A. E. BROUWER, A. M. COHEN, J. I. HALL AND H. A. WILBRINK

NEAR POLYGONS AND FISCHER SPACES

To Hans Cuypers and Ivonne Falarz on the occasion of their marriage

ABSTRACT. In this paper we exploit the relations between near polygons with lines of size 3 and Fischer spaces to classify near hexagons with quads and with lines of size three. We also construct some infinite families of near polygons.

1. NEAR POLYGONS

A *near polygon* is a connected partial linear space (X, L) such that given a point x and a line L there is a unique point on L closest to x (where distances are measured in the collinearity graph: two distinct points are adjacent when they are collinear). A near polygon of diameter n is called a *near* $2n$*-gon*, and for $n = 3$ a *near hexagon*. The concept of near polygon was introduced in Shult and Yanushka [20] as a tool in the study of systems of lines in a Euclidean space. A structure theory is developed in Shad and Shult [18] and Brouwer and Wilbrink [8]. Dual polar spaces were characterized by Cameron [11] as near polygons with 'classical point-quad relations'. (See also Shult [19] and Brouwer and Cohen [5].)

By Yanushka's lemma ([20, Prop. 2.5]), any quadrangle (in the collinearity graph of a near polygon) of which at least one side lies on a line with at least three points is contained in a unique geodetically closed subspace of diameter 2, necessarily a nondegenerate generalized quadrangle. Such a subspace is called a *quad*, and a near polygon is said to 'have quads' when any two points at distance 2 determine a quad containing them. When all lines have at least three points this is equivalent to asking that any two points at distance 2 have at least two common neighbours.

In this paper we construct some infinite families of near polygons, and classify near hexagons with lines of length 3 and with quads. This paper is a compilation of the three reports Brouwer *et al.* [6], Brouwer and Wilbrink [7], and Brouwer [4] together with the contributions of the third author, who was referee of [6].

Our main goal is the following theorem.

1.1. THEOREM. *Let* (X, L) *be a near hexagon with lines of size* 3 *and such that any two points at distance* 2 *have at least two common neighbours. Then*

Geometriae Dedicata **49**: 349–368, 1994.

(X, L) is finite, and is one of the eleven near hexagons with parameters as given below.

	v	$t+1$	t_2+1	*NPdim*	*Udim*	*Group*
(i)	759	15	3	22	23	M_{24}
(ii)	729	12	2	24	24	$3^6.2.M_{12}$
(iii)	891	21	5	20	22	$\mathrm{P\Gamma U}(6,2^2)$
(iv)	567	15	3, 5	20	21	$\mathrm{PO}_6^-(3)$
(v)	405	12	2, 3, 5	20	20	$3^5.\mathrm{Sym}(6)$
(vi)	243	9	2, 5	18	18	$(\mathbb{Z}_3\times \mathrm{AGL}(2,3)\times \mathrm{AGL}(2,3)).2$
(vii)	81	6	2, 5	12	12	$\mathrm{P\Gamma U}(4,2^2)\times \mathrm{Sym}(3)$
(viii)	135	7	3	8	15	$\mathrm{PO}_7(2)$
(ix)	105	6	2, 3	8	14	Sym(8)
(x)	45	4	2, 3	8	10	Sym(6) × Sym(3)
(xi)	27	3	2	8	8	Sym(3)wr Sym(3)

Here $v = |X|$, the number of points; $t + 1$ is the number of lines on each point; and $t_2 + 1$ is the number of common neighbours of two points at distance 2 (this need not be constant). The columns headed 'NPdim' and 'Udim' give the near-polygon embedding dimension and the universal embedding dimension. (The near-polygon embedding dimension of a near $2n$-gon with lines of size 3 is the 2-rank of the 0–1 matrix A_n with $(A_n)_{xy} = 1$ when x and y have distance n. The universal embedding dimension is v minus the 2-rank of the point–line incidence matrix. See also Ronan [16].)

A near polygon is called *regular* when its collinearity graph is distance-regular. Shult and Yanushka [20] classified the finite regular near hexagons with lines of size 3, and found that these either are generalized hexagons, or must have parameters as in (i)–(iii), (viii) or (xi). Moreover, they showed that there are unique examples with parameters (iii), (viii) and (xi). Uniqueness of the remaining two finite regular near hexagons was shown in Brouwer [2], [3] (they are obtained from the Steiner system $S(5, 8, 24)$ and from the extended ternary Golay code). The near hexagon (iv) on 567 vertices was first described in Aschbacher [1]; see also Kantor [14].

Given a collection $(X_i, \mathrm{L}_i)(i \in I)$ of near polygons, we can form their *direct product* (cf. [8]) with point set $X = \Pi_i X_i$ and as lines all subsets of X projecting to a single point in X_i for all $i \neq i_0$ and projecting in X_{i_0} onto an element of L_{i_0} (for some $i_0 \in I$). Thus, the unique generalized quadrangle of order $(s, 1)$ is the direct product of two lines of size $s + 1$. Examples (vii), (x) and (xi) are direct products of a quad and a line (so that (xi) is the direct product of three lines).

In the above theorem it is really necessary to require 'with quads'. On the one hand, even for generalized hexagons with lines of size 3 it is a famous open problem whether there can be any infinite examples. On the other hand, many 'degenerate' examples can be obtained by pasting together some lines and quads.

[Maybe a reasonable definition of 'degenerate' in this context is: becoming disconnected by the removal of a proper subspace. When $(X_i, \mathbf{L}_i)(i = 1, 2)$ are two near polygons meeting in a common proper subspace that is classical in both (cf. below), then $(X_1 \cup X_2, \mathbf{L}_1 \cup \mathbf{L}_2)$ is again a near polygon, and is degenerate in the above sense. In particular this holds when $X_1 \cap X_2$ is a point or a line. This construction can be extended to arbitrary families of near polygons.]

We shall talk about *n-things*, meaning things of cardinality n (as in n-sets, n-cliques, n-lines, n-quads, etc.).

2. Fischer spaces

A *Fischer space* is a linear space $(E, \mathbf{L})$ such that

(i) all lines have size 2 or 3, and
(ii) for any point x the map $\sigma_x: E \to E$ fixing x and all lines through x and interchanging the two points distinct from x on a line of size 3 through x is an automorphism.

One can show that $\sigma_x \sigma_y$ has order 2 or 3 whenever the line xy has size 2 or 3 (respectively). Alternatively, one may define Fischer spaces as partial linear spaces $(E, \mathbf{M})$ such that all lines have size 3, and such that any two intersecting lines determine a subspace isomorphic either to the unique Fischer space F_6 with six points and four 3-lines, or to the affine plane AG(2, 3) with nine points and twelve 3-lines. The connected components of a Fischer space are the connected components of the collinearity graph of this partial linear space. For more details on Fischer spaces, see Buekenhout [9]. In the following two sections we show how to obtain near polygons from Fischer spaces, and conversely Fischer spaces from near polygons.

3. Near polygons from Fischer subspaces of polar spaces

Let $(X, \mathbf{L})$ be a polar space of rank at least 2. For $x \in X$ we write $x^{\perp}$ for the set of all points collinear with x. If we let X^* be the set of maximal totally isotropic subspaces and $\mathbf{L}^*$ the set of subspaces of codimension 1 of the elements of X^*, then $(X^*, \mathbf{L}^*)$ (with containment as incidence) is a near polygon ('dual polar space').

Let $A \subseteq X$ and define $a^* = \{M \in X^* | a \in M\}$ for $a \in X$ and $A^* = \bigcup_{a \in A} a^*$. The set A^* has a graph structure induced by the collinearity graph of (X^*, L^*).

3.1. LEMMA. *A^* preserves distances, that is, two points of A^* have the same distance in A^* as in X^*.*

Proof. Let $M, N \in A^*$. If for some $a \in A$ we have $M, N \in a^*$, then (since a^* is a geodetically closed subspace of X^*) all geodesics (shortest paths) of X^* between M and N lie already in A^*. Otherwise, choose $a \in A \cap M$ and let $K = \langle a, a^\perp \cap N \rangle$, the smallest subspace containing a and $a^\perp \cap N$. Then $K \in a^*$ and K is adjacent to N, so that there is a geodesic in X^* from M to N over K, and this geodesic is in A^*. □

If A^* is a subspace of X^* (i.e. if A^* contains all elements of X^* on an element of L^* as soon as it contains two of them), then A^* is itself a near polygon. If our polar space is of type $\mathrm{O}^+_{2m}(q)$, then every subset of X^* is a subspace (since lines have size 2). If (X, L) is of type $\mathrm{Sp}(2m, q)$ or $\mathrm{U}(2m, q^2)$, then sufficient in order that A^* be a subspace is the condition

(*) *If a and a' are two points of A, noncollinear in (X, L), then the hyperbolic line $\{a, a'\}^{\perp\perp}$ is contained in A.*

Easy examples of sets A satisfying this condition are the intersections with X of subspaces of the projective space in which (X, L) is embedded, or arbitrary subsets of a totally isotropic subspace. Note that the resulting near polygons do not necessarily have quads.

As a special case, if (X, L) is the $\mathrm{Sp}(2d, 2)$ or $\mathrm{U}(2d, 2^2)$ polar space, then X carries in a natural way the structure of a Fischer space (with the hyperbolic lines as lines of size 3, and the orthogonal pairs as lines of size 2) and we find near polygons A^* for each Fischer subspace A of the Fischer space on X. In particular, Fischer subspaces of size 3 (a hyperbolic line), 18 (two orthogonal nondegenerate planes), 45 (points of weight 2 for the standard form $(x, y) = \Sigma x_i y_i^2$) and 126 of $\mathrm{U}(6, 2^2)$ yield near hexagons on 81, 243, 405 and 567 points, respectively. (For a discussion of the last case, cf. Kantor [13, p. 500].) Similarly, Fischer subspaces of size 3 (a hyperbolic line) and 28 (the complement of a hyperbolic quadric) in $\mathrm{Sp}(6, 2)$ yield near hexagons on 45 and 105 points. Thus, we find the examples (iv)–(vii) and (ix)–(x) of the theorem.

More generally, the complement of a hyperbolic quadric in $\mathrm{Sp}(2d, 2)$ yields a near $2d$-gon with quads (the quads being of types $\mathrm{GQ}(2, 1)$ and $\mathrm{GQ}(2, 2)$) and with $t + 1 = 2^d - 2$ lines on each point. An equivalent description of this near polygon is obtained by fixing a hyperbolic hyperplane H in $O(2d+1, 2)$ and then taking the subspace of the corresponding dual polar space on the maximal totally singular subspaces not contained in H.

Note that all near polygons found in the conclusion of our theorem are subspaces of the $U(6, 2^2)$ dual polar space, except for the examples (i) and (ii) associated with the Mathieu groups. Indeed, the $Sp(6, 2)$ dual polar space is a subspace of the $U(6, 2^2)$ dual polar space, and example (xi) is obtained as the collection of all 27 maximal totally isotropic subspaces meeting each of three mutually orthogonal hyperbolic lines.

4. Fischer spaces from near polygons with big subspaces

A geodetically closed proper subspace Y of a near polygon $(X, \mathbf{L})$ with the property that each point of X has distance at most 1 to Y is called a *big subspace* of X. [Note that in the near polygons A^* constructed in the previous section, the sets a^* are big subspaces.] If Y is a geodetically closed subspace of $(X, \mathbf{L})$, and $x \in X$, then $\pi x = \pi_Y x$ denotes the unique point in Y closest to x, if such a point exists; π is called the projection onto Y. Since Y is geodetically closed and X does not contain triangles of lines, πx is defined when $d(x, Y) \leqslant 1$ (and hence for arbitrary x when Y is big). The subspace Y is called *classical* when πx is defined for all x. Let us use $\sim$ to denote collinearity.

4.1. LEMMA. *Let $(X, \mathbf{L})$ be a near polygon, let Y be a big subspace, and let L be a line disjoint from Y. Then πL is a line, if π denotes the projection onto Y.*

Proof. Let $a, b \in L$. Since Y is geodetically closed, and $\pi a \sim a \sim b \sim \pi b$ is a path connecting πa and πb, we have $d(\pi a, \pi b) \leqslant 2$. Now πa has distance at most 2 to both b and πb so is adjacent to a point of the line $b\, \pi b$, which must be πb. This shows that πL is a clique, hence contained in a line L' in Y, but since each point of L' has distance at most 2 to a and b, it follows that $L' = \pi L$. □

4.2. LEMMA. *Let $(X, \mathbf{L})$ be a near polygon, and let Y and Z be geodetically closed subspaces with $Y \cap Z \neq \varnothing$. Then any line meeting both Y and Z also meets $Y \cap Z$. In particular, if Z is a big subspace of X, then $Y \cap Z$ is a big subspace of Y.*

Proof. Let $L = yz$ be a line with $L \cap Y = \{y\}$, $L \cap Z = \{z\}$, $y \neq z$. Let $x \in Y \cap Z$. Let $i = d(x, y)$, $j = d(x, z)$. If $i = j$, then there is a point $w \in L$ with $d(x, w) = i - 1$, and by geodetic closure of Y and Z we find $w \in Y \cap Z$, so we are done. Thus we may assume $j = i - 1$. But now again by geodetic closure $z \in Y$. □

4.3. PROPOSITION. *Let $(X, \mathbf{L})$ be a near polygon with lines of size* 3. *For any big subspace Y define an involution σ_Y by*

$$\sigma_Y(x) = \begin{cases} x & \text{if } x \in Y, \\ z & \text{if } \{x, y, z\} \text{ is a line meeting } Y \text{ in } y. \end{cases}$$

Then σ_Y is well defined and an automorphism of X. If Y and Z are two big subspaces then

(i) *if Y meets Z then σ_Y and σ_Z commute;*

(ii) *if $Y \cap Z = \varnothing$, then $W = \sigma_Y(Z) = \sigma_Z(Y)$ is a third big subspace, and $\sigma_W = \sigma_Y \sigma_Z \sigma_Y = \sigma_Z \sigma_Y \sigma_Z$.*

Proof. $\sigma = \sigma_Y$ is well defined since Y is geodetically closed. We have to show that σ preserves lines. This is clear for lines meeting Y, so let $L = \{a, b, c\}$ be a line disjoint from Y. The point $\sigma(a)$ is adjacent to some point of the line $b\,\pi(b)$ (by the near polygon property and 4.1), and since this line has only three points we find $\sigma(a) \sim \sigma(b)$. Thus $\sigma(L)$ is a line and σ is an automorphism. If Y' is another big subspace, then if $Y \cap Y' \neq \varnothing$ we see from the above lemma that σ leaves Y' invariant and hence commutes with $\sigma_{Y'}$. If $Y \cap Y' = \varnothing$, then $Y'' = \sigma(Y')$ is disjoint from both Y and Y', and $Y \cup Y' \cup Y''$ is isomorphic to the direct product $L \times Y$, where L is a 3-line. □

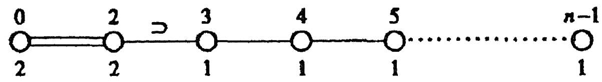
4.4. COROLLARY. *Let $(X, \mathbf{L})$ be a near polygon with lines of size* 3. *Let E be the collection of big subspaces of X, and let $\mathbf{L}_E$ be the collection of subsets $\{Y, Z, \sigma_Y(Z)\}$ of E. Then $(E, \mathbf{L}_E)$ is a Fischer space.* □

4.5. LEMMA. *Let Y be a big subspace of a near polygon $(X, \mathbf{L})$ with quads such that every quad meeting Y is the direct product of two lines. Then $X = Y \times L$ for some line L.*

Proof. If $\pi x = \pi x'$ for two nonadjacent points x, x', then the quad on the lines $x\,\pi x$ and $x'\,\pi x$ meets Y in one point only, contradiction. Thus, for $y \in Y$ we see that $\pi^{-1}(y)$ is a line L_y. For $y, z \in Y$ we have a natural 1–1 correspondence σ_z^y between L_y and L_z sending each point on one line to the closest point on the other line. (Indeed, if $d(y, z) = i$, and $a \in L_y$, then since quads exist there is a point $b \in L_z$ with $d(a, b) \leqslant i$. Now since Y is geodetically closed, the lines L_y and L_z must be parallel (cf. [8]).) Furthermore, if $y, z, w \in Y$ and $y \sim z$, then $\sigma_w^z \circ \sigma_z^y = \sigma_w^y$. It follows that we have a direct product. □

5. A FAMILY OF EXAMPLES WITH GROUP Sym $(2n)$

Let $\Gamma = \overline{T(2n)}$ be the complement of the triangular graph $T(2n)$. (That is, Γ is the graph with point set $\binom{2n}{2}$, the unordered pairs from a $2n$-set, two pairs being adjacent when they are disjoint, i.e. the commuting graph on the transpositions of Sym (2n), i.e. the Fischer space Sym (2n) together with its 2-lines.) We find a geometry of rank $n - 1$ with Buekenhout–Tits diagram



by taking as i-objects of our geometry the $(n - i)$-cliques in $\Gamma (i = 0, 2, 3, \ldots, n - 1)$. We call 0-objects *points* and 2-objects *lines*. We shall show that the points and lines are the points and lines of a near polygon.

5.1. LEMMA. *Let x and y be two points. The distance $d(x, y)$ of x and y in the collinearity graph on the points is $n - m$, where m is the number of connected components of $x \cup y$ regarded as a 2-factor of the complete graph K_{2n}.*

Proof. Points are n-cliques in the graph $\overline{T(2n)}$, i.e. are partitions of a 2n-set into pairs, i.e. are complete matchings of the complete graph K_{2n}. The union of two such matchings is a bipartite graph of valency 2 on 2n vertices and hence a union of 2k-circuits (here $k = 1$ is allowed). The claim of the lemma is that each such component contributes $k - 1$ to the distance $d(x, y)$. The proof is by induction on $d(x, y)$: If $d(x, y) = 0$ then our claim is true. Otherwise, let z be a neighbour of y such that $d(x, z) = d(x, y) - 1$. By induction we know that $x \cup z$ has precisely $n - d(x, z)$ components, and since y and z are collinear they have $n - 2$ pairs in common, so that y is obtained from z by replacing two pairs by two other pairs, covering the same four points. Clearly this can change the number of components by at most 1 so that $x \cup y$ has at least $n - d(x, y)$ components. But conversely, we have $d(x, y) \leqslant n - m$ since given a 2k-circuit $\cdots \sim p \sim q \sim r \sim s \sim \cdots$ with pq and rs in y and qr in x, we can find a neighbour z of y containing ps and qr, so that $x \cup z$ has one more component than $x \cup y$. □

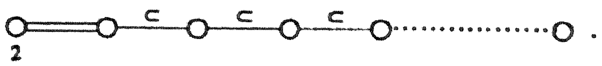
5.2. PROPOSITION. (X, L) = (0-objects, 2-objects) *is a near* $2(n - 1)$*-gon.*

Proof. Given a line L (i.e. a partial matching consisting of $n - 2$ edges) and a point x (i.e. a complete matching), consider $x \cup L$. This is a graph consisting of a number of closed circuits and two paths beginning and ending in a point not covered by L. There is a unique way of completing these two paths to two circuits, so there is a unique point on L closest to x. The diameter follows from the above lemma: $d(x, y)$ is maximal (and equals $n - 1$) when $x \cup y$ is connected, a 2n-circuit. □

This near polygon (let us call it H_{n-1}) has three points on each line, $\binom{n}{2}$ lines on each point, and the i-objects are geodetically closed sub near $2(i - 1)$-gons

$(i = 2, 3, \ldots, n-1)$. In particular, the 3-objects are generalized quadrangles of order 2. Not any two lines on a point determine a unique 3-object, but they do determine a unique quad: they determine a GQ(2, 2) quad whenever they have $n-3$ pairs in common, and a GQ(2, 1) quad (the direct product of two lines) otherwise.

More generally, every two points x, y at distance j determine a unique geodetically closed sub $2j$-gon $H(x, y)$; if $x \cup y$ is the union $\bigcup_i C_{2k_i}$ of $2k_i$-circuits, then $H(x, y)$ is the direct product $\Pi_i H_{k_i-1}$ of sub near $2(k_i-1)$-gons H_{k_i-1} which are k_i-objects of the geometry (where 1-objects are identified with 0-objects). The full geometry (with geodetically closed sub $2j$-gons as objects $(j = 0, 1, \ldots, n-2)$) has Buekenhout–Tits diagram



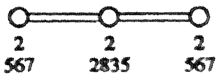
The derived geometry at a point is the geometry of subsets of an n-set.

For $n = 4$ we find example (ix) from the theorem again.

The near polygon H_{n-1} has a distance-preserving embedding in the $\mathrm{Sp}(2n-2, 2)$ dual polar space. [Indeed, consider the $\mathrm{Sp}(2n, 2)$ polar space defined on the vector space V over $\mathbb{F}_2$ with basis $\{e_i \mid 1 \leqslant i \leqslant 2n\}$ by the symplectic form $(x, y) = \Sigma_{i=1}^{2n} x_i y_i$ for $x = \Sigma x_i e_i$ and $y = \Sigma y_i e_i$. Let $e := \Sigma e_i$ and let $\phi: \Gamma \to V$ map the pair $\{i, j\}$ to the weight two vector $e_i + e_j$. Then ϕ determines a map $\bar{\phi}$ from the set of objects of the above geometry into the set of totally isotropic subspaces of the $\mathrm{Sp}(2n-2, 2)$ polar space on $W = e^{\perp}/\langle e \rangle$ (with the form inherited from V), sending the clique C to $\langle e, \phi(z) \mid z \in C \rangle / \langle e \rangle$. One checks immediately that $\bar{\phi}$ is a distance-preserving embedding of H_{n-1} into the $\mathrm{Sp}(2n-2, 2)$ dual polar space on W. For $n \leqslant 3$ this embedding is onto, i.e. an isomorphism. For $n = 4$ the image of H_3 in the $\mathrm{Sp}(6, 2)$ dual polar space consists of the 105 totally isotropic but not totally singular planes, for the quadratic form $Q(x) = \frac{1}{2}\,\mathrm{wt}(x)$.]

6. The Aschbacher near hexagon

Let us describe the near hexagon on 567 points in somewhat greater detail. Let V be a vector space of dimension 6 over $\mathbb{F}_3$ equipped with a nondegenerate quadratic form Q of Witt index 2. Let N be the set of 126 projective points of norm 1. The points and lines of the near hexagon are the 6-tuples and pairs, respectively, of mutually orthogonal points in N, with inclusion as incidence. Each line has size 3, and there are $\binom{6}{2} = 15$ lines on each point. Any two points at distance 2 determine a unique quad. The 126 27-quads are the points of N. The 567 15-quads correspond to the 6-tuples of mutually orthogonal points in N', the set of 126 projective points of norm 2. Each such 6-tuple B' in N' determines 15 elliptic lines that meet N in 15 pairs of points (the lines of the 15-quad) and any partition of B' into three pairs yields three pairs in N whose union is a point of the 15-quad. Points, lines and 15-quads yield a Buekenhout geometry with diagram



and the group $PO_6^-(3)$ has an outer automorphism interchanging N and N', yielding a polarity of this geometry. (For more details, see Kantor [14].) The 27-quads are big quads, and the corresponding Fischer space is the linear space induced on N by the lines of PG(5, 3): the elliptic lines become 2-lines and the tangents become 3-lines.

7. Classification

Let $(X, \mathbf{L})$ be a near hexagon with lines of size 3 and such that any two points at distance 2 have at least two common neighbours. Then, by Yanushka's lemma, any two points at distance 2, or any two intersecting lines, determine a unique quad, necessarily one of GQ(2, 1), GQ(2, 2) and GQ(2, 4) (Seidel [17], Cameron [10]).

Finiteness

Our first concern is proving that $(X, \mathbf{L})$ is finite. By Lemma 19 of [8] we find that each point is on the same number $t + 1$ of lines, and since the collinearity graph Γ of $(X, \mathbf{L})$ has finite diameter and lines have finite size, finiteness of X is equivalent to finiteness of $t + 1$.

For $A \subseteq X$, let us write $\Gamma_i(A)$ for the set of points at distance i from A in Γ. (And $\Gamma_i(a) = \Gamma_i(\{a\})$ for $a \in X$.)

We show finiteness by mimicking the reasoning in Cameron's classification of generalized quadrangles with lines of size three.

7.1. PROPOSITION. *Every near hexagon with quads and with lines of size* 3 *is finite.*

Proof. Fix a point $x \in X$, and label the remaining two points on each line on x arbitrarily with 0 and 1. This labelling induces a labelling of the vertices in $\Gamma_3(x)$ with 0–1 vectors indexed by the lines on x: for $z \in \Gamma_3(x)$ its label $\bar{z}$ has L-coordinate b when the point $\Gamma_2(z) \cap L$ has label b. Now if y and z are adjacent points in $\Gamma_3(x)$, then $\bar{y}$ and $\bar{z}$ have different L coordinates for all L

except for the finitely many lines L in the quad $\langle x, w \rangle$ where w is the third point of the line yz.

Now if $t + 1$ is infinite, it follows that $\Gamma_3(x)$ has infinite diameter: starting from y we can walk away, and after an odd number of steps all but a finite number of coordinates will have changed, while after an even number of steps all but a finite number of coordinates will be the same again; now while walking in $\Gamma_3(x)$ we can make the size of this finite exceptional set increase on each step, but after finitely many (say s) steps our exceptional set has size bounded by $5s$, so that our walk brings us arbitrarily far away from y in $\Gamma_3(x)$.

On the other hand, $\Gamma_3(x)$ has finite diameter: Let $y, z \in \Gamma_3(x)$. If $d(y, z) = 2$ in Γ, then consider the quad $Q = \langle y, z \rangle$. Now $Q \cap \Gamma_3(x)$ is either Q with an ovoid removed, or Q with all lines on a fixed point removed, and in any case is connected of diameter at most 3. Thus y and z have at most distance 3 in $\Gamma_3(x)$. And if $d(y, z) = 3$, then let M be a line on y, and $w \in M$, $d(w, z) = 2$. Now $\langle w, z \rangle \cap \Gamma_3(x)$ contains a common neighbour of w and z, so that y and z have at most distance 4 in $\Gamma_3(x)$.

It follows that $t + 1$ is finite. □

Local Spaces and Quads

Let $(X, \mathbf{L})$ be a near hexagon with quads, and let $x \in X$. The collection of lines and quads on x forms the collection of Points and Lines of a linear space, called the local space $\mathbf{L}_x$. Since a big quad never meets another quad in a single point, it follows that the Line corresponding to a big quad in $\mathbf{L}_x$ meets all other Lines in that local space.

Suppose that Q is a quad and that $\Gamma_2(Q)$ contains a point x. By [20, Prop. 2.6], the set $O_x = Q \cap \Gamma_2(x)$ is an ovoid in Q. On the other hand, GQ(2, 4) does not have ovoids (e.g. because no GQ(s, s^2) has ovoids, cf. Payne and Thas [15, 1.8.3]). This shows

(1) *Every quad of type GQ(2, 4) is big.*

If $\Gamma_2(Q)$ contains a line L, then we find a partition of Q into ovoids O_x for $x \in L$. On the other hand, GQ(2, 2) does not possess partitions into ovoids (it has six ovoids, and any two of them meet in a point). Thus:

(2) *For a quad Q of type* GQ(2, 2) *the set $\Gamma_2(Q)$ does not contain lines.*

(That is, $Q \cup \Gamma_1(Q)$ is a geometric hyperplane.)

(3) *If $(X, \mathbf{L})$ has a quad of type* GQ(2, 2) *that is not big, then* $10 \leqslant t + 1 \leqslant 15$.

[Indeed, let $x \in \Gamma_2(Q)$, where Q is a GQ(2, 2). Then each of the $t + 1$ lines on x lies in precisely one of the five quads $\langle x, z \rangle$ spanned by x and a point $z \in O_x$.

And these quads each have two or three lines on x. (Not five, since these quads meet Q in a single point and therefore cannot be big by Lemma 4.2.)]

(4) *Suppose that* (X, L) *has a quad* Q *of type* $\mathrm{GQ}(2, t_2)$, *and put* $v = |X|$, $q = |Q|$. *Then* $v \geqslant q + 2q(t - t_2)$ *with equality if and only if* Q *is big. In particular, if one quad of type* $\mathrm{GQ}(2, t_2)$ *is big, then all are.*

[Indeed, let Q be a quad of type $\mathrm{GQ}(2, t_2)$. Counting the neighbours of all points of Q we find $v \geqslant q + 2q(t - t_2)$ with equality when Q is big.]

7.2. LEMMA. *If* (X, L) *contains a quad* Q *of type* $\mathrm{GQ}(2, t_2)$, *then* $t \geqslant t_2 + 1$, *with equality if and only if* (X, L) *is the direct product of* Q *with a line, i.e. if and only if we are in case* (vii) *or* (x) *or* (xi) *of the theorem.*

Proof. Since X is connected and has diameter 3, some point of Q is on a line not contained in Q, so that $t \geqslant t_2 + 1$. In case of equality, Lemma 4.5 tells us that we have a direct product. □

7.3. LEMMA. *Suppose that* (X, L) *has a big quad* Q *of type* $\mathrm{GQ}(2, t_2)$. *If no larger quads occur, then* $t + 1 \leqslant t_2^2 + t_2 + 1$, *with equality if and only if* (X, L) *is a classical dual polar space, i.e. if and only if we are in case* (iii) *or* (viii) *or* (xi).

Proof. Let L be a line meeting Q in a point z and choose $x \in L \backslash \{z\}$. The t lines on x distinct from L lie in the $t_2 + 1$ quads on L, and each quad contributes at most t_2 to the total number of lines. In case of equality, all quads meeting a big quad are again big (by (4)), and we have a regular near hexagon, i.e. case (iii) or (viii) or (xi) (by Shult and Yanushka's classification). □

Now the classification will proceed as follows. Shult and Yanushka did the regular case, so we may assume that quads of different types occur. The three main cases are: (i) a 27-quad occurs (necessarily as a big quad); (ii) a 15-quad occurs as a big quad; and (iii) a 15-quad occurs as a non-big quad. We start with case (ii).

Big 15-*Quads*

Suppose that (X, L) has no 27-quads, but does have a big 15-quad. By the above we may assume that $t + 1 \in \{5, 6\}$. If $t + 1 = 5$, then $v = 75$ and all local spaces L_x have two 3-lines and four 2-lines, so that the total number of quads of type $\mathrm{GQ}(2, 1)$ equals 75.4/9, not an integer. Thus, $t + 1 = 6$, $v = 105$, and all local spaces L_x look like the Fischer space F_6. There are 28 big quads, and each big quad intersects 15 others. The Fischer space on the big quads has 28 points and 6 3-lines on each point. By Buekenhout [9] (or Fischer

[12]), this can be nothing but $\binom{8}{2}$, the Fischer space of the transpositions in Sym(8). Having identified the big quads with pairs from an 8-set such that intersecting quads correspond to disjoint pairs, we see that each point corresponds to a complete matching, and since there are as many points as complete matchings, each matching occurs. This identifies the near hexagon as the H_3 of Section 5, and we have case (ix) of the theorem.

Big 27-Quads

Now suppose that $(X, \mathbf{L})$ has at least one 27-quad. We may assume that $6 \leqslant t \leqslant 19$, and we have $v = 27(2t - 7)$.

Suppose $t+1 \geqslant 12$. If $\mathbf{L}_x$ contains a 5-Line, then each Point outside is on at least $(t-10)/2$ 5-Lines, which proves that every line that meets a 27-quad is contained in a 27-quad. Since we are assuming that there is at least one 27-quad, it follows that every line is in a 27-quad. If $t + 1 = 13$, then all local spaces $\mathbf{L}_x$ must contain three 5-Lines on a fixed Point, and 16 3-Lines, so that the total number of 15-quads equals

$$16v/15 = 16.27.(2t - 7)/15 = 16.27.17/15,$$

which is not an integer. Consequently, $t + 1 \neq 13$. If some line L is contained in all 27-quads it meets, then for $x \in L$ we see that the Point L of $\mathbf{L}_x$ is contained in all 5-Lines, and it follows that there are precisely three 5-Lines on L, and that $t+1=13$, impossible. Thus, for any flag (x, L) we can find a big quad on x not containing L. (This immediately implies that the group G generated by the involutions σ_Y for big quads Y is vertex-transitive: it suffices to show that two collinear points y, z are in the same G-orbit, but if $L = \{x, y, z\}$ is the line joining them and Y a big quad on x but not on L, then σ_Y moves y to z. However, we will not use the vertex-transitivity of G.)

Each Point of $\mathbf{L}_p$ is on at most two 5-Lines. For suppose $u \in \mathbf{L}_p$ is on at least three 5-Lines L, M, N. Since $t + 1 \neq 13$, we can find a Point $v \notin L \cup M \cup N$. Each of the four Lines joining v to the four Points of $L \backslash \{u\}$ must meet L, M, N, $\{v\}$ in distinct Points, hence is a 5-Line. Thus, v is on at least four 5-Lines. All lines not on v meet each of these and hence are 5-Lines, too. But then $\mathbf{L}_p \cong \mathrm{PG}(2,4)$ and $t + 1 = 21$, contrary to assumption.

Since each Point is on at least $(t - 10)/2$ 5-Lines, we have $t \leqslant 14$.

The Aschbacher Near Hexagon

If $t+1 = 15$, then each Point is on precisely two 5-Lines and there are six 5-Lines. We see that all local spaces $\mathbf{L}_p$ are isomorphic to the linear space obtained by taking as Points the points of a GQ(2, 2), and as Lines the 15 lines and 6 ovoids of this generalized quadrangle. We find $v = 567$, and the near hexagon has $6.567/27 = 126$ big quads. Let Δ be the graph on the big quads, adjacent when they have nonempty intersection. Then Δ is locally the collinearity graph of GQ(4, 2) and has $\mu = 27.2/3 = 18$. A nontrivial block of Δ for the Fischer group G cannot contain a Fischer 3-line since distinct points of Δ have different neighbourhoods in Δ (cf. Buekenhout [9, Prop. 13] or Fischer [12, §2]), and it cannot contain a Fischer 2-line since for each vertex a of Δ the stabilizer G_a is transitive on its neighbourhood $\Delta(a)$. Hence G acts primitively on Δ, and by Buekenhout [9, Prop. 16, 17], and the main theorem of Fischer [12] it follows that the Fischer group must be $O_6^-(3)$. Consequently, we may identify the big quads with the points x in PG(5, 3) for which $Q(x) = 1$, for a fixed quadratic form Q with Witt index 2; the 3-lines (of the Fischer space on the big quads) correspond to tangent lines, the 2-lines to elliptic lines. Each point of the near hexagon lies in 6 big quads (and these big quads meet each other so are joined by 2-lines), so points of the near hexagon correspond to orthonormal bases in the geometry. (Note that the elliptic lines have two points x with $Q(x) = 1$ and two with $Q(x) = -1$; the first pair and the second pair are both orthogonal.) Since there ae exactly 567 orthonormal bases, these are all the points. Thus, we have identified our near hexagon with the geometry that has as points the orthonormal bases, and as lines the orthonormal sets of size 2 (with inclusion as incidence). This is example (iv).

We have $t + 1 \neq 14$, since there is no suitable linear space on 14 points.

The Hall Near Hexagon

If $t+1 = 12$, then the local spaces $\mathbf{L}_p$ are all isomorphic (to a uniquely determined space $\mathbf{L}$ with three 5-Lines, nine 3-Lines and nine 2-Lines). We find $v = 405$, and the near hexagon has $3.405/27 = 45$ big quads. Let Δ be the graph on the big quads, adjacent when they have nonempty intersection. Then Δ has valency $k = 2.27/3 = 18$, and any two adjacent vertices of Δ have $\lambda = 3$ common neighbours. Any two nonadjacent vertices have either 9 or 18 common neighbours. [Indeed, let Q and R be two disjoint big quads. Each of the 27 lines meeting both lies in either one or two big quads. But it is not difficult to see that the stabilizer $G_{Q,R}$ of Q and R in the Fischer group G acts transitively on these 27 lines, so that we always have the same case, and $\mu(Q, R) = 27/3$ or $2.27/3$.] It follows that Δ is the 3-coclique extension of the collinearity graph of GQ(2, 2). There is (up to isomorphism) a unique way to turn Δ into a Fischer space, and it follows that we may identify the big quads with the isotropic points of weight 2 in a $U_6(2)$ geometry with standard form

$\Sigma\, x_i y_i^2$, such that the Fischer space structure is preserved. Each point of the near hexagon lies in three big quads, corresponding to pairwise orthogonal points in the $U_6(2)$ geometry, and distinct points of the near hexagon yield different triples of mutually orthogonal isotropic points of weight 2. But there are precisely 405 such triples, so that we have located the points. The lines of the near hexagon that are in two big quads correspond to the pairs of orthogonal isotropic points of weight 2. Two points $\{a, b, c\}$ and $\{a, d, e\}$ are collinear if and only if $\langle b, c\rangle \cap \langle d, e\rangle \neq \varnothing$. [Indeed, let $\{a, f, g\}$ be the third point of this line. Then $\{b, c, d, e, f, g\}$ induces a Fischer subspace of Δ, and from this our claim follows by a simple computation.] Thus, we have identified our near hexagon with the geometry that has as points the maximal totally isotropic subspaces of the $U_6(2)$ polar space that contain a weight 2 vector, collinear when they meet in a line. (The 45 27-quads, the 405 9-quads and the 243 15-quads are the isotropic points of weight 2, 4 and 6, respectively.) This is example (v).

There remain the cases with $7 \leqslant t + 1 \leqslant 11$. We shall show that if no 15-quad occurs then $t + 1 = 9$ (indeed, if $\mathbf{L}_p$ does contain a 5-Line but does not contain 3-Lines, then any two points off a 5-Line are joined by a 5-Line, and hence $t + 1 = 9$ and there are two 5-Lines) and there is a unique example—see the next section. There are no examples with a 15-quad, as we show now.

Consider a 15-quad R. We have $|R| = 15$, $|\Gamma_1(R)| = 30(t - 2)$ and $|\Gamma_2(R)| = 24(t - 6)$. There are $12(t - 6)(t + 1)$ lines meeting $\Gamma_2(R)$, and hence $(30(t - 2)t - 12(t - 6)(t + 1))/3 = 6t^2 + 24$ lines are contained in $\Gamma_1(R)$. For $x \in R$, let $n_{R,x}$ be the number of lines L contained in $\Gamma_1(R)$ and such that $x \in \pi_R L$. Then (for fixed R)

$$\sum_{x \in R} n_{R,x} = 3(6t^2 + 24)$$

so that $\bar{n}$, the average of $n_{R,x}$ over all incident pairs (R, x) equals $\frac{6}{5}(t^2 + 4)$. On the other hand, we can compute $n_{R,x}$ in the local space $\mathbf{L}_x$: Let m_i be the number of i-Lines in $\mathbf{L}_x$ meeting the Line R in a Point. (Of course m_i depends on the pair (R, x) under consideration.) Then

$$n_{R,x} = 2(m_2 + 4m_3 + 16m_5).$$

We shall derive a contradiction by first showing that the average over all R on x of $m_2 + 4m_3 + 16m_5$ is at most $\frac{3}{5}(t^2 + 4)$ for all points x, and then treating the case of equality.

If the local space $\mathbf{L}_x$ does not contain any 5-Lines, then

$$m_2 + 4m_3 \leqslant 2(m_2 + 2m_3) = 2.3(t - 2) \leqslant \tfrac{3}{5}(t^2 + 4)$$

with equality if and only if $t = 6$ and $\mathbf{L}_x$ is the Fano plane.

If the local space $\mathbf{L}_x$ contains a single 5-Line S, then the two Points of $R\backslash S$ are each on $t - 6$ 3-Lines distinct from R and on $10 - t$ 2-Lines. The Point $R \cap S$ must be on at least one 2-Line when t is odd. Thus, writing $t = 2[t/2] + \varepsilon$, we have

$$m_2 + 4m_3 + 16m_5 \leqslant 2(10 - t) + \varepsilon + 4(2(t - 6) \\ + \tfrac{1}{2}(t - 6 - \varepsilon)) + 16 = 8t - 24 - \varepsilon \leqslant \tfrac{3}{5}(t^2 + 4)$$

with equality if and only if $t = 6$.

If the local space $\mathbf{L}_x$ contains two 5-Lines, then $t = 9$ or $t = 10$. For $t = 9$ we find (for the unique possibility for $\mathbf{L}_x$) $m_2 + 4m_3 + 16m_5 = 7 + 4.3 + 16.2 = 51 = \frac{3}{5}.85$. For $t = 10$ any such local space $\mathbf{L}_x$ has nine 3-Lines and eight 2-Lines. For one choice of R (the third Line on the intersection of the two 5-Lines) we find $m_2 + 4m_3 + 16m_5 = 0 + 4.8 + 16.2 = 64$. For the other eight choices of R we find $m_2 + 4m_3 + 16m_5 = 4 + 4.6 + 16.2 = 60$. Thus, the average is $(64 + 8.60)/9$ which is smaller than $\frac{3}{5}.104$.

Summarizing: in all cases the average of $m_2 + 4m_3 + 16m_5$ over all R on x is too small, except in the cases $t = 6$ and $t = 9$.

If $t = 9$, there are two possible types of local spaces: the first has two 5-Lines and four 3-Lines, the second has 2-Lines only. This forces adjacent vertices to have isomorphic local spaces, so by connectivity of Γ we always have the same type. Since we are assuming that a 15-quad occurs, this is the first type, but then the total number of 15-quads if $4v/15 = 4.27.11/15$, not an integer. Thus $t \neq 9$.

If $t = 6$, there are three possible local spaces: the first has one 5-Line and one 3-Line, the second is the Fano plane and the third has 2-Lines only. As before we conclude by connectivity that all local spaces have the same type, and since we are assuming that there is a 27-quad, this is the first type. Now $v = 27.5 = 135$ and each point is in a unique 27-quad, so the big quads form a Fischer space on 5 points with 3-lines only. No such Fischer space exists. This settles the case where a 15-quad occurs.

No 15-Quads

Now suppose that 27-quads do occur, but 15-quads do not. Then $t + 1 = 9$ and each point is on two 27-quads (and sixteen 9-quads). We have $v = 243$, and the 18 big quads fall into two families $\mathbf{R}$ and $\mathbf{S}$ of size 9, each partitioning X. Each quad from one family meets each quad from the other family in a line. (The corresponding Fischer space is the disjoint union of two affine planes of order 3.) Since each line is contained in a big quad, and the maps $\phi_{S,T}\colon S \to T$ (for disjoint big quads S, T) defined by $\phi_{S,T}(s) \sim s$ are isomorphisms (preserving membership in a big quad R from the other family), the near hexagon $(X, \mathbf{L})$ is completely determined by one big quad from each of the two families, together with the spreads induced in each by intersecting it with the members of the other family.

But a spread in $GQ(2, 4)$ which is an affine plane of order 3 when regarded as a Fischer subspace dualizes to an ovoid in $GQ(4, 2)$ on which the hyperbolic lines induce the structure of $AG(2, 3)$. In the representation as isotropic points and totally isotropic lines of $U(4, 2^2)$ such ovoids are the sections with nontangent planes, and all are equivalent under $U(4, 2)$. Thus, we have a unique choice for the spread in $GQ(2, 4)$, and hence the near hexagon is uniquely determined. This is example (vi).

No Big Quads

In order to complete the classification we have to treat the case where all quads have size 9 or 15, and there is a 15-quad Q and a point x such that $d(x, Q) = 2$.

The $t + 1$ lines on x are contained in the 5 quads $\langle x, y\rangle$ with $y \in Q \cap \Gamma_2(x)$, and each of these quads contains either two or three lines on x, so

$$10 \leqslant t + 1 \leqslant 15,$$

and x is in $t - 9$ 15-quads (and in $14 - t$ 9-quads) meeting Q.

If $t + 1 = 15$, then we see in $\mathbf{L}_y$ that any Line disjoint from a 3-Line is itself a 3-Line, and it quickly follows that all Lines are 3-Lines. But then all quads on y, and, by connectivity, on any point, are 15-quads, contrary to assumption. Thus $t + 1 \neq 15$.

Next, let us show that there are constants a and b such that each point is contained in a 15-quads, and in b 9-point quads. Indeed, let the point p be in a resp. b quads of each type. Counting points around p we find

$$v = 1 + 2(t + 1) + (8a + 4b) + |\Gamma_3(p)|.$$

Counting pairs of Points in the local space $\mathbf{L}_p$ we find

$$3a + b = \tfrac{1}{2}t(t + 1).$$

Counting lines meeting $\Gamma_3(p)$ we find

$$\tfrac{1}{2}|\Gamma_3(p)|(t + 1) = 8a(t - 2) + 4b(t - 1) = (8a + 4b)(t + 1) - 4t(t + 1),$$

so that $|\Gamma_3(p)| = 8(2a + b - t)$. Combining these equations we find

$$v = 4t^2 - 2t + 3 + 4b,$$

so that a and b can be expressed in terms of v and t and are independent of the choice of p.

The total numbers of 9-quads and 15-quads are now

$$N_9 = \frac{1}{9}\,bv \quad \text{and} \quad N_{15} = \frac{1}{15}\,av.$$

In particular, these numbers are integers.

Another divisibility condition is obtained by counting quads meeting Q in a single point. Indeed, let y and z be two noncollinear points of Q. The graph with as vertices the 15-quads meeting Q in y or z only, where Q' is adjacent to Q'' if $Q \cap Q' \neq Q \cap Q''$ and $|Q' \cap Q''| = 1$, is bipartite and regular of valency $t - 10$. Since the noncollinearity graph of Q is connected, it follows that if $t > 10$ then the number of 15-quads Q' with $Q \cap Q' = \{y\}$ is independent of the choice of $y \in Q$. But the total number of 15-quads meeting Q in a single point is $\frac{1}{8}(t-9)|\Gamma_2(Q)|$. Thus:

If $t > 10$ then the number of 15-quads Q' with $Q \cap Q' = \{y\}$ equals $\frac{1}{120}(t-9)|\Gamma_2(Q)|$ for each $y \in Q$. In particular, this number is integral.

Since $|Q| = 15$ and $|\Gamma_1(Q)| = 30(t-2)$, we have

$$|\Gamma_2(Q)| = 4(t-2)(t-6) + 4b.$$

The conditions found thus far allow the following 15 possibilities for the parameters. Each of these will be ruled out below.

t	a	b	v	$\lvert\Gamma_2(Q)\rvert$	$\frac{1}{120}(t-9)\lvert\Gamma_2(Q)\rvert$	N_{15}	*Ruled out by*
9	2	39	465	240	0	62	B
9	5	30	429	204	0	143	B
9	7	24	405	180	0	189	C
9	10	15	369	144	0	246	C
9	12	9	345	120	0	276	C
10	9	28	495	240	2	297	B
10	15	10	423	168	7/5	423	D
11	2	60	705	420	7	94	A
11	7	45	645	360	6	301	A
11	12	30	585	300	5	468	B
11	17	15	525	240	4	595	E
12	6	60	795	480	12	318	A
12	16	30	675	360	9	720	A
13	6	73	945	600	20	378	A
13	21	28	765	420	14	1071	A

A. Let i_L be the number of 15-quads on the line L. Then

$$\sum_L 1 = \tfrac{1}{3}(t+1)v,$$

$$\sum_L i_L = 15N_{15} = av,$$

and

$$\sum_L i_L(i_L - 1) = N_{15}(5(a-1) - \tfrac{1}{24}(t-9)|\Gamma_2(Q)|).$$

The inequality $\Sigma_L (i_L - 3a/(t+1))^2 \geq 0$ reduces to

$$\tfrac{1}{3}(a - 1 - \tfrac{1}{120}(t-9)|\Gamma_2(Q)|) + 1 \geq \frac{3a}{t+1}$$

and rules out six of the above cases.

B. The set $H = Q \cup \Gamma_1(Q)$ is a geometric hyperplane (cf. Brouwer and Wilbrink [8, §(b)], and hence H meets any other 15-quad Q' in either all of Q' or in a 9-point subquadrangle or in three concurrent lines. It follows that $|Q' \cap \Gamma_2(Q)| \in \{0, 6, 8\}$. Let there be n_i 15-quads Q' with $|Q' \cap \Gamma_2(Q)| = i$. Then

$$n_0 + n_6 + n_8 = N_{15},$$

$$6n_6 + 8n_8 = a.|\Gamma_2(Q)|,$$

$$n_0 \geq 1 + 5(a-1) - \tfrac{1}{24}(t-9)|\Gamma_2(Q)|.$$

The resulting inequality

$$\tfrac{1}{8}a.|\Gamma_2(Q)| \leq N_{15} - 1 - 5(a-1) + \tfrac{1}{24}(t-9)|\Gamma_2(Q)|$$

rules out four more of the above cases.

C. If $t = 9$, then no two 15-quads meet in a single point, i.e. in each $\mathbf{L}_p$, no two 3-Lines are disjoint. But any collection of triples pairwise intersecting in one point either has a common point, or is a subcollection of the set of lines of a Fano plane. Thus, $a \leq 7$. Now suppose $a = 7$. Each local space $\mathbf{L}_p$ is a Fano plane together with three Points that are only on 2-Lines. Thus: each point p is on seven lines L that each are in three 15-quads and three 9-quads, and on three lines L that each are on nine 9-quads. We have (in the notation of B above):

$$n_0 + n_6 + n_8 = 189$$

$$6n_6 + 8n_8 = 1260$$

$$n_0 = 31 + \varepsilon,$$

where ε is the number of 15-quads in $\Gamma_1(Q)$. The only solution in nonnegative integers is $\varepsilon = 0$, $n_0 = 31$, $n_6 = 2$, $n_8 = 156$. Now consider points $z \in \Gamma_1(Q)$ such that the line zz' is in three 15-quads, where z' is the point of Q collinear with z. There are 120 such points z, 8 for each choice of z'. The four 15-quads on z not meeting Q each meet $\Gamma_1(Q)$ in three lines concurrent in z. Hence $n_8 \geqslant 4.120 = 480$. Contradiction. Thus, $t \neq 9$.

D. Count 9-quads meeting Q in a single point. We find

$$\tfrac{1}{4}(14 - t)|\Gamma_2(Q)| \leqslant 15b.$$

This rules out the possibility $(t, a) = (10, 15)$.

E. Remains the case $(t, a) = (11, 17)$. There are four 15-quads meeting Q in a given single point. Thus, in a local space $\mathbf{L}_p$, each 3-Line is disjoint from four 3-Lines, and meets the remaining twelve 3-Lines. Since no Point can be on more than five 3-Lines, it follows that each Point is on either no or on five 3-Lines. But the number of Points of the latter kind is $3a/5$, which is not an integer. This contradiction rules out the last possibility, so that there are no near hexagons satisfying the assumptions of this section.

REFERENCES

1. Aschbacher, M., 'Flag structures on Tits geometries', *Geom. Dedicata* **14** (1983), 21–32.
2. Brouwer, A. E., 'The uniqueness of the near hexagon on 729 points', *Combinatorica* **2** (1982), 333–340.
3. Brouwer, A. E., 'The uniqueness of the near hexagon on 759 points', in *Finite Geometries, T. G. Ostrom Conf. Pullman* 1981, pp. 47–60; Lecture Notes in Pure and Applied Math. 82 (N.L. Johnson, M. J. Kallaher and C. T. Long, eds), Marcel Dekker, New York, 1983.
4. Brouwer, A. E., 'The classification of near hexagons with lines of size three', Report R85-16, Instit. f. Elektroniske Systemer, Aalborg, DK (Sept. 1985).
5. Brouwer, A. E. and Cohen, A. M., 'Local recognition of Tits geometries of classical type', *Geom. Dedicata* **20** (1986), 181–199.
6. Brouwer, A. E., Cohen, A. M. and Wilbrink, H. A., 'Near polygons with lines of size three and Fischer spaces', Math. Centr. Report ZW191, Amsterdam (May 1983).
7. Brouwer, A. E. and Wilbrink, H. A., 'Two infinite sequences of near polygons', Math. Centr. Report ZW 194, Amsterdam (June 1983).
8. Brouwer, A. E. and Wilbrink, H. A., 'The structure of near polygons with quads', *Geom. Dedicata* **14** (1983), 145–176.
9. Buekenhout, F., 'La géometrie des groupes de Fischer', preprint, Univ. Libre de Bruxelles.
10. Cameron, P. J., 'Orbits of permutation groups on unordered sets, II', *J. London Math. Soc. (2)* **23** (1981), 249–264.
11. Cameron, P. J., 'Dual polar spaces', *Geom. Dedicata* **12** (1982) 75–85.
12. Fischer, B., 'Finite groups generated by 3-transpositions', *Invent. Math.* **13** (1971), 232–246.
13. Kantor, W. M., 'Generation of linear groups', in *The Geometric Vein (the Coxeter Festschrift)* (C. Davis, B. Grünbaum and F. A. Sherk, eds), Springer Verlag, Berlin, 1981, pp. 497–509.
14. Kantor, W. M., 'Some geometries that are almost buildings', *European J. Combin.* **2** (1981), 239–247.
15. Payne, S. and Thas, J. A., *Finite Generalized Quadrangles*, Pitman, New York (1985).

16. Ronan, M. A., 'Embeddings and hyperplanes of discrete geometries', *European J. Combin.* **8** (1987), 179–185.
17. Seidel, J. J., 'Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3', *Lin. Alg. Appl.* **1** (1968), 281–298.
18. Shad, S. A. and Shult, E. E., 'The near *n*-gon geometries', preprint, 1981.
19. Shult, E. E., 'Characterizations of the Lie incidence geometries', in *Surveys in Combinatorics (Invited papers for the Ninth British Combin. Conf.)*, London Math. Soc. Lecture Note Ser. 82 (E. Keith Lloyd, ed.), Cambridge Univ. Press, 1983, pp. 157–186.
20. Shult, E. E. and Yanushka, A., 'Near n-gons and line systems', *Geom. Dedicata* **9** (1980), 1–72.

Authors' addresses:

A. E. Brouwer and A. M. Cohen,
CWI, *Amsterdam*,
The Netherlands

and

Eindhoven Univ. of Technology,
Eindhoven,
The Netherlands

H. A. Wilbrink,
Eindhoven Univ. of Technology,
Eindhoven,
The Netherlands

and

J. J. Hall,
Michigan State University,
East Lancing, Michigan,
U.S.A.

(Received September 9, 1992)