets coincide otherwise. So r divides p because $E = T^{-p}E$ a.s. and the a.s.-invariant set $\bigcup_{0 \leqq i < r} T^{-i}E$ partitions $\mathcal{F}_\infty$. □

NOTE. It will be clear that also the T^p- and T^r-invariant σ-fields coincide.

COROLLARY 3.3. *If ξ is stationary ergodic with finite period p, then* tail(ξ_+) *and* tail(ξ_-) *coincide* a.s. *with the double tail σ-field, and so with the T^p-invariant σ-field.*

PROOF. By the approximation argument in Doob [6], pp. 458–9, each T^p-invariant event coincides a.s. with an event in tail (ξ_+), which of course is contained in the double tail σ-field. By Theorem 3.2 this a.s.-inclusion is an a.s.-equality. This proves the assertion for tail (ξ_+), which clearly is partitioned into atoms by $\bigcup_{0 \leqq i < r} T^{-i}E_+$ but now with $E_+ \in \text{tail}(\xi_+)$. The same argument applies to tail(ξ_-) also. □

Vanishing of coefficients in (1.2) imposes a strong property on the process. If e.g. $P_{\xi_-, \xi_+} = P_{\xi_-, (T\xi)_+}$ then ξ is a Bernoulli process in case ξ is ergodic because

$$P(\xi_- \in B_-, (T^n\xi)_+ \in B_+) = P(\xi_- \in B_-)P(\xi_+ \in B_+) \qquad \text{for } n = 1, 2, \dots .$$

This follows because the left-hand side is the same for $n = 1, n = 2, \ldots$ and its limit can be identified as the right-hand side by the ergodic theorem.

The results are discussed here from a probabilistic ("process") point of view, but there are important connections with an ergodic ("transformation") point of view.

Let T be an ergodic, measure preserving transformation with finite entropy on the unit interval, provided with a probability measure. Below we assume that $\mathcal{P}$ is a generating partition with finite entropy. Then

$$\xi_n(\omega) := i \qquad \text{if } T^n\omega \in P_i, \quad n \in \mathbf{Z},$$

determines a stationary process $\xi \equiv \xi^{\mathcal{P}}$, say with period $p = p^{\mathcal{P}}$. One would like to consider $p_T := \inf_{\mathcal{P}} p^{\mathcal{P}}$. Possibly nicer from the point of view of ergodic theory is $\bar{p}_T$, obtained as p_T, but with (1.2) in the definition of p replaced by the weaker requirement

$$\lim_{n\to\infty} E\bar{d}_n((\xi_1^n \mid \xi_-), (\xi_{k+1}^{k+n} \mid \xi_-)) = 0,$$

where $\xi_i^j := (\xi_i, \ldots, \xi_j)$ and for the $\bar{d}$-notation [22] is followed. Investigation of p_T is far from simple. One is interested in the invariant

$$\delta_{n+1}^k := \inf_{\mathcal{P}} \tfrac{1}{2} \| P_{\xi_-, (T^n\xi)_+} - P_{\xi_-, (T^{n+k}\xi)_+} \|$$

and particularly in when δ is attained. Here ξ should read $\xi^{\mathcal{P}}$. This is related to isomorphism problems. See also the skew product example below.

Assume T is a K-automorphism. Rohlin and Sinaǐ [20] proved that then both left and right tail σ-fields of $\xi^{\mathcal{P}}$ are trivial. Ornstein and Weiss [19] showed that one could always refine a finite $\mathcal{P}$ to a finite $\mathcal{Q}$ such that the double tail σ-field of $\xi^{\mathcal{Q}}$ is a.s. the entire σ-field, and then certainly $P^{\mathcal{Q}} = \infty$. The requirement that p_T is finite implies that there exists a partition $\mathcal{P}$ for which $\xi^{\mathcal{P}}$ has trivial double tail σ-field. Possibly one cannot find such $\mathcal{P}$ for certain K-automorphisms T.

4. Examples of periodicity

The first example shows that past and future can be curiously entertwined while $p = 1$. The second example suggests that periodicity may be a nice way to say more about skew products.

Throughout this section $S := (S_n)$ will be a random walk with independent, identically distributed increments (η_n) determined by

$$S_0 := 0; \quad S_n - S_{n-1} = \eta_n, \qquad n \in \mathbf{Z}. \tag{4.1}$$

EXAMPLE 4.1 (random walk). Suppose the increments of S have distribution

$$P(\eta_n \in I) = \int_I \frac{c}{1+|x|^\alpha}\,dx, \qquad 0<\alpha<1. \tag{4.2}$$

Then $(S_n)_{n\geqq 0}$ and $(S_{-n})_{n\geqq 0}$ are independent and by symmetry equally distributed. Moreover such a random walk is transient, i.e. any bounded set contains only finitely many S_n and (S_n) is "oscillating" making occasionally large jumps between left and right half axis (see [8], p. 204). As in [2] or [25] one can arrange $(S_n)_{n\in\mathbf{Z}}$ into an ascending sequence of random variables specified by

$$\cdots < S_{\sigma_{-1}} < S_{\sigma_0} = 0 < S_{\sigma_1} < \cdots$$

and its increments $\xi_n := S_{\sigma_n} - S_{\sigma_{n-1}}$, $n \in \mathbf{Z}$, form a stationary ergodic sequence. On the interval $(0,1)$ the measures

$$P(S_1 \in \cdot\,, S_1 > 0) \quad \text{and} \quad P(S_2 \in \cdot\,, S_2 > S_1 > 0) \tag{4.3}$$

have positive mass α in common. Similarly the measures

$$P(S_1 \in \cdot\,, S_1 > 0, (\eta_-, (T\eta)_+) \in \cdot\,)$$

and

$$P(S_2 \in \cdot\,, S_2 > S_1 > 0, (\eta_-, (T^2\eta)_+) \in \cdot\,)$$

also have mass α in common, because the vector of the form $(\tilde{\eta}_-, \tilde{\eta}_+)$ that is added to both of the expressions in (4.3) is independent of the other random variables of these expressions. Let $(\tilde{S}_n)$ denote in each of these cases the random walk with increments $(\tilde{\eta}_n)$. These Cauchy random walks are transient and miss $(0,1)$ with probability $\gamma > 0$. Then it follows that the distributions of

$$((S_{\sigma_n})_{n\leqq 0}, (S_{\sigma_n})_{n\geqq 1}) \quad \text{and} \quad ((S_{\sigma_n})_{n\leqq 0}, (S_{\sigma_n})_{n\geqq 2})$$

have mass at least $\alpha\gamma > 0$ in common and so ξ has period $p = 1$ by Theorem 2.1.

EXAMPLE 4.2 (skew product). Let S described by (4.1) be a random walk on the integers such that

$$\text{g.c.d.}\,\mathcal{L} = 1, \qquad \text{where } \mathcal{L} := \{i \in \mathbf{Z}: P(\eta_0 = i) > 0\}. \tag{4.4}$$

Assume ρ is a stationary ergodic sequence of real random variables such that ρ and η are independent and also P_ρ is non-atomic. The last assumption implies that ρ has no "recurring" patterns in the sense that

$$P(\rho = T^k\rho) = 0 \qquad \text{for } k \neq 0. \tag{4.5}$$

The shift T_ξ associated with the process

$$\xi_n = (\eta_n, \rho_{S_n}) \tag{4.6}$$

will be studied here. It is a factor of a skew product that is defined here as the shift T_ξ but with (ρ_k) replaced by $(T^k\rho)$. In case S_n visits all integers a.s. the ξ-sequence determines the ρ-sequence a.s. and one observes that both these T_ξ-shifts are isomorphic. From a general theorem in Kakutani [13] ergodicity of ξ is known by (4.4). We shall also use the following inequality.

PROPOSITION 4.1. *Under the conditions above we have*

$$\| P_{\xi_-,(T^n\xi)_+} - P_{\xi_-,(T^{n+k}\xi)_+} \| \leqq \| P_{S_{n+1}} - P_{S_{n+k+1}} \|, \tag{4.7}$$

and equality holds if the random walk is recurrent.

The invariant δ of section 3 may be useful in the recurrent case.

PROOF. Let us first note that for random variables X' and X'' on a common probability space with the same space of values, we have the "coupling" property

$$\| P_{X'} \wedge P_{X''} \| \geqq P(X' = X''). \tag{4.8}$$

By Schwarz [22] equality can be attained on a suitable probability space for any pair of marginal distributions. There and in the later result of [2] coupling arguments as below can be found.

By the Markov property, the right-hand side in (4.7) equals

$$\| P_{S_-,(T^nS)_+} - P_{S_-,(T^{n+k}S)_+} \|.$$

Denote this as $\| P_{X'} - P_{X''} \|$ and let q be the mass that these probability measures have in common. Similarly as mentioned above we can construct a probability space such that equality holds in (4.8), i.e. with probability q

$$S'_- = S''_- \quad \text{and} \quad (T^nS')_+ = (T^{n+k}S'')_+. \tag{4.9}$$

We may suppose additionally that there is given a process $\rho' \equiv \rho''$ independent of these random walks and distributed as ρ. By (4.9) we have, with the obvious notation, with probability at least q

$$\xi'_- = \xi''_- \quad \text{and} \quad (T^n\xi')_+ = (T^{n+k}\xi'')_+$$

which implies (4.7) by (4.8) for the ξ-processes and (2.5).

To prove the second assertion we let $\| P_{X'} - P_{X''} \|$ denote now the left-hand side of (4.7). Suppose these measures have mass q' in common. We can construct a probability space with processes ξ' and ξ'' marginally distributed as ξ, such that the event A for which

$$\xi'_- = \xi''_- \quad \text{and} \quad (T^n\xi')_+ = (T^{n+k}\xi'')_+$$

has probability q'. To do this one first constructs the random variables above as before and then extends the probability space to get all of ξ' and ξ'', with the right marginals. On A there holds

$$\rho'_{S'_n} = \rho''_{S''_n}, \quad S'_n = S''_n, \qquad n \leqq 0.$$

By recurrence of S'_- and S''_- on $\mathbf{Z}$ we have $\rho'_k = \rho''_k$ for all $k \in \mathbf{Z}$ on A. Also

$$\rho'_{S'_n} = \rho''_{S''_{n+k}}, \quad S'_n = S''_{n+k}, \qquad n \geqq 1,$$

and, again by recurrence, writing $Z = S''_{n+k+1} - S'_{n+1}$

$$\rho'_k = \rho''_{k+Z} \qquad \text{for all } k \in \mathbf{Z} \text{ on } A.$$

By (4.6) we should have $Z = 0$ on A and so

$$q' \leqq P(Z = 0) \leqq \| P_{S_{n+k+1}} \wedge P_{S_{n+1}} \|$$

by (4.8) for the S-variables. This proves the converse of (4.7). The study in [14] of (4.10) makes a deep use of a "recurrent pattern" argument as above. □

From the 0–2 law of theorem 7(d) in [17] or, in case equality holds, from Theorem 2.1 it follows that the right-hand side of (4.7) converges for $n \to \infty$ iff there is some n, i for which

$$P(S_n = i),\ P(S_{n+k} = i) > 0$$

or also iff k divides

$$p' := \text{g.c.d.}\{i - j:\ i, j \in \mathscr{L}\}.$$

So by Proposition 4.1 the period p of ξ is at most p' and equals p' if the random walk is recurrent.

To study the double tail σ-field of ξ it is sufficient by Theorem 3.2 to investigate much weaker properties of the shift T_ξ. This shift is a factor of the skew product referred to above. Let T_ρ be the shift on ρ-sequence space. Following the argument in Adler and Shields [1] it can be concluded easily from Kakutani [13] that the skew product is weakly mixing under P_ξ iff the family $\{T^i_\rho \times T^j_\rho\}_{i,j \in \mathscr{L}}$ is ergodic under $P_\rho \times P_\rho$ or equivalently if this holds for

$\{T_\rho^{p'} \times \mathrm{id}, \mathrm{id} \times T_\rho^{p'}\}$, and for this it is necessary and sufficient that $T_\rho^{p'}$ is ergodic under P_ρ. Hence by Theorem 3.2 the process ξ has trivial (double) tail σ-field if the $T^{p'}$-invariant σ-field of ρ is trivial. This improves Meilijson [16] somewhat and indicates the use of periodicity.

Let us now discuss some specific examples. The literature on skew products considers only transformations but the choice of the process ξ that is meant below will be clear in each case. Examples with $p = 1$ and ρ deterministic were discussed by Shields [23], who discusses a process that is not absolutely regular (weak Bernoulli) and by Feldman [7]. The case where η and ρ are Bernoulli processes with

$$P(\eta_0 = \pm 1) = P(\rho_0 = \pm 1) = \tfrac{1}{2} \tag{4.10}$$

was studied by Kalikow [14] and has $p = 2$ whereas ξ has a trivial double tail σ-field. The transformations associated with the last two examples are not Bernoulli shifts.

5. Absolute regularity

Let us note first that an absolutely regular process ξ has period 1 because

$$\tfrac{1}{2}\| P_{\xi_-,(T^n\xi)_+} - P_{\xi_-,(T^{n+1}\xi)_+} \| \leqq \beta_n + \beta_{n+1} \downarrow 0 \qquad \text{as } n \to \infty.$$

Proof of Theorem 2.2. Suppose $\beta_n < 1$ for some $n \geqq 1$. Then ξ has finite period. To see this note that for $i = n$ the measure $\mu_i := P_{\xi_-,(T^i\xi)_+}$ by (2.5) has mass $\alpha := 1 - \beta_n$ in common with $\mu := P_{\xi_-} \times P_{\xi_+}$, and also μ_i for $i > n$ has at least mass α in common with μ (by stationarity of ξ_+). Because μ is finite not all μ_i can be mutually disjoint and so ξ has finite period.

We will assume that ξ is a coordinate process. At the end of section 3 we have seen that tail(ξ_+) and tail(ξ_-) are partitioned into r atoms of the form $\{(T^i\xi)_+ \in E_+\}$ and $\{(T^i\xi)_- \in E_-\}$ respectively, $0 \leqq i < r$, that coincide a.s. for each i and are T^r-invariant. We write these sets also as $\{\xi_\pm \in T^{-i}E_\pm\}$. Let

$$P^i(\cdot) := P(\cdot \mid T^{-i}(E_- \times E_+)).$$

The measures $P^i_\pm := P^i_{\xi_\pm}$ are concentrated on $T^{-i}E_\pm$. Using (2.5) and the decomposition $P = (1/r)\Sigma_{0 \leqq i < r} P^i$ we have

$$1 - \beta_n = \left\| \left(\frac{1}{r}\sum_i P^i\right) \wedge \left(\frac{1}{r^2}\sum_{j,k} P^j_- \times P^k_+\right) \right\|_{\mathscr{F}_{0,n}}$$

where $\mathscr{F}_{m,n} := \sigma((T^m\xi)_-, (T^n\xi)_+)$. The measure P^i is concentrated on

$T^{-i}(E_- \times E_+)$ and $P_-^j \times P_+^k$ on $T^{-j}E_- \times T^{-k}E_+$, so they can have mass in common only if $i = j = k$. Thus one observes

$$1 - \beta_n = \frac{1}{r} \sum_i \left\| P^i \wedge \frac{1}{r} P_-^i \times P_+^i \right\|_{\mathscr{F}_{0,n}}. \tag{5.1}$$

For some n we have $\beta_n < 1$ and some term, say the ith, in the sum above is positive. Because β_n is non-increasing we may assume r divides n. Let us now compare for this i

$$P^i \quad \text{and} \quad P_-^i \times P_+^i. \tag{5.2}$$

The process $\tilde{\xi}_n := (\xi_{nr+i})_{0 \leqq i < r}$ is stationary and has trivial right and left tail under P^i. As in Bradley [5] the measures (5.2) on $\bigcap_n \mathscr{F}_{-n,n}$ are ergodic, measure preserving under T^r and by Lemma 3.1

$$\lim_{n \to \infty} \| P^i - P_-^i \times P_+^i \|_{\mathscr{F}_{-n,n}} = 0. \tag{5.3}$$

Because T is measure preserving under P this holds for any i and we may replace $\mathscr{F}_{-n,n}$ by $\mathscr{F}_{0,2n}$. From (5.1) it follows now that

$$1 - \beta_n \uparrow \frac{1}{r} \qquad \text{as } n \to \infty.$$

We saw that $\tilde{\xi}$ is absolutely regular under P^i and so its period is 1. Thus the period p of ξ divides r. Because $r \leqq p$ we have $r = p$. □

NOTE 5.1. From the argument leading to (5.1) and (5.3) it follows that under (1.1) on $\mathscr{F}_\infty := \bigcap_n \sigma((T^{-n}\xi)_-, (T^n\xi)_+)$ there holds

$$P_{\xi_-, \xi_+} = f P_{\xi_-} \times P_{\xi_+} \tag{5.4}$$

with $f = 0$ outside $\bigcup_{0 \leqq i < p} T^{-i}E_- \times T^{-i}E_+$ and $f = p$ on this set. So if the measures in (5.4) are equivalent on sequence space, provided with any σ-field containing $\mathscr{F}_\infty$, then clearly $p = 1$ and the process ξ is absolutely regular. Ledrappier [15] obtained a similar result for finite valued processes and gives several examples.

NOTE 5.2. If one is only interested in Theorem 2.2, then one could also show that tail (ξ_+) has an atom using [5], lemma 1. Then some of the considerations using aperiodicity would become superfluous.

Results as above can also be used to study processes that are not stationary. Suppose $\tilde{\xi} = (\tilde{\xi}_n)_{n \geqq 1}$ is any process such that

$$P_{\tilde{\xi}} \ll P_\xi \tag{5.5}$$

where $\xi = (\xi_n)_{n\geqq 1}$ is stationary and has trivial tail. Then

$$\lim_{n\to\infty} \| P_{(T^n\xi)_+} - P_\xi \| = 0. \tag{5.6}$$

This follows by using the martingale argument as in Lemma 3.1. The reader will note that (5.5) can be relaxed to the requirement that the mass of the P_ξ-singular component of $P_{(T^n\xi)_+}$ vanishes asymptotically.

6. Absence of periodicity for continuous time

We discuss a way in which Theorem 2.2 can be extended to continuous time such that no periodicity occurs. We require a light measurability condition.

The process (ξ_t) will have its sample paths in the space $\Gamma^{\mathbf{R}}$ provided with a shift invariant σ-field $\mathcal{D}$. Here Γ is any set. If $x \in \Gamma^{\mathbf{R}}$ is a sample path and I an interval denote by x_I the restriction of x to I. Let $\mathcal{D}_I$ be the σ-field consisting of all $D \in \mathcal{D}$ such that if two sample paths x and y coincide on I then $y \in D$ if $x \in D$. We assume D is generated by all $\mathcal{D}_I$ for finite intervals I, and also that for $D \in \mathcal{D}$

$$f(t, x) := 1_D(T_t x)$$

is jointly measurable in t and x.

Assume $\xi := (\xi_t)$ is stationary, i.e. its distribution on $(\Gamma^{\mathbf{R}}, \mathcal{D})$ is shift invariant. It has the continuity property

$$\lim_{t\to 0} P(\{\xi \in D\} \triangle \{T_t\xi \in D\}) = 0. \tag{6.1}$$

To see this note that by stationarity the probability above coincides for each s with

$$\int |f(s,x) - f(s+t,x)| \, P(\xi \in dx).$$

Average over $s \in [0, h]$ and apply Fubini. The assertion (6.1) follows by using that because $f(\cdot, x)$ is measurable for all x

$$\frac{1}{h}\int_0^h |f(s,x) - f(s+t,x)| \, ds \to 0 \qquad \text{as } t \to 0.$$

Denote $\xi_- := \xi_{(-\infty,0]}$ and $\xi_+ := \xi_{(0,\infty)}$ and write

$$\beta_t := \beta(\xi_-, (T_t\xi)_+), \qquad t \geqq 0.$$

Under the measurability conditions above we have

THEOREM 6.1. *If ξ is stationary ergodic then* $\lim_{t\to\infty} \beta_t = 0$ *or* 1.

PROOF. Let ξ^h for any $h>0$ be the discrete time process

$$\xi^h_n := \xi_{(nh,(n+1)h]}, \qquad n \in \mathbf{Z}.$$

We may define $\operatorname{tail}(\xi_+) := \operatorname{tail}(\xi^h_+)$ because $\operatorname{tail}(\xi^h_+)$ is the same for all $h>0$. Assume $\beta_t < 1$ for some $t>0$. Because β_t is non-increasing we may assume h divides t. By Theorem 2.2, ξ^h has finite period and for any atom $\{\xi \in E\}$ in $\operatorname{tail}(\xi_+)$ either the atom $\{T^h\xi \in E\}$ coincides or is disjoint with $\{\xi \in E\}$ a.s. So the function

$$f(h) = P(\{\xi \in E\} \triangle \{T^h\xi \in E\})$$

has values 0 or $2P(\xi \in E)$. By (6.1) this function is continuous and because $f(0)=0$ it vanishes. So $\{\xi \in E\}$ is a.s. invariant and by ergodicity has probability 1. So ξ^h is absolutely regular with period 1 and hence $\beta_t \downarrow 0$. □

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