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The Locally Icosahedral Graphs

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ABSTRACT

There are precisely three locally icosahedral graphs, namely the point graph of the 600-cell on 120 vertices, and quotients of this graph on 60 and 40 vertices, respectively.

THE 600-CELL

The 600-cell is a regular polytope in $\mathbb{R}^4$ with 120 vertices, 720 edges, 1200 (triangular) 2-faces and 600 (tetrahedral) 3-faces. Its Schläfli symbol is {3,3,5}; cf. Coxeter [2], Sec. 8.5. It is the (unique) thin building of type

$$H_4 = \underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} \overset{5}{-} \underset{4}{\circ}$$

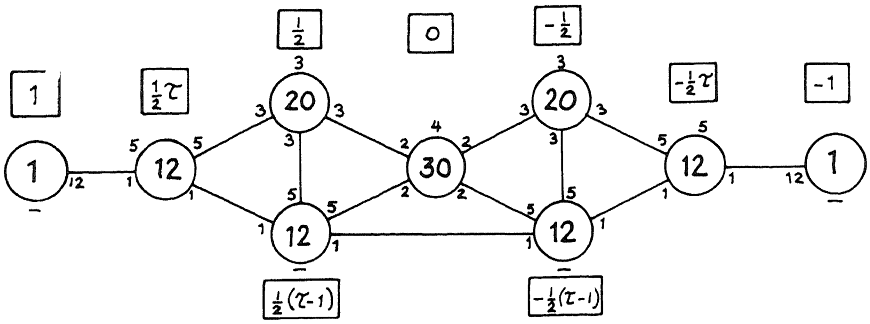
(where vertices are 1-objects, etc.). The vertices and edges of the 600-cell form a graph Q that is locally an icosahedron, i.e., for each vertex x of Q the induced graph on the collection of neighbors of x is isomorphic to the (graph of the vertices and edges of the) icosahedron. In this note we shall determine all locally icosahedral graphs.

The vertices of Q may be described using quaternions (Witt [4]; cf. Bourbaki [1], Ch. VI, Sec. 4, Exercise 12): take the 8 vertices ±1, ±i,

$\pm j$, $\pm k$, the 16 vertices $(1/2)(\pm 1 \pm i \pm j \pm k)$ and the 96 vertices obtained from $(1/2)(\pm \tau \pm i \pm (\tau - 1)j)$ using even permutations of $(1,i,j,k)$. Here $\tau = 2 \cos \frac{\pi}{5} = \frac{1}{2}(\sqrt{5} + 1)$ is the golden ratio, root of $\tau^2 = \tau + 1$. Note that this set of 120 quaternions forms a subgroup [isomorphic to $SL_2(5)$] of the multiplicative group of quaternions of unit norm. Define an inner product on Q by $(x,y) = (1/2)(x\bar{y} + y\bar{x})$, where the bar denotes quaternion conjugation. This is the ordinary euclidean inner product when Q is viewed as a subset of the four dimensional euclidean space with basis $(1,i,j,k)$: $(x,y) = \Sigma_{i=1}^4 x_i y_i$ for $x = x_1 + x_2 i + x_3 j + x_4 k$, $y = y_1 + y_2 i + y_3 j + y_4 k$. Two vertices x,y are adjacent iff $(x,y) = (1/2)\tau$. If we fix a vertex $x \in Q$ then the point set of Q is partitioned into the nine sets

$$Q_\alpha(x) = \{y \in Q \mid (x,y) = \alpha\} \text{ with } \alpha \in \{\pm 1, \pm\tfrac{1}{2}\tau, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}(\tau - 1), 0\}$$

The stabilizer of x in Aut Q is transitive on each set $Q_\alpha(x)$. The following diagram shows the cardinalities of the $Q_\alpha(x)$, and for any $y \in Q_\alpha(x)$ how many neighbors y has in $Q_\beta(x)$.



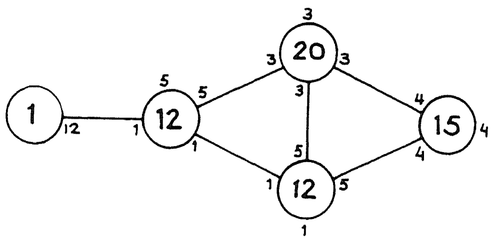
The group of automorphisms of Q has order $120^2 = 14400$ and is generated by the orthogonal reflections σ_a: $x \mapsto -a\bar{x}a$ $(a \in Q)$. It consists of the $(1/2)120^2$ transformations $\sigma_{a,b}$: $x \mapsto \bar{a}xb$ of determinant 1 $(a,b \in Q)$ and the $(1/2)120^2$ transformations $\overline{\sigma_{a,b}}$ of determinant -1. (Note that $\sigma_{a,b} = \sigma_{c,d}$ iff either $a = c$, $b = d$, or $a = -c$, $b = -d$.) Its center is $\{\pm 1\}$ of order 2. In other words, Aut Q $\simeq [SL_2(5) \circ SL_2(5)] \cdot 2$ where the $\circ$ denotes central product.

LOCALLY ICOSAHEDRAL GRAPHS

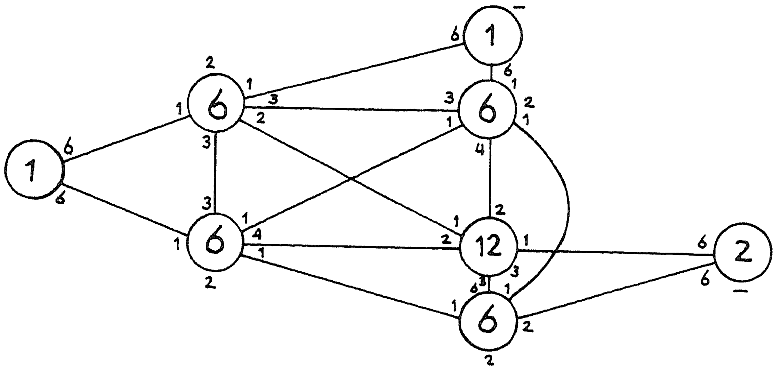
Let G be a connected locally icosahedral graph on ν vertices. We may give G the structure of a geometry of type H_4 (cf. Tits [3]) by taking as i-objects complete subgraphs of cardinality i ($1 \le i \le 4$) and as incidence (symmetrized) inclusion. Note that this geometry is thin, i.e., each flag of corank one is in precisely two maximal flags.

By Tits [3], Theorem 1, it follows that $G \simeq Q/A$, where Q is the thin building of type H_4 and A is a group of automorphisms of Q satisfying condition (Q1) of [3]. From (Q1) and the fact that both G and Q are locally icosahedral it follows immediately that two vertices (1-objects) of Q in the same A-orbit have distance at least 4 and hence inner product at most $-1/2$.

Suppose $\sigma \in A$ fixes a vertex $x \in Q$. Then σ must fix each neighbor of x, and hence all of Q, i.e., $\sigma = 1$. Let x_1, x_2, x_3 be three vertices in the same A-orbit, then $(x_i,x_j) \le -1/2$ for $1 \le i,j \le 3$ and $(x_1 + x_2 + x_3, x_1 + x_2 + x_3) \le 3 - 6(1/2) = 0$, whence $(x_i,x_j) = -1/2$ for $1 \le i,j \le 3$ and $x_1 + x_2 + x_3 = 0$. This shows immediately that $|A| \le 3$. We shall see that each of the possibilities $|A| = 1$, 2, or 3 leads to a quotient unique up to isomorphism. This is clear for $|A| = 1$. If $|A| = 2$ and $1 \ne \sigma \in A$ then $(x,\sigma x)$ cannot be $-1/2$ otherwise σ would fix the plane π on $\{0,x,\sigma x\}$, but $\pi \cap Q$ is a regular hexagon in π and σ must fix two of its vertices, contradiction. Similarly $(x,\sigma x)$ cannot be $-(1/2)\tau$, otherwise we find that σ fixes two vertices of the regular decagon $\pi \cap Q$. Consequently, $\sigma = -1$. And in fact $Q/\langle -1\rangle$ is locally icosahedral on 60 vertices and its automorphism group is isomorphic to $[\mathrm{Alt}(5) \times \mathrm{Alt}(5)] \cdot 2$ and is transitive (rank 5) on the vertex set; it has diagram



Finally, if $|A| = 3$ and $1 \ne \sigma \in A$ then $\det \sigma = 1$ so $\sigma = \sigma_{a,b}$ for certain $a,b \in Q$. Since $\sigma^3 = 1$ we may take $\bar{a}^3 = b^3 = 1$. Conjugating σ with $\overline{\sigma_{1,1}}$ we get $\sigma_{b,a}$ and conjugation by $\sigma_{c,d}$ yields $\sigma_{\bar{c}ac,\bar{d}bd}$. Since $PSL_2(5) \simeq \mathrm{Alt}(5)$ has only one conjugacy class of elements of order 3 we may assume that σ is one of the elements $\sigma_{1,1}$, $\sigma_{1,a}$, or $\sigma_{a,a}$. But $\sigma_{1,1}$ has order 1 and $\sigma_{a,a}$ fails the condition $(1,\sigma 1) = -(1/2)$. Thus $\sigma = \sigma_{1,a}$ is (up to conjugacy) the unique possibility. And in fact $Q/\langle\sigma_{1,a}\rangle$ is locally icosahedral on 40 vertices and its automorphism group is isomorphic to $SL_2(5) \circ Z_4$ and is transitive on the vertex set; it has diagram



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