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Trilinear alternating forms on a vector space of dimension 7

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ABSTRACT

For vector spaces of dimension at most 7 over fields of cohomological dimension at most 1 (including algebraically closed fields and finite fields) all trilinear alternating forms and their isotropy groups are determined.

1. Introduction.

Let E be a vector space over the field F of dimension $n < \infty$. Whereas the problem of classifying bilinear alternating forms on E is well known and very elementary, the classification of trilinear alternating forms seems tractable for small values of n only. By the classification of r-linear alternating forms on E we mean the determination of equivalence classes of these forms with respect to the following equivalence relation:

Two r-linear forms f,f' on E are said to be *equivalent* if there is a linear transformation $g \in GL(E)$, the general linear group on E, such that

$$f(x_1,...,x_r) = f'(gx_1,...,gx_r) \qquad (x_1,...,x_r \in E). \tag{1}$$

Letting $g \in GL(E)$ act on the vector space of r-linear forms f on E by means of

$$(g \cdot f)(x_1,...,x_r) = f(g^{-1}x_1,...,g^{-1}x_r) \qquad (x_1,...,x_r \in E), \tag{2}$$

we can rewrite (1) as $g \cdot f = f'$. We shall often write gf instead of $g \cdot f$. The equivalence classes are the $GL(E)$-orbits under this action. An r-linear form f on E is called *alternating* if for all $x_1,...,x_r \in E$ with $x_i = x_j$ for at least two distinct i,j $(1 \leq i,j \leq r)$, we have $f(x_1,...,x_r) = 0$. The vector space of all r-linear alternating forms on E will be denoted by $\mathrm{Alt}_r(E)$. The classification of r-linear alternating forms on E consists of the determination of all $GL(E)$-orbits in $\mathrm{Alt}_r(E)$. We shall write $G = GL(E)$, and $G_f = \{g \in G \mid gf = f\}$ for any $f \in \mathrm{Alt}_r(E)$; we shall refer to

G_f as the *stabilizer (in G) of f*. Obviously, G_f is an algebraic group defined over F; in fact, it is a Zariski closed subgroup of G. If K is an extension field of F, then any r-linear form f on E has a unique extension to an r-linear form f^K on $E\otimes_F K$. Two r-linear forms f,f' on E are called *K-equivalent* whenever their extensions f^K, $(f')^K$ are equivalent. We shall denote by $\overline{F}$ the algebraic closure of F and we shall write $\overline{E} = E\otimes_F\overline{F}$, $\overline{G} = GL(\overline{E})$, and $\overline{f} = f^{\overline{F}}$.
Now, dim $\overline{G}$ and dim $\overline{G_f}$ are well-defined integers and the orbit $\overline{Gf}$ of $\overline{f}$ has the structure of an algebraic variety of dimension dim $\overline{G}$ − dim $(\overline{Gf})$, see BOREL [1]. Since dim $\overline{G} = n^2$ and dim $\mathrm{Alt}_r(\overline{E})=\binom{n}{r}$, this yields

$$\dim\ \overline{G_f} \geqslant n^2 - \binom{n}{r}. \tag{3}$$

If $\overline{G}$ has finitely many orbits in $\mathrm{Alt}_r(\overline{E})$, there must be a form $\overline{f}$ in $\mathrm{Alt}_r(\overline{E})$ for which equality holds in (3). But equality implies $n^2 \geqslant \binom{n}{r}$. Hence, if $\mathrm{Alt}_r(\overline{E})$ consists of finitely many equivalence classes, either $r\leqslant 2$, or $r=3$ and $n\leqslant 8$. This illustrates why the problem of classifying trilinear alternating forms differs from its bilinear counterpart.
If $r=2$, there are finitely many G-orbits, see DIEUDONNÉ [4]. We shall see in the sequel that for arbitrary F this is no longer true if $r=3$ and $n=7$. However, if F is algebraically closed, $r=3$ and $n\leqslant 7$, there are finitely many G-orbits. The classification in this case was carried out by J.A. SCHOUTEN [7] in 1931 for $F=\mathbb{C}$, and, independently, by CRESP [3] in 1976 for F algebraically closed of characteristic $\neq 2,3$. GUREVICH [5] gives an answer to the classification problem with $F=\mathbb{C}$, $r=3$ and $n=8$. In this paper, we present a relatively short proof of the classification for F algebraically closed of arbitrary characteristic, $r=3$ and $n=7$. Furthermore, we compute the stabilizers in G of representatives of each of its orbits and derive the classification for $n\leqslant 7$ and $r=3$ over certain non-algebraically closed fields – including all finite fields – by use of (noncommutative) first order Galois cohomology. In particular for a finite field F, it turns out that the number of projective $GL(n,F)$-orbits on $\bigwedge^3 F^n$ is 4 if $n=6$, and 11 if $n=7$.

After finishing this work, it has come to our knowledge that MIGLIORE [10] has also studied trilinear alternating forms on F^7 for finite fields F.

The outline of the paper is as follows. The main results are stated in Section 2. In the hope that for finite fields this classification will be of some relevance to finite geometry and combinatorics, we have included two corollaries of this classification. Section 2 ends with an elementary but basic lemma for the classification over an algebraically closed field. The classification over these fields appears in Section 4, while Section 3 is devoted to the computation of the stabilizers of the forms given in Section 2. Finally, in Section 5 the necessary Galois cohomology is introduced and an account is given of the classification over a field of cohomological dimension at most 1.

2. Notation and Statement of Results.

In this section, we let $E = F^n$ be the standard vector space over the field F of dimension n, consisting of column vectors, and denote by $e_1,...,e_n$ the standard basis vectors. The dual E^* of E will be interpreted as a vector space of row vectors. Thus, $e_1^\top,...,e_n^\top$, where $x^\top$ is the transposed of $x \in E$, is a dual basis of $e_1,...,e_n$, i.e., is a basis of E^* with

$$e_i^\top e_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

By well-known theory, cf. BOURBAKI [2], the space $\mathrm{Alt}_r(E)$ may be identified with the skew-symmetric tensor product $\overset{r}{\wedge} E^*$ of r copies of E^*. In fact, we shall identify the two by viewing $x_1^\top \wedge x_2^\top \wedge ... x_r^\top$ for $x_1,...,x_r \in E$ as elements of $\mathrm{Alt}_r(E)$ by means of

$$x_1^\top \wedge \cdots \wedge x_r^\top(u_1,...,u_r) = \sum_{\sigma \in S_r} sg(\sigma) \prod_{i=1}^{r} x_i^\top u_{\sigma(i)} \quad (u_1,...,u_r \in E) \tag{4}$$

where S_r stands for the symmetric group on r symbols and $sg(\sigma)$ is the sign of $\sigma \in S_r$. Obviously,

$$\{e_{i_1}^\top \wedge \cdots \wedge e_{i_r}^\top \mid 1 \leqslant i_1 < i_2 < ... < i_r \leqslant n\}$$

is a basis of $\mathrm{Alt}_r(E)$.

The action of $g \in G = GL(E)$ on $\mathrm{Alt}_r(E)$ defined in (2) is determined by

$$g(x_1^\top \wedge \cdots \wedge x_r^\top) = x_1^\top g^{-1} \wedge \cdots \wedge x_r^\top g^{-1} \quad (x_1, \ldots, x_r \in E), \tag{5}$$

where $x^\top h$ for $x \in E$ and $h \in G$ is the element of E^* satisfying $(x^\top h)v = x^\top (h\, v)$ for all $v \in E$. In order to simplify notation, we shall often abbreviate $e_i^\top$ to $\underline{i}$. Also, we shall write $\underline{ij}$ for $\underline{i} \wedge \underline{j}$ and $\underline{ijk}$ for $\underline{i} \wedge \underline{j} \wedge \underline{k}$, where $1 \leqslant i,j,k \leqslant n$. We are now ready to formulate the main results.

2.1. Theorem. *Let E be a vector space of dimension 7 over an algebraically closed field F. Then any nonzero trilinear alternating form on E is equivalent to exactly one of the nine forms f_i for $i = 1,...,9$ of Table 1. The stabilizer G_i of f_i in $G = GL(E)$ has the structure given in Table 1.*

Here, F_+^m, for $m \in \mathbb{N}$, denotes a group with composition series of length m all of whose m factors are isomorphic to the additive group of the field F, F^* denotes the multiplicative group of F, and μ_3 the subgroup $\{x \in F \mid x^3 = 1\}$. Furthermore, λ, μ are such that

$$P_\lambda(x) = \begin{cases} X^2 - \lambda & \text{if } F \text{ has odd characteristic} \\ X^2 + \lambda X + 1 & \text{otherwise} \end{cases}$$

and $P_\mu(X) = X^3 - \mu$ are irreducible in $F[X]$. We note that Table 1 contains only a rough description of the isomorphism type of G_f. A full description can be found in Section 3.

Table 1
Alternating trilinear forms on a
vector space of dimension 7 over a perfect field F.

name	form	description of G_f
f_1	$\overline{123}$	$F_\star^{12}.(SL(3,F).GL(4,F))$
f_2	$\overline{123}+\overline{145}$	$F_\star^{14}.(GL(2,F).(Sp(4,F).F^\star))$
f_3	$\overline{123}+\overline{456}$	$(F_\star^{6}.(SL(3,F).SL(3,F).F^\star)).\mathbb{Z}_2$
f_4	$\overline{162}+\overline{243}+\overline{135}$	$F_\star^{14}.(GL(3,F).F^\star)$
f_5	$\overline{123}+\overline{456}+\overline{147}$	$(F_\star^{10}.(GL(2,F).GL(2,F))).\mathbb{Z}_2$
f_6	$\overline{152}+\overline{174}+\overline{163}+\overline{243}$	$F_\star^{12}.(SL(3,F).F^\star)$
f_7	$\overline{146}+\overline{157}+\overline{245}+\overline{367}$	$F_\star^{8}.((GL(2,F).GL(2,F))\,/\,F^\star)$
f_8	$\overline{123}+\overline{145}+\overline{167}$	$F_\star^{6}.(Sp(6,F).F^\star)$
f_9	$\overline{123}+\overline{456}+\overline{147}+\overline{257}+\overline{367}$	$G_2(F).\mu_3$
$f_{10,\lambda}$	$\overline{123}+\lambda(\overline{156}+\overline{345}+\overline{426})$ if char $F\neq 2$ $\overline{126}+\overline{153}+\overline{234}+\lambda(\overline{156}+\overline{345}+\overline{426})+(\lambda^2+1)\overline{456}$ if char$F=2$	$(F_\star^{6}.(F^\star.SL(3,K))).\mathbb{Z}_2$
$f_{11,\lambda}$	$f_{10,\lambda}+\overline{147}$	$(F_\star^{10}GL(2,K)).\mathbb{Z}_2$
$f_{12,\mu}$	μf_9	$G_2(F).\mu_3$

A form $f\in \mathrm{Alt}_3(E)$, where $E=F^n$, can conveniently be represented by a diagram in the following way. A *diagram* consists of *vertices* and *lines* which are labelled 3-cycles of vertices. The vertices are the coordinates $1,...,n$. A line is a 3-cycle (i,j,k) with $\lambda=f(e_i,e_j,e_k)\neq 0$ whose label is λ. In a picture of a diagram the lines (i,j,k) for which $i<j<k$ does not hold are suppressed. If $f(e_i,e_j,e_k)=1$, the label is not drawn at all. See Figure 1 for an example.
A perfect field is said to have cohomological dimension at most 1 if the Galois group of its algebraic closure has cohomological dimension at most 1; cf. SERRE [8], where it is also shown that finite fields have cohomological dimension at most 1.

2.2. Theorem. *If E is a vector space of dimension 7 over a perfect field F of cohomological dimension at most 1, then any nonzero trilinear form is equivalent to one of the forms f_i $(1\leqslant i\leqslant 9)$, $f_{j,\lambda}$ $(j=10,11)$, $f_{12,\mu}$ of Table 1. Moreover, the only pairs of equivalent forms in Table 1 occur among the pairs $f_{j,\alpha}, f_{j,\beta}$, where either $j=10,11$ and $P_\alpha(X)$, $P_\beta(X)$ define isomorphic extension fields of F in the respective cases, or $j=12$ and the polynomials $X^3-\alpha$, $X^3-\beta$ define isomorphic cubic extension fields of $F(\sqrt[3]{-1})$ and $\alpha^{-1}\beta$ has a cube root in $F(\sqrt[3]{-1})$.*

Table 2
The number of forms in $\mathrm{Alt}_3(F^7)$, F a finite field of order q, equivalent to one of the forms in Table 1.

form	$\mid G / G_{f_i} \mid$	degree
f_1	$(q^7-1)(q^5-1)(q^3+1)/(q-1)^2$	13
f_2	$(q^7-1)(q^5-1)(q^4+q^2+1)(q^2+q+1)q^2$	20
f_3	$\frac{1}{2}(q^7-1)(q^5-1)(q^3+1)(q^2+1)q^9$	26
f_4	$(q^7-1)(q^6-1)(q^5-1)(q^3+q^2+q+1)q^4$	25
f_5	$\frac{1}{2}(q^7-1)(q^6-1)(q^5-1)(q^2+1)(q^2+q+1)q^9$	31
f_6	$(q^7-1)(q^6-1)(q^5-1)(q^4-1)q^6$	28
f_7	$(q^7-1)(q^6-1)(q^5-1)(q^3-1)(q^2+1)q^{11}$	34
f_8	$(q^7-1)(q^5-1)(q^3-1)q^6$	21
f_9	$\epsilon(q^7-1)(q^5-1)(q^4-1)(q^3-1)(q-1)q^{15}$	35
$f_{10,\lambda}$	$\frac{1}{2}(q^7-1)(q^5-1)(q^3-1)(q^2-1)q^9$	26
$f_{11,\lambda}$	$\frac{1}{2}(q^7-1)(q^6-1)(q^5-1)(q^3-1)(q-1)q^9$	31
$f_{12,\mu}$	$\frac{1-\epsilon}{2}(q^7-1)(q^5-1)(q^4-1)(q^3-1)(q-1)q^{15}$	35
f_{12,μ^2}	$\frac{1-\epsilon}{2}(q^7-1)(q^5-1)(q^4-1)(q^3-1)(q-1)q^{15}$	35

2.3. Corollary. *Let E,F be as in Theorem 2.2. If H is a subgroup of $GL(E)$ which is irreducible on E and fixes a nonzero trilinear alternating form on E, then H is a subgroup of the Chevalley group of type G_2 over F extended by* $\mu_3 = \{x \in F^* \mid x^3 = 1\}$.

Proof. By inspection of the stabilizers of Table 1 and their action on E (see Section 3), it is immediate that G_{f_9}, $G_{f_{12,\mu}}$ are the only stabilizers of trilinear alternating forms which are irreducible on E. Since H is irreducible and fixes a nonzero element of $\mathrm{Alt}_3(E)$, it must be a subgroup of a conjugate of G_{f_9}, $G_{f_{12,\mu}}$, and hence of $G_2(F)\mu_3$.

2.4. Corollary. *Let F be a finite field of order q. Then the number of forms in* $\mathrm{Alt}_3(E)$, *with $E = F^7$, equivalent to one of the forms in Table* 1 *is as given in Table* 2.

Proof. Given $f \in \mathrm{Alt}_3(E)$, the number in question is $\mid G \mid / \mid G_f \mid$, where $G = GL(E)$ has order $q^{21} \prod_{i=1}^{7} (q^i - 1)$. Since G_f is conjugate to G_{f_i} if f is equivalent to f_i, and the right hand side is known by Table 1, the result follows by straightforward computation.

Here, $\epsilon = 1$ if $gcd(q-1,3) = 1$ and $\epsilon = \frac{1}{3}$ otherwise; λ,μ are as in Table 1. (Observe that if $gcd(q-1,3) = 1$, then $f_{12,\mu}$ and f_{12,μ^2} do not occur.)

We finish this section by introducing the basic tools of the proof of the above theorems. Let E, F,G be as before and $r \geq 1$. For $f \in \mathrm{Alt}_r(E)$ we define the *kernel*

$$f_3^g = a_3\underline{12} - a_2\underline{13} + a_1\underline{23} + a_6\underline{45} - a_5\underline{46} + a_4\underline{56}.$$

It readily follows that $R_0(f_3) = <e_7>$ and that

$$R_{\leqslant 1}(f_3) = <e_1, e_2, e_3, e_7> \ \cup \ <e_4, e_5, e_6, e_7>.$$

Suppose $g \in G_3$. Then g either stabilizes each of the two linear subspaces whose union is $R_{\leqslant 1}(f_2)$, or interchanges them. Moreover, g stabilizes $<e_7>$. Hence, after multiplication of g by an element of A, we may assume that g is of diagonal form with entries $(\lambda, 1, 1, \mu, 1, 1, 1)$ for some $\lambda, \mu \in F^*$. As in 3.1, it follows that $\lambda = \mu = 1$, so that $g \in A$.

3.4. *The form* $f_4 = \underline{162} + \underline{243} + \underline{135}$ *has stabilizer* $G_4 = A$, *where*

$$A = \left\{ \begin{pmatrix} h & 0_{3\times 3} & 0_{3\times 1} \\ r & h(\det h^{-1}) & 0_{3\times 1} \\ & a & \lambda \end{pmatrix} \;\middle|\; \begin{matrix} h \in GL(3,F), \ \lambda \in F^* \\ a \in F^{1\times 6}, \ r \in F^{3\times 3} \\ \text{trace } r = 0 \end{matrix} \right\},$$

so $G_4 \cong F_*^{14} : (GL(3,F) \times F^*)$.

Proof. A simple computation shows that $R_0(f) = <e_7>$ and that $R_1(f) = <e_4, e_5, e_6> \setminus \{0\}$. Suppose $g \in G$. Then g stabilizes $R_0(f)$ and $R_1(f) \cup \{0\}$, so after multiplication by an element of A, we may restrict attention to g of the form

$$g = \begin{pmatrix} & a & 0_{3\times 4} \\ \alpha & 0\ 0 & \\ & 0_{3\times 3} & 1_4 \end{pmatrix}$$

for some $\alpha \in F$ and $a = (a_{ij}) \in GL(3,F)$. Now,

$$\begin{aligned} g^{-1} f_4 = {} & (a_{11}a_{22} - a_{12}a_{21})\underline{162} + (a_{11}a_{23} - a_{13}a_{21})\underline{163} + \\ & (a_{12}a_{23} - a_{13}a_{22})\underline{263} + \alpha(a_{23}a_{32} - a_{22}a_{33})\underline{123} + \\ & (a_{22}a_{31} - a_{21}a_{32})\underline{124} + (a_{23}a_{31} - a_{21}a_{33})\underline{134} + \\ & (a_{23}a_{32} - a_{22}a_{33})\underline{234} + (a_{11}a_{33} - a_{13}a_{31})\underline{135} + \\ & (a_{12}a_{33} - a_{13}a_{22})\underline{235}, \end{aligned}$$

by straightforward use of (5), so that $g \cdot f_4 = f_4$ implies $\alpha = 0$ and $a = \pm I_3$. It follows that $g \in A$.

3.5. *The form* $f_5 = \underline{123} + \underline{456} + \underline{147}$ *has stabilizer* $G_5 = A$, *where*

$$A = \left\{ \left(\begin{array}{ccccccc} \det h_1^{-1} & \multicolumn{3}{l}{0_{1\times 3}} & & & \\ \alpha_2 & \multicolumn{2}{c}{h_1} & -\lambda_1 & & & \\ \alpha_1 & & & \lambda_2 & \multicolumn{3}{c}{0_{4\times 3}} \\ 0 & 0 & 0 & \det h_2^{-1} & & & \\ -\lambda_3 & 0 & 0 & \alpha_4 & & & \\ -\lambda_4 & 0 & 0 & \alpha_3 & h_2 & & 0_{2\times 1} \\ \alpha_5 & \lambda_2 & \lambda_1 & \alpha_6 & \lambda_4 & \lambda_3 & \det h_1 h_2 \end{array} \right) \middle| \begin{array}{c} h_1, h_2 \in GL(2,F) \\ \lambda_1,\dots,\lambda_4 \in F \\ \alpha_1,\dots,\alpha_6 \in F \end{array} \right\} . < \pi_{(14)(25)(36)} > ,$$

so $G_5 \cong (F_{+}^{10} : (GL(2,F) \times GL(2,F))).\mathbb{Z}_2$.

Proof. By computation, $R_{\leqslant 1}(f_5) = \; <e_5, e_6, e_7> \; \cup \; <e_2, e_3, e_7>$ and $R_{\leqslant 2}(f_5) = \; <e_1, e_2, e_3, e_5, e_6, e_7> \; \cup \; <e_2, \dots, e_7>$. Suppose $g \in G_5$. Then either g stabilizes each of the two 3-dimensional subspaces in $R_{\leqslant 2}(f_5)$, or interchanges them. Hence g stabilizes their linear span $<e_2, e_3, e_5, e_6, e_7>$ as well as their intersection $<e_7>$. After multiplication of g by a suitable element of A, we may, therefore, assume that g stabilizes $<e_1, e_4>$. But then g either interchanges $<e_1>, <e_4>$ or stabilizes each of them (as it stabilizes $R_{\leqslant 2}(f_5) \cap <e_1, e_4>$). In the former case, multiply by $\pi_{(14)(25)(36)} \in A$, so as to reduce study to the case where $g<e_i> \; = \; <e_i>$ for each i $(i = 1,4)$. In fact, by multiplication with a suitable diagonal element of A, we may even assume $ge_i = e_i$ for $i = 1,4$. Since $R_{\leqslant 1}(f_5) \cap \ker f_5^{e_1} = \; <e_5, e_6>$ and $R_{\leqslant 1}(f_5) \cap \ker f_5^{e_4} = \; <e_2, e_3>$, these subspaces must be g-invariant. It readily follows that g belongs to A.

3.6. *The form* $f_6 = \underline{152} + \underline{174} + \underline{163} + \underline{243}$ *has stabilizer* $G_6 = A$, *where* $A = A_1 A_2$ *with*

$$A_1 = \left\{ \left(\begin{array}{ccccccc} \lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \lambda_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ \lambda_4 & 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda_5 & 0 & \lambda_4 & \lambda_3 & \lambda_1 & 0 & 0 \\ \lambda_6 & 0 & 0 & \lambda_2 & 0 & \lambda_1 & 0 \\ \lambda_7 & 0 & 0 & 0 & 0 & 0 & \lambda_1 \end{array} \right) \middle| \begin{array}{c} \lambda_1 \in F^* \\ \lambda_2 \dots, \lambda_7 \in F \end{array} \right\}$$

$$A_2 = \left\{ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & h & (a^{\mathrm{T}})^{-1} \end{array} \right) \middle| \begin{array}{c} h \in F^{3\times 3},\ a \in SL(3,F) \\ h = h^{\mathrm{T}} \end{array} \right\} ,$$

so $G_6 \cong F_{+}^{12} : (SL(3,F) \times F^*)$.

Proof. Notice that $R_{\leqslant 1}(f_6)=<e_5,e_6,e_7>$ and $R_{\leqslant 2}(f_6)=<e_2,...,e_7>$. Suppose $g \in G_6$. After multiplication of g by an element of A_1 from the left, we may assume that $ge_1=e_1$. Since g stabilizes $<e_5,e_6,e_7>$, after multiplication of g by an element of A_2, we may restrict attention to the case where g has the form

$$\begin{bmatrix} 1 & 0\ \ 0\ \ 0 & \\ 0 & & \\ 0 & a & 0_{4\times 3} \\ 0 & & \\ \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0 & \alpha_4 & \alpha_3\\ 0 & 0 & \alpha_2 \\ 0 & 0 & 0\end{matrix} & \begin{matrix}\alpha_1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix} \end{bmatrix}$$

for some $a \in GL(3,F)$, with $\alpha_1 \in F^*$, and $\alpha_2,\alpha_3,\alpha_4 \in F$. By a computation using (5), we obtain

$$\begin{aligned} f_6 = g^{-1}f_6 = {} & \alpha_1\underline{15}(\underline{2}a^{\mathrm{T}}) + \alpha_3\underline{14}(\underline{2}a^{\mathrm{T}})+\alpha_4\underline{13}(\underline{2}a^{\mathrm{T}})+ \\ & \underline{17}(\underline{4}a^{\mathrm{T}})+\underline{16}(\underline{3}a^{\mathrm{T}})+\alpha_2\underline{14}(\underline{3}a^{\mathrm{T}}) + \\ & (\det a)\underline{243}. \end{aligned}$$

It follows that $\underline{3}a^{\mathrm{T}} = \underline{3}$, $\underline{4}a^{\mathrm{T}} = \underline{4}$ and $\underline{2}a^{\mathrm{T}} = \alpha_1^{-1}\underline{2}$, so that

$$a = \begin{bmatrix} \alpha_1^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$\underline{243} = f_6 - \underline{152} - \underline{174} - \underline{163} = \underline{14}(\alpha_1^{-1}\alpha_3\underline{2} + \alpha_2\underline{3})+\underline{13}(\alpha_4\underline{2})+(\det a)\underline{243}.$$

This implies $\alpha_1 = (\det a)^{-1}=1$ and $\alpha_2 = \alpha_3 = \alpha_4=0$, so that $g=1 \in A$, indeed.

3.7. *The form* $f_7 = \underline{146} + \underline{157} + \underline{245} + \underline{367}$ *has stabilizer* $G_7=A$ *where* $A=A_1A_2$ *with*

$$A_1 = \left\{ \begin{bmatrix} 1_3 & \begin{matrix} \alpha_1 & \beta_1 & \beta_3 & -\alpha_3 \\ \alpha_2 & \beta_2 & \beta_1 & -\alpha_1 \\ \alpha_3 & \beta_2 & \gamma_3 & \delta_3 \end{matrix} \\ 0_{4\times 3} & 1_4 \end{bmatrix} \middle| \ \alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma_3,\delta_3 \in F \right\}, \textit{ and}$$

$$A_2 = \left\{ \left. \begin{pmatrix} \frac{\det h}{(\alpha\delta - \beta\gamma)} \begin{bmatrix} \alpha\delta + \beta\gamma & \alpha\gamma & \beta\delta \\ 2\alpha\beta & \alpha^2 & \beta^2 \\ 2\gamma\delta & \gamma^2 & \delta^2 \end{bmatrix} & 0_{3\times 4} \\ 0_{4\times 3} & \begin{matrix} \delta h^{-1} & \begin{matrix} 0 & \gamma \\ -\gamma & 0 \end{matrix} \\ \begin{matrix} 0 & -\beta \\ \beta & 0 \end{matrix} & \alpha h^{\mathrm{T}} \end{matrix} \end{pmatrix} \right| \begin{matrix} h \in GL(2,F) \\ \text{and} \\ \alpha,\beta,\gamma,\delta \in F, \\ \alpha\delta \neq \beta\gamma \end{matrix} \right\}$$

so $G_7 \cong F_\star^8 : ((GL(2,F)\times GL(2,F)) / F^\star)$.

Proof. A direct computation leads to the relations

$$R_{\leqslant 1}(f_7) = \{x_1e_1+x_2e_2+x_3e_3 \mid x_1^2 = x_2x_3\}, \text{ and}$$

$$R_{\leqslant 2}(f_7) = \{\sum_{i=1}^{7} x_ie_i \mid x_4x_6+x_5x_7=0\}.$$

Suppose $g \in G_7$. Since $<e_1,e_2,e_3>$ is the linear subspace of E spanned by $R_{\leqslant 2}(f_7)$, it is left invariant by g. Write $W_0 = <e_4,e_5,e_6,e_7>$.
Let W be a 4-dimensional subspace of E with $W \cap <e_1,e_2,e_3> = \{0\}$ and $f_7(W,W,W)=0$. Then there is an element $r \in G$ of the form

$$\begin{bmatrix} 1_3 & a \\ 0_{3\times 4} & 1_4 \end{bmatrix} \quad \text{with} \quad a=(a_{ij}) \in F^{3\times 4},$$

such that $W = rW_0$. The requirement $f_7(W,W,W)=0$ implies

$$\begin{cases} 0=g^{-1}f_7(e_4,e_5,e_6)=-a_{12}+a_{23} \\ 0=g^{-1}f_7(e_4,e_5,e_7)=a_{11}+a_{24} \\ 0=g^{-1}f_7(e_4,e_6,e_7)=a_{31}+a_{14} \\ 0=g^{-1}f_7(e_5,e_6,e_7)=a_{32}-a_{13} \ . \end{cases}$$

Hence, $r \in A_1$. Thus, up to multiplication of g by an element of A_1, we may assume that g stabilizes W_0. Consider the group H generated by the elements of A_2 restricted to W_0. This is a subgroup of index 2 in the full subgroup of the linear group on W_0 preserving the quadric

$$\left\{ \sum_{i=4}^{7} x_ie_i \mid x_4x_6+x_5x_7=0 \right\},$$

see DIEUDONNÉ [4]. In fact, H is isomorphic to $(GL(2,F)\times GL(2,F))/F^\star$, the central product of two copies of $GL(2,F)$. Multiplication of g by a suitable element of A_2, therefore leads to the case where we have either $g\mid_{W_0} = 1$ or

$g \mid W_0 = \pi_{(46)}$. However, $\pi_{(46)}$ cannot be extended to a linear transformation on E preserving f_7 (i.e., to a member of G_7), for, denoting such an extension by h and writing $he_2 = x_1e_1 + x_2e_2 + x_3e_3$, we would then have

$$1 = h^{-1}f(e_2,e_4,e_5) = f(he_2,e_6,e_5) = \sum_{i=1}^{3} x_i f(e_i,e_6,e_5) = 0,$$

which is absurd. The conclusion is that we are left with the case where $g \mid W_0 = 1$. Explicit verification of $f_7 = g^{-1}f_7$ by means of (5), yields that $g = 1$, so that $g \in A$.

3.8. *The form* $f_8 = \underline{123} + \underline{145} + \underline{167}$ *has stabilizer* $G_8 = A$ *with* $A = A_1A_2$, *where*

$$A_1 = \left\{ \begin{pmatrix} \lambda^{-1} & 0\;0\;0\;0\;0\;0 \\ & \lambda\;0\;0\;0\;0\;0 \\ & 0\;1\;0\;0\;0\;0 \\ a & 0\;0\;\lambda\;0\;0\;0 \\ & 0\;0\;0\;1\;0\;0 \\ & 0\;0\;0\;0\;\lambda\;0 \\ & 0\;0\;0\;0\;0\;1 \end{pmatrix} \;\middle|\; a \in F^6 \times 1,\ \lambda \in F^* \right\}, \text{ and}$$

$$A_2 = \left\{ \begin{pmatrix} 1 & 0_{1\times 6} \\ 0_{1\times 6} & h \end{pmatrix} \;\middle|\; \begin{array}{l} h \in GL(6,F) \\ h \text{ preserves } \underline{23} + \underline{45} + \underline{67} \end{array} \right\}.$$

Thus $G_8 \cong F^6{:}(Sp(6,F).F^*)$.

Proof. Observe that $R_{\leqslant 1}(f_8) = \langle e_2,...,e_7 \rangle$. Let $g \in G_8$. Since g stabilizes $R_1(f_8)$, we may assume that G fixes e_1 (after multiplication by a suitable element of A_1). But then g fixes $f_8^{e_1}$, whence $g \in A_2$, so $g \in A$.

3.9. *The form* $f_9 = \underline{123} + \underline{456} + \underline{147} + \underline{257} + \underline{367}$ *has stabilizer* $G_9 = A$, *with* $A = G_2(F)\mu_3$, *where* $G_2(F)$ *is the Chevalley group of type* G_2 *over* F, *and* $\mu_3 = \{\alpha \in F \mid \alpha^3 = 1\}$.

Proof. Direct computation shows

$$R_{\leqslant 2}(f_9) = R_2(f_9) \cup \{0\} = \{\sum_{i=1}^{7} x_ie_i \in E \mid x_7^2 = x_1x_4 + x_2x_5 + x_3x_6\}.$$

Let H denote the full subgroup of G preserving the quadric $R_2(f_9)$ and set $H_1 = H \cap SL(E)$. Then H_1 preserves the quadratic form

$$Q(x) = -x_7^2 + x_1x_4 + x_2x_5 + x_3x_6 \quad (x = (x_1,...,x_7) \in E)$$

and each $h \in H$ is of the form λh_1 for some $\lambda \in F^*$ and some $h_1 \in H_1$. Moreover, $G_2(F)$ is the subgroup of H_1 preserving f_9, cf. SPRINGER [9]. In fact, one can show that the bilinear multiplication $\cdot$ on E defined by $f_9(x,y,z)= B(x\cdot y,z)$ for all $x,y,z \in E$, where B is the bilinear form associated with Q, is the restriction to the orthoplement of unity in the 8-dimensional split algebra of the octonions. In order to see this, take the 8-dimensional split algebra of the octaves to be the set of matrices of the form

$$\begin{bmatrix} \alpha & v \\ w & \beta \end{bmatrix} \quad \text{where } \alpha,\beta \in F \text{ and } v,w \in F^3$$

supplied with the usual matrix addition, with multiplication

$$\begin{bmatrix} \alpha & v \\ w & \beta \end{bmatrix} \begin{bmatrix} \alpha' & v' \\ w' & \beta' \end{bmatrix} = \begin{bmatrix} \alpha\alpha'-v\cdot w' & \alpha v' + \beta' v + w\times w' \\ \alpha' w + \beta w' + v\times v' & \beta\beta' - v'\cdot w \end{bmatrix}$$

where $\cdot$ and $\times$ stand for the standard inner and outer product on F^3, respectively, and with quadratic form Q given by

$$Q\left(\begin{bmatrix} \alpha & v \\ w & \beta \end{bmatrix}\right) = \alpha\beta + v\cdot w \ .$$

The unity of this algebra is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; its orthoplement is spanned by the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ e_{i+1} = \begin{bmatrix} 0 & \epsilon_i \\ 0 & 0 \end{bmatrix}, \ e_{i+4} = \begin{bmatrix} 0 & 0 \\ \epsilon_i & 0 \end{bmatrix}$$

where $i=1,2,3$ and $(\epsilon_i)_i$ is the standard basis of F^3. A straightforward computation shows that the restrictions of Q and $(x,y,z) \mapsto B(x\cdot y,z)$ to the orthoplement of the unity are as indicated above when taken with respect to $e_1,\ldots,e_7$.
We need two properties of the group A in order to finish the proof. They are formulated in the following lemma.

Lemma. (i) *The group $G_2(F)$, viewed as subgroup of G preserving f_9 and B is transitive on the set of unordered pairs of mutually non-orthogonal (with respect to B) totally isotropic* 1*-dimensional subspaces of E.*
(ii) *If $g \in G$ and $\mu \in F^*$ satisfy $gf_9 = \mu f_9$, then there is $\lambda \in F^*$ with $\mu = \lambda^{-3}$ and $g \in \lambda A$.*

Statement (i) is a well-known property of $G_2(F)$, see for instance SPRINGER [9].
Before proving part (ii) of the lemma, let us show how it leads to $G_9=A$. Suppose $g \in G_9$. Write $g=\lambda h$ with $\lambda \in F^*$ and $h \in H_1$. Then $f_9=gf_9 = \lambda^{-3}(hf_9)$, so $hf_9 = \lambda^3 f_9$. By part (ii) of the lemma, we have $h \in \mu_3\, \lambda^{-1}A = \lambda^{-1}A$, whence $g=\lambda h \in A$, as wanted.
It remains to prove part (ii) of the lemma. To this end, let g,μ be as in the hypothesis. Notice that g preserves $R_2(f_9)$, the quadric defined by Q. By part (i) of the lemma, we may assume that $g\langle e_1\rangle = \langle e_1\rangle$ and $g\langle e_4\rangle = \langle e_4\rangle$ (after multiplication of g by a suitable member of A). Thus, g preserves $e_1^{\perp}$ and

$e_4^\perp$, where $\perp$ is taken with respect to B. The following subspaces of E are preserved by g: $R_0(f_9^{e_1}) \cap e_1^\perp \cap e_4^\perp = \langle e_5,e_6\rangle$, $R_0(f_9^{e_4}) \cap e_1^\perp \cap e_4^\perp = \langle e_2,e_3\rangle$, and $(R_0(f_9^{e_1})+R_0(f_9^{e_4}))^\perp = \langle e_7\rangle$. Therefore, g must have the form

$$\begin{bmatrix} \alpha_1 & & & & \\ & a & & & \\ & & \alpha_4 & & \\ & & & b & \\ & & & & \alpha_7 \end{bmatrix} \quad \text{where } \alpha_1,\alpha_4,\alpha_7 \in F^* \text{ and } a,b \in GL(2,F).$$

Since

$$\begin{bmatrix} \det a^{-1} & & & & \\ & a & & & \\ & & \det a & & \\ & & & a^{-1} & \\ & & & & 1 \end{bmatrix} \quad \text{for } a \in GL(2,F),$$

represents an element of $G_2(F)$, we may restrict attention to the case where $ge_2=e_2$ and $ge_3=e_3$. But then g preserves $e_2^\perp \cap \langle e_5,e_6\rangle = \langle e_6\rangle$ and $e_4^\perp \cap \langle e_5,e_6\rangle = \langle e_5\rangle$, so g has diagonal form $(\alpha_1,1,1,\alpha_4,\alpha_5,\alpha_6,\alpha_7)$. From $gf_9 = \mu f_9$, we get $\mu^{-1} = \alpha_1 = \alpha_4\alpha_5\alpha_6 = \alpha_1\alpha_4\alpha_7 = \alpha_5\alpha_7 = \alpha_6\alpha_7$, whence $\alpha_1 = \mu^{-1}$, $\alpha_4^3 = \mu$, $\alpha_5 = \alpha_6 = \alpha_4\mu^{-1}$, $\alpha_7 = \alpha_4^{-1}$. Set $\lambda = \alpha_4^{-1}$. Then $\mu = \lambda^3$ and g has diagonal form $(\lambda^3,1,1,\lambda^{-1},\lambda^2,\lambda^2,\lambda) = \lambda(\lambda^2,\lambda^{-1},\lambda^{-1},\lambda^{-2},\lambda,\lambda,1)$, whence $g \in \lambda A$.

3.10. *Fix $\lambda \in F$ such that the polynomial*

$$P_\lambda(X) = \begin{cases} X^2 - \lambda & \text{if } F \text{ has odd characterictic} \\ X^2 + \lambda X + 1 & \text{if } F \text{ has even characteristic} \end{cases},$$

is irreducible, and set $K = F(\alpha)$, where α is a root of $P_\lambda(X)$. Then, the form

$$f_{10,\lambda} = \begin{cases} \underline{123} + \lambda(\underline{156} + \underline{345} + \underline{426}) & \text{if } F \text{ has odd characteristic,} \\ \underline{126} + \underline{153} + \underline{234} + \lambda(\underline{156} + \underline{345} + \underline{426}) + (\lambda^2+1)\underline{456} & \text{otherwise} \end{cases}$$

is K-equivalent but not equivalent to f_3. Furthermore, let σ denote the generator of the Galois group of K over F, and set

$$g = \begin{bmatrix} 1_3 & -\alpha 1_3 & 0_{3\times 1} \\ 1_3 & \alpha 1_3 & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & \frac{1}{4} \end{bmatrix} \quad \text{if } F \text{ has odd characteristic,}$$

$$g = \begin{bmatrix} \alpha & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha+\lambda & 0 & 0 & \alpha\lambda+\lambda^2+1 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 & 0 & \alpha\lambda^{-1}+1 & 0 \\ \alpha+\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \alpha\lambda+1 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 & 0 & \alpha\lambda^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{-1} \end{bmatrix} \quad \textit{otherwise,}$$

The stabilizer of $f_{10,\lambda}$ *is* $G_{10,\lambda}=g^{-1}Ag$,

$$\textit{where } A = \left\{ \begin{bmatrix} k & 0_{3\times 3} & 0_{6\times 1} \\ 0_{3\times 3} & k^\sigma & \\ a & a^\sigma & \rho \end{bmatrix} \middle| \begin{array}{l} k \in SL(3,K) \\ a \in K^3 \\ \rho \in F^* \end{array} \right\} < \pi_{(14)(25)(36)} >.$$

In particular, $G_{10,\lambda} \cong (F_*^6 :(F^* \times SL(3,K))).\mathbb{Z}_2$.

Proof. Since

$$g \cdot f_{10,\lambda}^K = \begin{cases} 4f_3^K & \text{if } F \text{ has odd characteristic,} \\ f_3^K & \text{otherwise,} \end{cases}$$

$f_{10,\lambda}$ is K-equivalent to f_3. Put $G_3^K = G_{f_3}^K$. For $h \in G_3^K$, its conjugate $g^{-1}hg$ belongs to G if and only if it is fixed by σ, i.e., if and only if

$$h^\sigma = (g^\sigma g^{-1})h(g^\sigma g^{-1})^{-1}. \tag{8}$$

But $g^\sigma g^{-1} = \begin{bmatrix} 0_{3\times 3} & 1_3 & 0_{6\times 1} \\ 1_3 & 0_{3\times 3} & \\ 0_{1\times 6} & & 1 \end{bmatrix}$, so by use of 3.3, we readily

find that the elements $h \in G_3^K$ satisfying (8) are contained in A. The conclusion is that any $x \in G_{10,\lambda}$ is of the form $x = g^{-1}hg$ with $h \in A$, so that $G_{10,\lambda} = g^{-1}Ag$.

It may be worthy of note that the restriction of $f_{10,\lambda}$ to $<e_1,...,e_6>$ can be obtained as the restriction to $<e_1,e_2,e_3> \otimes K$ (viewed as a vector space over F) of trace$_{K/F}(f_1^K)$.

3.11. *Let* λ, $P_\lambda(X)$, $K=F(\alpha)$, σ, g *be as in* 3.10. *The form* $f_{11,\lambda}=f_{10,\lambda} + \underline{147}$ *is K-equivalent but not equivalent to* f_5. *The stabilizer of* $f_{11,\lambda}$ *is* $G_{11,\lambda} = g^{-1}Ag$, *where*

$$A = \left\{ \left(\begin{array}{ccccccc} \det k^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & & & -\lambda_1 & 0 & 0 & 0 \\ & \multicolumn{2}{c}{k} & & & & \\ \alpha_1 & & & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\det k^{-1})^\sigma & 0 & 0 & 0 \\ -\lambda_1^\sigma & 0 & 0 & \alpha_2^\sigma & & & 0 \\ & & & & \multicolumn{2}{c}{k^\sigma} & \\ \lambda_2^\sigma & 0 & 0 & \alpha_1^\sigma & & & 0 \\ \alpha_5 & \lambda_2 & \lambda_1 & \alpha_5^\sigma & \lambda_2^\sigma & \lambda_1^\sigma & \det(k^\sigma k) \end{array} \right) \;\middle|\; \begin{array}{c} k \in GL(2,K) \\ \lambda_1, \lambda_2, \alpha_1, \alpha_2, \alpha_5 \in K \end{array} \right\} < \pi_{(14)(25)(36)} >.$$

In particular, $G_{11,\lambda} \cong (F_\star^{10}.GL(2,K)).\mathbb{Z}_2$.

Proof. Since $g \cdot f_{11,\lambda} = \mu f_5$, with $\mu = 4$ if F has even characteristic and $\mu = 1$ otherwise, $f_{11,\lambda}$ is K-equivalent to f_5.

Put $G_5^K = G_{f_5}^K$, and recall that this group has been described in §3.5. For

$$h = \left(\begin{array}{ccccccc} \det g_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & & & -\lambda_1 & 0 & 0 & 0 \\ & \multicolumn{2}{c}{k} & & & & \\ \alpha_1 & & & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \det g_2^{-1} & 0 & 0 & 0 \\ -\lambda_1^\sigma & 0 & 0 & \alpha_4 & & & 0 \\ & & & & \multicolumn{2}{c}{g_2} & \\ \lambda_2^\sigma & 0 & 0 & \alpha_5 & & & 0 \\ \alpha_5 & \lambda_2 & \lambda_1 & \alpha_6 & \lambda_4 & \lambda_3 & \det(g_1 g_2) \end{array} \right) \in G_5^K,$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in K$ and $g_1, g_2 \in GL(2,K)$, we have $g^{-1} h g \in G_{11,\lambda}$ if and only if $h^\sigma \in G$, i.e., if and only if $h \in A$. This settles 3.11.

3.12. *Suppose* $K = F(\alpha)$ *is a cubic field extension of* F *with* $\alpha^3 = \mu$ *for some* $\mu \in F^*$. *Then* $f_{12,\mu} = \mu f_9$ *and* $f_{12,\mu^2} = \mu^2 f_9$ *are mutually inequivalent forms, both inequivalent but* K*-equivalent to* f_9, *and the stabilizer* G_9 *of* f_9 *coincides with both the stabilizer* $G_{12,\mu}$ *of* $f_{12,\mu}$ *and* G_{12,μ^2}.

Proof. This is immediate from the lemma in 3.9.

4. The Classification over an Algebraically Closed Field.

In this section, we prove Theorem 2.1. Thus, we assume that $F = \overline{F}$ is an algebraically closed field, and set $E = F^7$.

The following result will be used frequently.

4.1. Lemma. *Suppose D is a linear subspace of E of dimension $m \geqslant 3$. If $f \in \mathrm{Alt}_3(D)$ and $e_1, e_2, e_3 \in D$ satisfy $f(e_1, e_2, e_3) \neq 0$, then e_1, e_2, e_3 can be extended to a basis $e_1, \ldots, e_m$ of D such that*

$$f(e_1, e_2, e_3) \neq 0 \text{ and } f(e_i, e_j, e_k) = 0 \text{ for } 1 \leqslant i \leqslant j \leqslant 3 \text{ and } 4 \leqslant k \leqslant m. \quad (9)$$

Proof. The intersection of the kernels of the three linear functionals $x \mapsto f(e_i, e_j, x)$ for $1 \leqslant i < j \leqslant 3$ is a linear subspace of D of dimension $m-3$. Taking $e_4, \ldots, e_m$ to be a basis of this subspace, we obtain $e_1, \ldots, e_m$ as stated.

Notice that the relations in (9) are invariant under linear transformations preserving the subspaces $< e_1, e_2, e_3 >$ and $< e_4, \ldots, e_m >$. Let $f \in \mathrm{Alt}_3(E)$. Write $C = \ker f$ and let D be a complement of C in E, so that $E = C \oplus D$. Let m be the dimension of D. Now, $G = GL(E)$ is transitive on ordered pairs of subspaces of given dimensions forming a decomposition of E. Hence, for any basis $e_1, \ldots, e_7$, we can replace f by an equivalent form whose kernel is $C = < e_{m+1}, \ldots, e_7 >$ (complemented by $D = < e_1, \ldots, e_m >$).

From now on, let $f \in \mathrm{Alt}_3(E)$.

4.2. Lemma. *Suppose $f \neq 0$. Then $\mathrm{rk} f \geqslant 3$. If, moreover, $\mathrm{rk} f \leqslant 4$, then $\mathrm{rk} f = 3$ and f is equivalent to f_1.*

Proof. Set $r = \mathrm{rk} f$, and let D be a complement of $\ker f$ with basis $e_1, \ldots, e_r$. If $r < 3$, then $f|_D = 0$, so $f = 0$, a contradiction. Hence $r \geqslant 3$. If $r \leqslant 4$, then $f|_D$ is equivalent to $f_1|_D$, whence f is equivalent to f_1 by Lemma 4.1.

4.3. Lemma. *If $\mathrm{rk} f = 5$, then f is equivalent to f_2.*

Proof. Let $e_1, \ldots, e_5$ be a basis of a complement D of $\ker f$ such that (9) is satisfied for $f|_D$. Since $\mathrm{rk} f = 5$, one of the values $f(e_i, e_4, e_5)$, for $i = 1,2,3$, must be nonzero. Without loss of generality, we may assume $f(e_1, e_4, e_5) = 1$. Thus,

$$f = \underline{123} + \underline{145} + \lambda \underline{245} + \mu \underline{345} \quad \text{for some } \lambda, \mu \in F.$$

Taking $g \in G$ with $g\underline{1} = \underline{1} - \lambda \underline{2} - \mu \underline{3}$ and $g\underline{i} = \underline{i}$ for $i > 1$, we get $gf = f_3$, whence the lemma.

4.4. Lemma. *Suppose $\lambda \in F^*$, $\mu \in F$, and let*

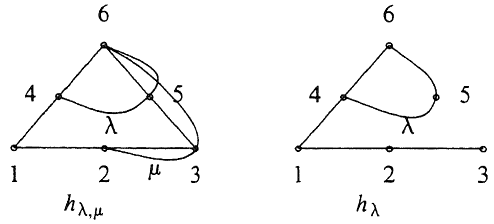
$$h_{\lambda,\mu} = \underline{123} + \underline{146} + \underline{356} + \lambda \underline{456} + \mu \underline{236}, \quad \text{and}$$

$$h_\lambda = \underline{123} + \underline{146} + \lambda \underline{456}.$$

Then $h_{\lambda,\mu}$ and h_λ are both equivalent to f_3.

Proof. Take $g \in G$ with $g \cdot \underline{4} = \underline{4} - \lambda^{-1} \underline{3}$, $g \cdot \underline{2} = \underline{2} - (\lambda^{-1} - \mu) \underline{6}$, $g \cdot \underline{i} = \underline{i}$ for $i = 1,3,5,6,7$. Then $g \cdot h_{\lambda,\mu} = h_\lambda$. It therefore remains to prove that h_λ is equivalent to f_3. Take $g_1 \in G$ with $g_1 \cdot \underline{5} = \underline{5} - \lambda^{-1} \underline{1}$ and $g_1 \cdot \underline{i} = \underline{i}$ for $i \neq 5$. Then $g_1 \cdot h_\lambda = \underline{123} + \lambda \underline{456}$, which is clearly equivalent to f_3.

Figure 1.
The diagrams of the forms $h_{\lambda,\mu}$ and h_λ.



4.5. Lemma. *For $\lambda,\mu \in F$, suppose $f = \underline{145} + \underline{246} + \underline{356} + \mu\underline{123} + \lambda\underline{456}$. Then f is equivalent to f_4 whenever $\mu=0$ or $\lambda^2\mu = -4$, and equivalent to f_3 otherwise.*

Proof. If $\mu=0$, it is readily checked that f is equivalent to f_4. Thus, let $\mu\neq 0$. There is $\chi \in F^*$ with $\chi^2 = \mu$ (since F is algebraically closed). Now, $h \in G$, determined by $h\cdot\underline{4} = \chi\underline{4}$, $h\cdot\underline{i} = \chi^{-1}\underline{i}$ for $i=1,2$, $h\underline{j} = \underline{j}$ for $j=3,5,6,7$ applied to f yields $h\cdot f = \underline{145} + \underline{246} + \underline{356} + \underline{123} + \lambda\chi\underline{456}$. Therefore, without loss of generality, we assume that $\mu=1$. Let $\alpha, \beta, \gamma, \delta, \zeta, \eta \in F$ satisfy $\alpha\beta, \gamma\delta, \zeta\eta \neq 1$, and take $g \in G$ with $g\cdot\underline{7} = \underline{7}$ and

$$g\cdot 1 = \underline{1} + \alpha\underline{6},\ g\cdot\underline{2} = \underline{2} + \gamma\underline{5},\ g\cdot\underline{3} = \underline{3} + \xi\underline{4}, \qquad (10)$$
$$g\cdot\underline{6} = \underline{6} + \beta\underline{1},\ g\cdot\underline{5} = \underline{5} + \delta\underline{2},\ g\cdot\underline{4} = \underline{4} + \eta\underline{3}.$$

Then

$$g\cdot f = \begin{cases} (1-\beta\gamma + \beta\zeta - \gamma\zeta + \beta\lambda)\underline{145} + \\ (\beta - \delta - \beta\delta\zeta + \zeta - \beta\lambda\delta)\underline{124} + \\ (\eta - \beta\gamma\eta + \beta - \gamma + \beta\lambda\eta)\underline{135} + \\ (-\delta\eta + \beta\eta - \beta\delta + 1 - \beta\delta\lambda\eta)\underline{123} + \\ (\alpha - \gamma + \zeta - \zeta\gamma\alpha + \lambda)\underline{456} + \\ (-\alpha\delta + 1 - \zeta\delta - \alpha\zeta - \lambda\delta)\underline{246} + \\ (1 + \alpha\eta - \gamma\eta - \alpha\gamma + \lambda\eta)\underline{356} + \\ (\eta - \delta - \alpha\eta\delta + \alpha - \lambda\eta\delta)\underline{236}. \end{cases} \qquad (11)$$

First of all, assume that F has odd characteristic. Putting $\eta = \delta = \alpha = 0$ and $\beta = \gamma = -\xi = \frac{1}{2}\lambda$, we get the form $g\cdot f = (1+\frac{1}{4}\lambda^2)\underline{145} + \underline{123} + \underline{246} + \underline{356}$. If $\lambda^2+4=0$, then the form is equivalent to f_4. Otherwise, rescaling $\underline{1}$ reduces the proof to the case where $\lambda=0$. Thus we may assume $f = \underline{145} + \underline{246} + \underline{356} + \underline{123}$. Now, applying the transformation g of (10) again, but now with $\beta = \delta = \eta = 1$ and $\alpha = \gamma = \zeta = -1$, we get by (11) that $g\cdot f = 2(\underline{135} + \underline{246})$, so that f is equivalent to f_3. This settles the case of odd characteristic. From now on in this proof, assume that F has even characteristic. Take $\psi \in F^*$ such that $\psi^2 + \lambda\psi + 1 = 0$, and consider the transformation g of (10), but now with $\alpha = \delta = \eta = 0$ and $\beta = \gamma = \zeta = \psi$. Then $g\cdot f = \underline{123} + \underline{246} + \underline{356} + \lambda\underline{456}$. Thus, f is equivalent to f_4 if $\lambda=0$, and equivalent to f_3 otherwise according to Lemma 4.4.

4.6. Lemma. *If* $\mathrm{rk} f = 6$, *then* f *is equivalent to either* f_3 *or* f_4.

Proof. Let $e_1,\ldots,e_6$ be a basis of a complement D of $\ker f$ such that (9) is satisfied for $f|_D$. First, suppose that $f(e_4,e_5,e_6)\neq 0$. Then, up to rescaling $\underline{6}$, we have

$$f = \underline{123} + \underline{456} + \sum_{i=1}^{3}(\alpha_i \underline{i45} + \beta_i \underline{i46} + \gamma_i \underline{i56}) \quad \text{for } \alpha_i,\beta_i,\gamma_i \in F.$$

Applying a suitable transformation in E preserving the subspace of E^* spanned by $\underline{1}$, $\underline{2}$, $\underline{3}$ and the subspace spanned by $\underline{4}$, $\underline{5}$, $\underline{6}$, we can easily reduce study to the case where $f = \underline{123} + \underline{456} + \alpha\underline{145} + \beta\underline{246} + \gamma\underline{356}$ for $\alpha,\beta,\gamma \in F$. If at least two of α,β,γ are zero, we are done. Without loss of generality, suppose $\beta\gamma\neq 0$. After rescaling by $g \in G$ with $g\cdot\underline{4} = \beta\underline{4}$, $g\cdot\underline{5} = \gamma\underline{5}$ and $g\cdot\underline{6} = \beta^{-1}\gamma^{-1}\underline{6}$, we may assume that $f = \underline{123} + \underline{456} + \lambda\underline{145} + \underline{356} + \underline{246}$ for some $\lambda \in F$. If $\lambda = 0$, then f is equivalent to f_3 by Lemma 4.4, and if $\lambda \neq 0$, the desired result follows from Lemma 4.5 after rescaling of $\underline{1}$ by λ^{-1}. Next, suppose that $f(e_4,e_5,e_6)=0$. Since $\mathrm{rk} f = 6$, without loss of generality we may assume that $f(e_1,e_4,e_5)=1$. Replacing e_2,e_3 by a basis of the kernel of the linear functional $x \mapsto f(x,e_4,e_5)$ on $\langle e_1,e_2,e_3\rangle$, if necessary, we get $f(e_2,e_4,e_5)=f(e_3,e_4,e_5)=0$. Replacing e_6 by $e_6+f(e_1,e_5,e_6)e_4 - f(e_1,e_4,e_6)e_5$, we also have $f(e_1,e_5,e_6)= f(e_1,e_4,e_6)= 0$. Thus we are left with the case where $f = \underline{123} + \underline{145} + \alpha\underline{346} + \beta\underline{356} + \gamma\underline{246} + \delta\underline{256}$, for some $\alpha,\beta,\gamma,\delta \in F$. Since $\mathrm{rk} f = 6$, at least one of $\alpha,\beta,\gamma,\delta$ must be nonzero. Thanks to symmetry, we only need consider $\beta \neq 0$. After rescaling, $\underline{6}$, we may put $\beta=1$. Let $g_1 \in G$ be such that $f\underline{3} = \underline{3} - \delta\underline{2}$, $g_1\underline{5} = \underline{5} - \alpha\underline{4}$ and $g_1\underline{i} = \underline{i}$ for $i \neq 3,5$. Then

$$g\cdot f = \underline{123} + \underline{145} + \underline{356} + (\gamma - \alpha\delta)\underline{246}.$$

Thus, either $\gamma = \alpha\delta$ and f is equivalent to f_4, or $\gamma \neq \alpha\delta$ and after rescaling $\underline{2}$, we arrive at the form discussed in Lemma 4.5 (with $\lambda=0$). Hence f is equivalent to f_3 or f_4 in all cases.

If f and f' are forms on E and D,D' are linear subspaces of E, then $f|_D$ is said to be *equivalent to* f' *on* D', if there is $g \in G = GL(E)$ with $g(D)= D'$ and $(gf)|_{D'} = f'|_{D'}$. Furthermore, given the standard basis $e_1,\ldots,e_7$ of E, we shall write W_i for the subspace spanned by $\{e_j \mid j \neq i\}$.

4.7. Lemma. *If* $\mathrm{rk} f = 7$ *and* E *has a 6-dimensional subspace* W *such that* $f|_W$ *is equivalent to* f_3 *on* W_7, *then* f *is equivalent to one of* f_5, f_7, f_9.

Proof. Let $e_1,\ldots,e_6$ be a basis of W such that $f|_W = (\underline{123} + \underline{456})|_W$. Choose $e_7 \in E$ such that $f(e_i,e_j,e_7)=0$ whenever i,j satisfy $1\leqslant i,j\leqslant 3$ or $4\leqslant i,j\leqslant 6$ (cf. Lemma 4.1). Then

$$f = \underline{123} + \underline{456} + \sum_{i=1}^{3}\sum_{j=4}^{6}\alpha_{ij}\,\underline{ij7} \quad \text{for some } \alpha_{ij} \in F.$$

By use of a linear transformation stabilizing the linear subspaces of E spanned by $\underline{1},\underline{2},\underline{3}$, by $\underline{4},\underline{5},\underline{6}$, and by $\underline{7}$ respectively, we can restrict attention to the case where

$$f = \underline{123} + \underline{456} + \underline{367} + \alpha\underline{257} + \beta\underline{147} \text{ for some } \alpha,\beta \in F.$$

If $\alpha = \beta = 0$, then f is equivalent to f_5. Without harming generality, assume $\beta \neq 0$. Rescaling $\underline{1}$ and $\underline{2}$ by β^{-1} and β respectively, we may take $\beta = 1$. If $\alpha = 0$, then f is equivalent to f_7. If $\alpha \neq 0$, then choose $\gamma \in F$ with $\gamma^3 = \alpha$ and rescale $\underline{1},\underline{2},\underline{3},\underline{7}$ by $\gamma,\gamma^{-2},\gamma,\gamma^{-1}$, respectively. The resulting form is f_9. This proves the lemma.

4.8. Lemma. *Suppose* $\operatorname{rk} f = 7$ *and* E *has no* 6*-dimensional subspace* U *such that* $f|_U$ *is equivalent to* f_3 *on* W_7. *If there is a subspace* W *such that* $f|_W$ *is equivalent to* f_4 *on* W_7, *then* f *is equivalent to* f_6.

Proof. Let W be as stated. There is a basis $e_1,\dots,e_6$ of W such that $f|_W = \underline{125} + \underline{134} + \underline{236}$. Choose $e_7 \in E$ such that $f(e_i,e_j,e_7) = 0$ for i,j, 1,2,3. Then

$$f = \underline{125} + \underline{134} + \underline{236} + \sum_{i=1}^{6}\sum_{j=4}^{6}\alpha_{ij}\underline{ij7} \text{ for some } \alpha_{ij} \in F.$$

Replacing f by $g{\cdot}f$, where $g \in G$ is given by $g{\cdot}\underline{1} = \underline{1} - \alpha_{25}\underline{7}$, $g{\cdot}\underline{2} = \underline{2} - \alpha_{36}\underline{7}$, $g{\cdot}\underline{3} = \underline{3} + \alpha_{26}\underline{7}$, and $g{\cdot}\underline{i} = \underline{i}$ for $i = 4,5,6,7$ we have that $\alpha_{25} = \alpha_{26} = \alpha_{36} = 0$. First of all, assume that $\alpha_{15} \neq 0$. Then $\alpha_{16} \neq 0$, because otherwise $f|_{W_4} = (\underline{125} + \underline{236} + \alpha_{15}\underline{157} + \alpha_{35}\underline{357} + \alpha_{56}\underline{567})|_{W_4}$ would be equivalent to f_3 on W_7. (Take $h \in GL(W_4)$ with $h.\underline{1} = \underline{1} - \alpha_{35}\alpha_{15}^{-1}\underline{3}$, $h{\cdot}\underline{6} = \underline{6} - \alpha_{35}\alpha_{15}^{-1}\underline{5}$, and $h{\cdot}\underline{i} = \underline{i}$ for $i = 1,2,3,5,7$, and apply the 6-dimensional analogue of Lemma 4.4 to $h{\cdot}(f|_{W_4})$).
We claim that $\alpha_{56} = 0$. For if $\alpha_{56} \neq 0$, then $g_1 \in G$ with $g_1{\cdot}\underline{1} = \underline{1} - \alpha_{56}\alpha_{16}^{-1}\underline{5}$, $g_1{\cdot}\underline{4} = \underline{4} - \alpha_{35}\alpha_{56}^{-1}\alpha_{16}\underline{7}$, and $g_1{\cdot}\underline{i} = \underline{i}$ for $i \neq 1,4$ applied to f yields

$$(g_1{\cdot}f)|_{W_4} = (\underline{125} + \underline{236} + \alpha_{15}\underline{157} + \alpha_{16}\underline{167} + \gamma\underline{137})|_{W_4}$$

for some $\gamma \in F$. This implies by Lemma 4.4 that $f|_{g^{-1}W_4}$ is equivalent to f_3 on W_7, contradictory to the assumption. Thus, $f = \underline{125} + \underline{236} + \alpha_{35}\underline{357} + \alpha_{16}\underline{167} + \alpha_{15}\underline{157} + \underline{134} + \sum_{i=1}^{6}\delta_i\underline{i47}$, for certain $\delta_i \in F$. Consider $f|_{W_4}$, once again. In view of Lemma 4.4 and the assumption, we have $\alpha_{35} \neq 0$, and in view of Lemma 4.5, we have $\alpha_{15}^2 = 4\alpha_{35}\alpha_{16}$. In particular, F has odd characteristic and $\alpha_{15} \neq 0$. Applying $g_2 \in G$ with $g_2{\cdot}\underline{1} = \underline{1} - \frac{1}{2}\alpha_{15}\alpha_{16}^{-1}\underline{3}$, $g_2{\cdot}\underline{2} = \underline{2} + \frac{1}{2}\alpha_{15}\underline{7}$, $g_2{\cdot}\underline{6} = \underline{6} - \frac{1}{2}\alpha_{15}\alpha_{16}^{-1}\underline{5}$, $g_2{\cdot}\underline{7} = \alpha_{16}^{-1}\underline{7}$, and $g_2{\cdot}\underline{i} = \underline{i}$ for $i \neq 1,2,6,7$ to f leads to the form

$$f' = g_2.f = \underline{125} + \underline{236} + \ + \underline{134} + \underline{167} + \sum_{i=1}^{6}\delta_i\underline{i47}, \text{ with } \delta_i \in F.$$

Now, consider $(f')|_{W_1} = (\underline{236} + (\sum_{i=2}^{6}\delta_i\underline{i})\underline{47})|_{W_1}$.

Clearly, this form is equivalent to f_3 on W_7 if it is nondegenerate on W_1. Hence $\delta_5=0$. Next, consider

$$(f')|W_6 = (\underline{125} + \underline{1}(\underline{3} + \delta_1\underline{4})(\underline{4} + \underline{7}) + \delta_2\underline{24}(\underline{4} + \underline{7}) + \delta_3(\underline{3} + \delta_1\underline{4})\underline{4}(\underline{4} + \underline{7}))\,|\,_{W_6}.$$

By Lemma 4.4, this form is equivalent to f_3 on W_7, unless $\delta_3=0$. Hence, $\delta_3=0$. Consider $(f')|W_3 = (\underline{125} + (\underline{6} + \delta_1\underline{4})\underline{71} + \delta_2\underline{247} + \delta_6(\underline{6} + \delta_1\underline{4})\underline{47})|W_3$. Again, by Lemma 4.4 and the assumption, we get $\delta_6=0$. Consider $f'\,|\,_{W_5} = (\underline{236} + \underline{134} + \underline{167} + \delta_1\underline{147} + \delta_2\underline{247})|W_5$. If $\delta_2=0$, then $\delta_1=0$ by Lemma 4.5; if $\delta_1=0$, then $\delta_2=0$ by Lemma 4.4. But $\delta_1 = \delta_2=0$ yields $f' = \underline{125} + \underline{236} + \underline{134} + \underline{167}$, which is equivalent to f_6. Thus, the case where $\delta_1,\delta_2 \neq 0$ remains. But then, according to Lemma 4.5, we must have $\delta_1^2 = -4\delta_2$. Applying $g_3 \in G$ given by $g_3{\cdot}\underline{2} = \underline{2} + 2\delta_1^{-1}\underline{1}$, $g_3{\cdot}\underline{3} = \underline{3}-2\delta_1^{-1}\underline{7}$, $g_3{\cdot}\underline{4} = \underline{4}-2\delta_1^{-1}\underline{6}$, $g_3{\cdot}\underline{7}=-4\delta_1^{-2}\underline{7}$ and $g_3{\cdot}\underline{i} = \underline{i}$ for $i=1,5,6$ to f', we obtain $g_3{\cdot}f' = \underline{125} + \underline{134} + \underline{236} + \underline{247}$, which is clearly equivalent to f_6. This settles the case where $\alpha_{15} \neq 0$.

Thus, for the rest of the proof, we may assume $f=\underline{125} + \underline{134} + \underline{236} + \sum_{i=1}^{6}\sum_{j=4}^{6}\alpha_{ij}\underline{ij7}$, with $\alpha_{15} = \alpha_{25} = \alpha_{26} = \alpha_{36}=0$.

Suppose that $\alpha_{14} \neq 0$. If $\alpha_{34}=0$, then a permutation of basis elements (according to (3,1,2)(4,5,6) on the indices) reduces this case to the previous one. If $\alpha_{34} \neq 0$, then apply $g_4 \in G$ with $g_4{\cdot}\underline{1} = \underline{1} - \alpha_{34}\underline{7}$ and $g{\cdot}\underline{i} = \underline{i}$ for $i \neq 1$ to f so as to obtain $g_4{\cdot}f = f - \alpha_{34}\underline{347} - \alpha_{34}\underline{257}$; the permutation (4,5)(2,3) on the basis elements then leads $g_4{\cdot}f$ to a form with $\alpha_{15} \neq 0$ and $\alpha_{25} = \alpha_{26} = \alpha_{36} = 0$, a situation that has been dealt with before. Thus we may assume $\alpha_{14}=0$. Similarly, we may assume $\alpha_{34}=0$. Now,

$$f=\underline{125} + \underline{134} + \underline{236} + \alpha_{16}\underline{167} + \alpha_{24}\underline{247} + \alpha_{35}\underline{357} + \alpha_{45}\underline{457} + \alpha_{46}\underline{467} + \alpha_{56}\underline{567}$$

If $\alpha_{46} \neq 0$, then $g_5 \in G$ given by $g_5{\cdot}\underline{1} = \underline{1} - \alpha_{56}\alpha_{46}^{-1}\underline{5}$, $g_5{\cdot}\underline{2} = \underline{2} + \alpha_{56}\alpha_{24}\alpha_{46}^{-1}\underline{7}$, $g_5{\cdot}\underline{4} = \underline{4} - \alpha_{45}\alpha_{46}^{-1}\underline{5}$, $g_5{\cdot}\underline{6} = \underline{6} - \alpha_{45}\alpha_{16}\alpha_{46}^{-1}\underline{7}$, $g_5{\cdot}\underline{i} = \underline{i}$ for $i=3,5,7$, is such that

$$(g_5f)|W_3 = (\underline{125} + \alpha_{16}\underline{167} + \alpha_{24}\underline{247} + \alpha_{46}\underline{467})|W_3$$

is equivalent to f_3 on W_7, cf. Lemma 4.4. Hence, $\alpha_{46}=0$, and by symmetry, also, $\alpha_{56} = 0$. Consequently,

$$f=\underline{125} + \underline{134} + \underline{236} + \alpha_{16}\underline{167} + \alpha_{24}\underline{247} + \alpha_{35}\underline{357}.$$

Since $\operatorname{rk} f=7$, at least one of $\alpha_{16},\alpha_{24},\alpha_{35}$ is nonzero. If exactly one of them is nonzero, we are done. Suppose therefore, that at least two of them, say α_{16} and α_{24} (without loss of generality) are nonzero. Suppose that F has even characteristic. Then $g_6 \in G$ determined by $g_6{\cdot}\underline{1} = \underline{1} + \alpha_2\mu\underline{2}$, $g{\cdot}\underline{4} = \underline{4} + \alpha_1\mu\underline{6}$, $g_6{\cdot}\underline{7} = \underline{7} + \mu\underline{3}$, and $g{\cdot}\underline{i}=\underline{i}$ for $i=2,3,5,6$, where $\mu \in F^*$ is such that

$\mu^2 + \mu + (\alpha_1\alpha_2)^{-1} = 0$, satisfies

$$(g_6 f)|_{W_5} = (\underline{134} + \alpha_1\underline{167} + \alpha_2\underline{247} + \alpha_2\mu\underline{234})|_{W_5}.$$

The latter form is equivalent to f_3 on W_7 by Lemma 4.4. This contradiction shows that F has odd characteristic. But then $f|_{W_5}$ is equivalent to f_3 on W_7 by Lemma 4.5, leading to the final contradiction. This ends the proof of Lemma 4.8.

4.9. Lemma. *Suppose* $\mathrm{rk} f = 7$ *and* E *has no* 6*-dimensional subspace* U *such that* $f|_U$ *has rank* 6. *Then* f *is equivalent to* f_8.

Proof. In view of the assumption, there must be a 5-dimensional subspace W of E such that $f|_W$ is equivalent to f_2 on $<e_1,...,e_5>$. Take such a subspace and let $e_1,..,e_7$ be a basis of E such that $W = W_6 \cap W_7$ and $f|_W = (\underline{123} + \underline{145})|_W$. Now, $f(e_6,e_i,e_j) = f(e_7,e_i,e_j) = 0$ for $i,j = 1,...,5$, for otherwise E has a 6-dimensional subspace V on which $f|_V$ has rank 6. If $f(e_2,e_6,e_7) \neq 0$, then $f|_{W_3}$ clearly has rank 6, a contradiction. Hence, $f(e_2,e_6,e_7) = 0$, and similarly, $f(e_3,e_6,e_7) = f(e_4,e_6,e_7) = f(e_5,e_6,e_7) = 0$. Since $\mathrm{rk} f = 7$, we must have $f(e_1,e_6,e_7) \neq 0$, and f is equivalent to f_8.

Now, Lemmas 4.7, 4.8, and 4.9 classify all forms $f \in \mathrm{Alt}_3(E)$ with $\mathrm{rk} f = 7$. Thus, together with Lemmas 4.2, 4.3, and 4.6, this completes the proof of Theorem 2.1.

5. The Classification over Perfect Fields of Cohomological Dimension at most 1.

Let E be a vector space of dimension $n < \infty$ over a perfect field F, and let $\bar{F}$ be the algebraic closure of F. Galois cohomology provides a tool to keep track of the inequivalent forms for E which become equivalent when extended to forms $\bar{f}$ on $\bar{E} = E \otimes_F \bar{F}$. We shall first recall some basic notions from SERRE [8].

5.1. Definitions. For L a Galois field over F (within $\bar{F}$), denote by Γ_L the Galois group of $\bar{F}$ over L, and set $\Gamma = \Gamma_F$. The latter is a *profinite group,* i.e., it is compact and discontinuous when topologized in such a way that the groups Γ_L for L an extension field of F (contained in $\bar{F}$) form the collection of open subgroups of Γ. Set $\bar{G} = GL(\bar{E})$ and suppose A is a subgroup of G topologized by the discrete topology. Then Γ has a natural action from the right on A given by

$$a^\sigma = (a_{ij}^\sigma)_{1 \leqslant i,j \leqslant n} \quad \text{for} \quad a = (a_{ij})_{1 \leqslant i,j \leqslant n} \in A \quad \text{and} \quad \sigma \in \Gamma.$$

A *cocycle* of Γ *with values in* A is a continuous map $\Gamma \to A$, denoted by $\sigma \mapsto a_\sigma$ ($\sigma \in \Gamma$), such that $a_{\sigma\tau} = a_\sigma^\tau a_\tau$ for any $\sigma, \tau \in \Gamma$. Two such cocycles a and a' are called *cohomologous* if there is $b \in A$ with $a_\sigma' = b^\sigma a_\sigma b^{-1}$ for every $\sigma \in \Gamma$. This defines an equivalence relation on the set of all cocycles of Γ with values in A. The set of equivalence classes is called the *first cohomology of* Γ *in* A and denoted by $H^1(\Gamma,A)$. If A is abelian, $H^1(\Gamma,A)$ can be given a group structure and higher cohomology groups $H^i(\Gamma,A)$ can be defined. Of course, there is always a distinguished element $1 \in H^1(\Gamma, A)$, namely the class corresponding to the trivial cocycle $\sigma \mapsto 1$ ($\sigma \in \Gamma$). We recall that a perfect field F is said to have *cohomological dimension at*

most 1 if the cohomological dimension of the Galois group Γ is at most 1. (This means that for each prime p and each discrete simple Γ-module A with $pA=0$, we have $H^2(\Gamma, A)=0$.) The reader is referred to SERRE [8] for further details. Here, we just mention that finite fields have cohomological dimension at most 1.
Suppose $f \in \mathrm{Alt}_3(E)$. We are interested in the set $T(F,f)$ of equivalence classes of forms $f' \in \mathrm{Alt}_3(E)$ such that $\overline{f'}$ and $\overline{f}$ are equivalent forms in $\mathrm{Alt}_3(\overline{E})$.

5.2. Theorem. *Let F be a perfect field, and suppose $f \in \mathrm{Alt}_3(E)$. Retain the above notation, and put $A=\overline{G}_{\overline{f}}$. Then the following holds:*

(i) *There is a bijective map $\theta: T(F,f) \to H^1(\Gamma, A)$, given by $\theta(f')\sigma = g^\sigma g^{-1}$ ($f' \in T(F,f)$, $\sigma \in \Gamma$), where $g \in \overline{G}$ is such that $g \cdot f' = f$.*

(ii) *If F has cohomological dimension at most 1, and if A^0 denotes the connected component of unity in A, viewed as a linear algebraic group over $\overline{F}$, then A^0 is a normal subgroup of A preserved by Γ, and the natural map $H^1(\Gamma, A) \to H^1(\Gamma, A/A^0)$ is bijective.*

Proof. (i) See SERRE [8] Chapter III, Proposition 1.
(ii) See SERRE [8] Chapter III, Theorem 1, Conjecture 1, Supplement 2, and Corollary 3.

We are now ready for the

5.3. Proof of Theorem 2.2. Suppose $f \in \mathrm{Alt}_3(E)$, $f \neq 0$, and set $A=\overline{G}_{\overline{f}}$.
If $\overline{f}$ of the form $\overline{f}_i$ for some i $(1 \leq i \leq 9)$ with $i \neq 3,5,9$, then A is connected, cf. Table 1, so f is equivalent to f_i by the above theorem.
Suppose $\overline{f} = \overline{f}_3$ (in other words, $G \cdot f \in T(F,f_3)$). Then $A/A^0 \cong \mathbb{Z}_2$ by Table 1 and Γ acts trivially on A/A^0. It is easy to see that $H^1(\Gamma, A/A^0)$ coincides with the set of continuous group morphisms $\Gamma \to A/A^0$. But any such morphism is related to a subgroup Δ of Γ of index 2, whence to a quadratic extension K of Γ (determined by $\Delta = \Gamma_K$). By Theorem 5.2 and § 3.10, there is a unique quadratic extension K of F (within $\overline{F}$) such that f is equivalent to $f_{10,\lambda}$ for each $\lambda \in F$ such that $P_\lambda(X)$ is irreducible in $F[X]$ and reducible in $K[X]$. The same argument holds for $\overline{f} = \overline{f}_5$, with $f_{11,\lambda}$ instead of $f_{10,\lambda}$.
Finally, suppose $\overline{f} = \overline{f}_9$. If F has characteristic 3, then A is connected, as $\mu_3 = \{1\}$, so f is equivalent to f_9 by Theorem 5.2. Suppose from now on that the characteristic of F is distinct from 3. If $A/A^0 = \{1\}$, there is nothing to prove. So, suppose $A/A^0 \neq \{1\}$. Then, by Table 1, we get $A/A^0 \cong \mathbb{Z}_3 \cong \mu_3$ and there is $\sigma \in \Gamma$ acting on A/A^0 as it does on $\mu_3 \subseteq \overline{F}$. Write $\mu_3 = \{1, \omega, \overline{\omega}\}$.
Assume, first of all, that $\mu_3 \subseteq F$. Then the action of Γ on μ_3 is trivial, so a cocycle is a continuous group morphism $\Gamma \to \mu_3$ whose kernel is a normal subgroup of Γ of index 1 or 3. Of course, index 1 only occurs in the case of the trivial cocycle. Thus, any nontrivial cocycle has a kernel Δ of index 3 in Γ corresponding to a Galois extension of F of degree 3 with group Γ/Δ. Applying this to the cocycle $\theta(f)$ of Theorem 5.2, this means that for f there is a unique cubic Galois extension K of F

such that f is K-equivalent to f_9. Let $\gamma \in \Gamma$ be such that $\Gamma = \Delta \cup \Delta\gamma \cup$ There are two cocycles in $H^1(\Gamma, A)$ whose kernel is Δ; they are determined by ω and $\gamma \mapsto \omega^2$, respectively. Hence, there are exactly two classes of forms $T(F,f_9)$ which are K-equivalent to f_9 but not equivalent to f_9. Writing $K = F$ where $\alpha^3 = \mu \in F$ (see LANG [6] Chapter 8, Theorem 10), we can represent th by $f_{12,\mu}$ and f_{12,μ^2}, respectively (cf. § 3.12).

Suppose, next, that μ_3 is not contained in F. Let X be the kernel of the action on μ_3. Then X has index 2 in Γ. Given a cocycle $\sigma \mapsto a_\sigma$ $(\sigma \in \Gamma)$, call the gr $\Delta_a = \{\sigma \in \Gamma \mid a_\sigma = 1\}$ *the kernel of the cocycle.* Clearly, Δ_a has index 3 in Γ, un it is the kernel of the trivial cocycle. Suppose that Δ_a has index 3 in Γ. T $\Delta_a \cap X$ is a subgroup of X of index 3, which is normal in X since it is the ke of the group morphism $\gamma \mapsto a_\gamma$ from X to μ_3. Thus, there is $\sigma \in X$ such that partitioned by $(\Delta_a \cap X)\sigma^i$ for $i = 0,1,2$, whence $\Gamma = \Delta_a \cup \Delta_a\sigma \cup \Delta_a\sigma^2$. N $\Delta_a \cap X$ has index 2 in Δ_a, and we have $\tau \in \Delta_a$ such that $\omega^\tau = \overline{\omega} = \omega^2$ and Δ_a $(\Delta_a \cap X) \cup (\Delta_a \cap X)\tau$. Moreover, the cocycles $\gamma \mapsto a_\gamma \omega^\gamma \omega^{-1}$ and $\gamma \mapsto a_\gamma \overline{\omega}^\gamma$ are cohomologous to a and have kernels $\Delta_a \cap X\{1, \tau\sigma^2\}$ and $\Delta_a \cap X\{1,$ respectively. Since these kernels are open subgroups of Γ, it follows $(\tau\sigma)^2 \in \Delta_a \cap X$. Furthermore, $\Delta_a \cap X = \cap_c \{\gamma \in \Gamma \mid c_\gamma = 1\}$, where c runs all cocycles cohomologous to a, is a normal subgroup of Γ, so $\Gamma/(\Delta_a \cap X) \cong$ the symmetric group on 3 letters. Writing K for the extension of F correspondin $\Delta_{\theta(f)}$, we obtain that $K(\omega)$ is Galois over F with group $< \sigma|_{K(\omega)}, \tau|_{K(\omega)} >$ S_3, that f is $K(\omega)$-equivalent to f_9 and that the $K(\omega)$-equivalence class contai f_9 consists of exactly three equivalence classes of forms on E. (For, there are ex ly three classes of cocycles b with $\Delta_b \cap X = \Delta_{\theta(f)} \cap X$, cf. Theorem 5.2.) By 3 the forms f_9, λf_9, $\lambda^2 f_9$, where $\lambda \in F(\omega)$ is such that $X^3 - \lambda$ is irreducible $F(\omega)$ but reducible over $K(\omega)$, are mutually $F(\omega)$-inequivalent, and $K(\omega)$-equiva to f_9. Let $\alpha \in K$ satisfy $\alpha^3 = \lambda$. Then $\alpha^\tau \alpha^{-1}$ is fixed by σ (for $\alpha^\sigma = \alpha\omega^i$ for s $i \in \{1,2\}$, and $\alpha^{\tau\sigma} = \alpha^{\sigma^{-1}\tau} = \alpha^\tau \omega^{2i}$), whence $\alpha^\tau \alpha^{-1} \in F(\omega)$. By Hilbert 90, LANG [6], there is $\beta \in F(\omega)$ such that $\beta^\tau \beta^{-1} = \alpha^\tau \alpha^{-1}$. Set $\mu = \lambda\beta^{-3}$. Thi an element of F. As for $i = 0,1,2$ the forms $\mu^i f_9$ are clearly $F(\omega)$-equivalen $\lambda^i f_9$, they are mutually inequivalent forms on E which are $K(\omega)$-equivalent to Since f is $K(\omega)$-equivalent to f_9, it follows that f is, up to equivalence, one of $\mu^i f_9 = f_{12,\mu^i}$ $(i = 1,2)$. Also from the above we have that for $\lambda, \mu \in F$ the elen $\lambda^{-1}\mu$ has a cube root in $F(\omega)$ if and only if f_9 is equivalent to $\lambda^{-1}\mu f_9$, which o ously amounts to the equivalence of λf_9 and μf_9. This establishes Theorem 2.2.

5.4. Concluding remark. It is not hard to generalize part of Theorem 2.2 to case of an arbitrary perfect field F by use of SERRE [8]. For, by Proposition 6, position 3 and Lemma 1 of Chapter III [loc. cit.], we have $H^1(\Gamma, \overline{U}) = 0$ whenev is one of the groups $(\overline{F})_*^m$, $Sp(2m,\overline{F})$, $GL(m,\overline{F})$ $(m \in \mathbb{N})$, and using the exact quences of Propositions 6 and 8 of Chapter I [loc. cit.], we get $H^1(\Gamma, SL(m,\overline{F}))$ so that

$$H^1(\Gamma, G_i) = 0 \quad \text{if} \quad i = 1,2,4,6,8 \quad \text{and}$$

$$H^1(\Gamma, G_i) \cong H^1(\Gamma, \mathbb{Z}_2) \quad \text{if} \quad i = 3,5.$$

This implies that for $i = 1,2,3,4,5,6,8$, any trilinear form which is $\bar{F}$-equivalent to f_i must be equivalent to either f_i or one of $f_{j,\lambda}$ (where $j = 10, 11$, if $i = 3, 5$, respectively).

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