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On the Hellinger Type Distances for Filtered Experiments

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We study the Hellinger type distances $\rho_p(P_T,\tilde{P}_T)$ on a filtered space. Here $p \geq 2$ is an arbitrary number and P_T and $\tilde{P}_T$ are two probability measures stopped at a random time T. We give lower and upper bounds for $\rho_p(P_T,\tilde{P}_T)$ in predictable terms.

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1. Introduction

1.1. Let $(\Omega,F,\mathbf{F})$ be a stochastic basis, i.e. a measurable space with a filtration $\mathbf{F}=(F_t)_{t\geq 0}$ such that $\bigvee_{t\geq 0}F_t=F_\infty=F$. Given two probability measures P and $\tilde{P}$ define a probability measure Q by $Q=(P+\tilde{P})/2$. Suppose that $\mathbf{F}$ satisfies the usual assumptions with respect to Q. Consider then the optional projections of the measures $P,\tilde{P}$ and Q with respect to $\mathbf{F}$. We will denote these optional valued processes by $P,\tilde{P}$ and Q, respectively. If T is a $\mathbf{F}$-stopping time, then P_T is the restriction of the measure P to the sub- σ-field F_T of F; define $\tilde{P}_T$ and Q_T similarly. Since the measures P_T are absolutely continuous with respect to the measure Q_T, we can define (Q,F)-martingales ζ and $\tilde{\zeta}$ by

$$\zeta_T = dP_T/dQ_T \text{ and } \tilde{\zeta}_T = d\tilde{P}_T/dQ_T. \tag{1.1}$$

The collection $(\Omega,F,\mathbf{F},P,\tilde{P})$ is called the *binary experiment.*

In the present paper the following distances between stopped measures P_T and $\tilde{P}_T$ are studied

$$\rho_p(P_T,\tilde{P}_T) = \{E_Q|\zeta_T^{1/p}-\tilde{\zeta}_T^{1/p}|^p\}^{1/p} \tag{1.2}$$

where $p\geq 2$. Recall that if $p=2$ then $\rho_2(P_T,\tilde{P}_T)$ is called the Hellinger distance. For more details on such kind of distances see Liese and Vajda (1987). Note that the distances are independent of a particular choise of the dominating measure Q.

1.2. With the binary experiment $(\Omega,F,\mathbf{F},P,\tilde{P})$ we associate the Hellinger process by

$$h = (1/2)((\zeta_- \tilde{\zeta}_-)^{-2}\cdot\langle\zeta^c\rangle + (\sqrt{1+x/\zeta_-} - \sqrt{1-x/\tilde{\zeta}_-})^2 * \nu^{\zeta,Q}) \tag{1.3}$$

Here $\nu^{\zeta,Q}$ is the compensator of the jump measure of the process ζ. It is known that the Hellinger process controls the Hellinger distance in the sense of Jacod and Shiryaev (1987), Section V.4 (see also Valkeila and Vostrikova (1986)). In particular,

$$\rho_2^2(P_T,\tilde{P}_T) \leq 2\sqrt{E_P h_T}. \tag{1.4}$$

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To control ρ_p^p also for $p>2$, along with the Hellinger process (1.3) we introduce the process

$$k(p) = |(1+x/\zeta_-)^{1/p}-(1-x/\tilde{\zeta}_-)^{1/p}|^p * \nu^{\zeta,Q} \tag{1.5}$$

where $p \geq 2$. As is shown in this paper (see Theorem 3.2 below), for each even integer $p \geq 2$ there is a constant $C_p>0$ such that

$$\rho_p^p(P_T,\tilde{P}_T) \leq C_p \ E_P(h_T^{p/2} + k_T(p)); \tag{1.6}$$

for $p=2$, in particular

$$\rho_2^2(P_T,\tilde{P}_T) \leq \ 8E_P h_T \tag{1.7}$$

(cf. (1.4)).

1.2. This paper is organized as follows. In Section 2 more details can be found on the quantities introduced above. In particular the key Burkholder type inequality (2.6) is presented.

The first of two theorems, presented in Section 3 gives upper and lower bounds for ρ_p in terms of the expectation with respect to the measure Q.

In the case where the processes h and $k(p)$ are not necessarily deterministic, it is useful to have bounds in terms of the expectation with respect to the measure P: for an upper bound see Theorem 3.2 below. This upper bound is given in a slightly more general form then (1.6), useful for an application in Section 4, Theorem 4.2.

In Sections 4 and 5 applications to sequences of binary experiments and to a parametric family of experiments are discussed (see (4.1) and (5.1) below). In Theorem 4.1, in particular, we give necessary and sufficient conditions for the convergence to a limiting Gaussian experiment, alternative to those of JACOD and SHIRYAEV (1987), Theorems X.1.12 and X.1.64.

Finally, in Section 5 we demonstrate how to evaluate, based on (1.6), certain modulus of continuity (see (5.4) below) needed in various statistical applications (see, e.g. IBRAGIMOV and HAS'MINSKII (1981), KUTOYANTS (1984), DZHAPARIDZE (1986), VALKEILA and VOSTRIKOVA (1987) and VOSTRIKOVA (1988)).

2. CERTAIN PROPERTIES OF ρ_p AND RELATED PROCESSES

2.1. We assume that $(\Omega,F,\mathbf{F})$ is as described above. Moreover, we assume $F_0=\{\varnothing,\Omega\}$ Q - a.s. For unexplained notation in below we refer to JACOD (1979), JACOD and SHIRYAEV (1987) and LIPTSER and SHIRYAEV (1988).

Let $\mathfrak{D}$ be the space of right-continuous functions with left-hand limits on $\mathbf{R}_+=[0,\infty[$. We can take such versions of the density processes ζ and $\tilde{\zeta}$ that their paths are in $\mathfrak{D}$, and

$$\zeta + \tilde{\zeta} = 2, \ <\zeta^c> \ = \ <\tilde{\zeta}^c>, \ \Delta\zeta = -\Delta\tilde{\zeta}, \text{ and } <\zeta^c,\tilde{\zeta}^c> \ = \ -<\zeta^c> \tag{2.1}$$

(here and elsewhere below the angle brackets process is understood as a (Q,F)-compensator). This follows from the special choice of the dominating measure Q.

Note that the jump measure μ^{ζ} of the (Q,F)-uniformly integrable martingale ζ, as well as its (Q,F)-compensator $\nu^{\zeta,Q}$ involved in (1.3) and (1.5), only charges the set $\{(\omega,t,x):\zeta_{t-}(\omega)>0, \tilde{\zeta}_{t-}(\omega)>0, -\zeta_{t-}(\omega)\leq x \leq \tilde{\zeta}_{t-}(\omega)\}$; see JACOD and SHIRYAEV (1987), Theorem IV.1.33.

Note also, that the processes $k(p)$, $p \geq 2$, related to the discontinuous part of ζ only, exist since $k(p) \leq 2h$ (see the next paragraph), and that $k(p)$, $p \geq 2$, as well as h, are independent of the measure Q (JACOD and SHIRYAEV (1987), Theorem IV.1.22).

By the easily verified inequality

$$|u^{1/q}-v^{1/q}|^q \leq |u^{1/p}-v^{1/p}|^p, \tag{2.2}$$

valid for each $u,v \geq 0$ and $1<p \leq q$ we get the following facts: (i) for $p \geq 2$ we have $k(p) \leq k(2) \leq 2h$, (ii) the process $k(p)$ decreases as p increases and (iii) as $p \to \infty$

$$k(p) \equiv k(p;P,\tilde{P}) \to h(0;P,\tilde{P}) + h(0;\tilde{P},P) \quad (Q - \text{a.s.}),$$

where

$$h(0;P,\tilde{P}) = \tilde{\lambda} 1_{\{\lambda=0\}} * \nu^{\zeta,Q}, \; k(0;\tilde{P},P) = \lambda 1_{\{\tilde{\lambda}=0\}} * \nu^{\zeta,Q}$$

with

$$\lambda = 1 + x/\zeta_-, \; \tilde{\lambda} = 1 - x/\tilde{\zeta}_-; \tag{2.3}$$

see JACOD and SHIRYAEV (1987), IV.1.57, and also IV.1.36 for the definition of the Hellinger process of order $\alpha \in (0,1)$:

$$h(\alpha) \equiv h(\alpha;P,\tilde{P}) = \frac{\alpha(1-\alpha)}{2}\left[\frac{1}{\zeta} + \frac{1}{\tilde{\zeta}}\right]^2 \circ <\zeta^c> + \phi_\alpha(\lambda,\tilde{\lambda}) * \nu^{Q,\zeta}$$

with

$$\phi_\alpha(u,v) = \alpha u + (1-\alpha)v - u^\alpha v^{1-\alpha}.$$

Obviously, $h \equiv h(1/2)$. Note also that for any even integer $p > 2$

$$k(p) = -\sum_{k=1}^{p-1}(-1)^k \binom{p}{k} h(k/p)$$

due to the binomial formula and properties of $h(\alpha)$.

2.2. By (2.2) ρ_p^p decreases too as p increases.

Besides,

$$\rho_p^p(P,\tilde{P}) \to \tilde{P}(\zeta = 0) + P(\tilde{\zeta} = 0) \text{ as } p \to \infty.$$

For the variational distance $\|P - \tilde{P}\| = \rho_1(P,\tilde{P})$, in particular, we have (cf. JACOD and SHIRYAEV (1987), V.4.8, and LIESE and VAJDA (1987), Ch. 2)

$$\|P - \tilde{P}\| \geq \rho_p^p(P,\tilde{P}), \quad c_p\|P - \tilde{P}\| \leq \rho_p(P,\tilde{P}), \quad p \geq 1$$

where the second inequality is obtained by Jensen's inequality applied to the left-hand side inequality

$$\frac{1}{p}|\zeta - \tilde{\zeta}| \leq |\zeta^{1/p} - \tilde{\zeta}^{1/p}| \leq \frac{1}{2^{1-1/p}}|\zeta - \tilde{\zeta}|, \; p \geq 1. \tag{2.4}$$

The last relation is easily verified by taking into consideration that $\zeta + \tilde{\zeta} = 2$.

2.3. As the process $\zeta^{1/p} - \tilde{\zeta}^{1/p}$ is a martingale if only $p = 1$, the relation (2.4) allows us to estimate bounds of ρ_p^p by applying Burkholder-type inequalities. Namely, there are universal constants c_p and C_p such that for a stopping time T

$$c_p E_Q[\zeta]_T^{p/2} \leq \rho_p^p(P_T,\tilde{P}_T) \leq C_p E_Q[\zeta]_T^{p/2}; \tag{2.5}$$

see, e.g., LIPTSER and SHIRYAEV (1988), Section 1.9, Theorem 7.

Furthermore, usual considerations establishing Burkholder-type inequalities (see LENGLART, LEPINGLE and PRATELLI (1980), and LIPTSER and SHIRYAEV (1988)) allows us to replace (2.5) by

$$c_p E_Q\{<\zeta>_T^{p/2} + ((\Delta\zeta)_T^*)^p\} \leq \rho_p^p(P_T,\tilde{P}_T) \leq C_p E_Q\{<\zeta>_T^{p/2} + ((\Delta\zeta)_T^*)^p\}$$

or, taking into consideration that $|\Delta\zeta|_T^p \leq |x|^p * \mu_T^\zeta \leq [\zeta]_T^{p/2}$, by

$$c_p E_Q\{<\zeta>_T^{p/2} + |x|^p * \nu_T^{\zeta,Q}\} \leq \rho_p^p(P_T,\tilde{P}_T) \leq C_p E_Q\{<\zeta>_T^{p/2} + |x|^p * \nu_T^{\zeta,Q}\} \tag{2.6}$$

with some other constants c_p and C_p.

3. Main results.

3.1. The inequalities (1.6) and (1.7) easily follow from the corresponding statements of Theorem 3.2 below. The proof of this theorem is based on the following statements of independent interest (note that here, in contrast with Theorem 3.2 below, p is not necessarily positive even integer):

Theorem 3.1. *Let S and T be stopping times, $S \leqslant T$. For $p \geqslant 2$ there are constants c_p and C_p such that*

$$\rho_p^p(P_T,\tilde{P}_T) \geqslant c_p E_Q\{(X_-^2 \circ h)_T^{p/2} + (X_-^p \circ k(p))_T\} \tag{3.1}$$

and

$$\rho_p^p(P_T,\tilde{P}_T) \leqslant C_p E_Q\{(X_- \circ h)_S^{p/2} + (X_- \circ k(p))_S\} + 2Q(S<T) \tag{3.2}$$

with $X = \zeta\tilde{\zeta}$.

If the measures P and $\tilde{P}$ correspond to processes with independent increments, then the processes h and $k(p)$ can be assumed to be deterministic (for more details see Jacod and Shiryaev (1987), Theorem IV.4.24). In this particular case we have

Corollary 3.1. *Suppose that the processes h and $k(p)$ and the stopping time T are deterministic. Then we can replace (3.1) and (3.2) with the following inequalities:*

(i) $\rho_p^p(P_T,\tilde{P}_T) \geqslant c(p;T,h)(h_T^{p/2} + k_T(p))$

and

(ii) $\rho_p^p(P_T,\tilde{P}_T) \leqslant C_p(h_T^{p/2} + k_T(p))$.

Proof of Theorem 3.1. In view of (2.6) it suffices to apply the following lemmas, the first two of which give the corresponding estimates of the expectations of two terms involved in (2.6), and the third one leads to the upper bound of form (3.2).

Lemma 3.1. *Let $X = \zeta\tilde{\zeta}$ and let h be given by (1.3). Then*

$$\frac{1}{2} X_-^2 \circ h \leqslant \langle\zeta\rangle \leqslant 2X_- \circ h$$

(cf. Jacod *and* Shiryaev (1987), Lemma V.4.26).

Proof. In view of (1.3) and the easily verified facts that $X \leqslant 1$ and

$$\langle\zeta\rangle = \langle\zeta^c\rangle + (X_-(\frac{\lambda-\tilde{\lambda}}{2}))^2 * \nu^{\zeta,Q}$$

by (2.3), it suffices to verify only that

$$X_-(\sqrt{\lambda}-\sqrt{\tilde{\lambda}})^2 \leqslant X_-(\lambda-\tilde{\lambda})^2 \leqslant 4(\sqrt{\lambda}-\sqrt{\tilde{\lambda}})^2, \tag{3.3}$$

by taking into consideration that $\sqrt{\lambda}+\sqrt{\tilde{\lambda}} \geqslant 1$, and that $X_-(\sqrt{\lambda}+\sqrt{\tilde{\lambda}})^2 \leqslant 4$ due to Shwartz' inequality and the identities: $\zeta+\tilde{\zeta}=2$ and $\zeta_-\lambda+\tilde{\zeta}_-\tilde{\lambda}=2$.

The inequalities (3.3) can easily be extended to the case $p \geqslant 2$:

$$X_-^{p-1}(\lambda^{1/p}-\tilde{\lambda}^{1/p})^p \leqslant X_-^{p-1}(\lambda-\tilde{\lambda})^p \leqslant 4^{p-1}(\lambda^{1/p}-\tilde{\lambda}^{1/p})^p,$$

and this gives

LEMMA 3.2. *For $p \geq 2$*

$$(1/2)^p X^p_- \circ k(p) \leq |x|^p * \nu^{\zeta,Q} \leq 2^{p-2} X_- \circ k(p)$$

LEMMA 3.3. *For two stopping times $S \leq T$, and $p \geq 1$*

$$\rho_p^p(P_T, \tilde{P}_T) \leq \rho_p^p(P_S, \tilde{P}_S) + 2Q(S<T)$$

PROOF. For $p=1$ see JACOD and SHIRYAEV (1987), p. 280. The general case is treated analogously.

PROOF OF COROLLARY 3.1. (ii) is obvious. To prove (i) observe that the function $f_t = E_Q\sqrt{X_t}$ is decreasing, in fact $f_t = \mathcal{E}_t(-h)$ where $\mathcal{E}$ is Dolean-Dade's exponential (as it satisfies $f_t = 1-(f_- \cdot h)_t$ in accordance with JACOD and SHIRYAEV (1987), IV.1.20), and this and Jensen's inequality entail

$$\inf_{s \leq t} E_Q(X_{s-})^p \geq \inf_{s \leq t} (E_Q\sqrt{X_{s-}})^{2p} \geq f_t^{2p}$$

Hence (ii) takes place with $c(p;T,h) = c_p(\mathcal{E}_T(-h))^{2p}$.

REMARK 3.1. In the simplest case $p=2$ we have the following representation

$$\rho_2^2(P_T, \tilde{P}_T) = 2E_Q(X_-^{1/2} \circ h)_T \tag{3.4}$$

(see VALKEILA and VOSTRIKOVA (1986)). Comparing (3.4) and (3.1) for $p=2$, with $2E_Q(X_-^2 \circ h)$ on the right-hand side (constants here and in the next paragraph are defined by (2.4) with $p=2$) we see that the lower bound obtained is quite crude; cf. also Corollary 3.1, Assertion (i).

As for the upper bound (3.2) for $p=2$ and $S=T$, with $4E_Q(X_- \circ h)$ on the right-hand side, it is simply derived from (3.4) by the following considerations:

$$\begin{aligned}
\rho_2^2 &\geq \frac{1}{2} E_Q |\zeta^{1/2} - \tilde{\zeta}^{1/2}|^4 \\
&\geq E_Q |\zeta_-^{1/2} - \tilde{\zeta}_-^{1/2}|^2 \circ |\zeta^{1/2} - \tilde{\zeta}^{1/2}|^2 \\
&= 2\rho_2^2 - 2E_Q X^{1/2} \circ |\zeta^{1/2} - \tilde{\zeta}^{1/2}|^2 \\
&= 2\rho_2^2 - 4E_Q X_- \circ h
\end{aligned}$$

Here we have first used the inequality $|\zeta^{1/2} - \tilde{\zeta}^{1/2}|^2 \leq 2$, then Ito's formula and, finally, (3.4).

REMARK 3.2. By JACOD and SHIRYAEV (1987), Lemma I.3.12, we have

$$E_Q(X_- \circ h)_T \leq 2E_Q(\zeta_- \circ h)_T = 2E_Q\zeta_T h_T = 2E_P h_T,$$

since $\tilde{\zeta} \leq 2$, and this gives (1.7). Thus the upper bound here can be given in terms of the expectation with respect to the measure P. For the general result see the following theorem.

3.2.

THEOREM 3.2. *Let S and T be stopping times, $S \leq T$. For a positive even integer p there are constants C_p and B_p such that*

$$\begin{aligned}
\rho_p^p(P_T, \tilde{P}_T) \leq\ & C_p E_P(h_S^{p/2} + k(p)_S) \\
&+ B_p P^{1/p}(S<T)
\end{aligned}$$

PROOF. In view of (3.2) it suffices to show that

$$E_Q(X_-\circ k(p)_T) \leqslant 2E_P k(p)_T, \tag{3.5}$$

$$E_Q(X_-\circ h)_T^{p/2} \leqslant pE_P h_T^{p/2} \tag{3.6}$$

and

$$\rho_p^p(P_T,\tilde{P}_T) - \rho_p^p(P_S,\tilde{P}_S) \leqslant B_p P^{1/p} \ (S<T). \tag{3.7}$$

Since $\tilde{\zeta} \leqslant 2$, (3.5) follows from JACOD and SHIRYAEV (1987), lemma I.3.12.

To prove (3.6) apply the same lemma, along with the considerations of LIPTSER and SHIRYAEV (1988), Lemma I.9.6: for $A = X_-\circ h$ we have

$$\begin{aligned} E_Q A_T^{p/2} &\leqslant \frac{p}{2} E_Q \int_0^T A_s^{\frac{p}{2}-1} dA_s \\ &\leqslant \frac{p}{2} E_Q \int_0^T X_{s-} h_s^{\frac{p}{2}-1} dh_s \\ &\leqslant pE_Q \int_0^T \zeta_{s-} h_s^{\frac{p}{2}-1} dh_s = pE_P \int_0^T h_s^{\frac{p}{2}-1} dh_s \leqslant pE_P h_T^{p/2}. \end{aligned}$$

For (3.7) see VOSTRIKOVA (1987), Theorem 2.2.

REMARK 3.3. The method for establishing (3.7) developed by VOSTRIKOVA (1987) in the course of proving her Theorem 2.2 amounts in justifying the equality of the left-hand side of (3.7) to

$$E_P\left[1_{\{S<T\}} \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (Z_T^{k/p} - Z_S^{k/p})\right]$$

with $Z = \tilde{\zeta}/\zeta$, using here Hölder's inequality with exponents $1/p$ and $p-1/p$ and, finally, evaluating the factor

$$\left(E_P \sum_{k=1}^{p-1} (-1)^k (Z_T^{k/p} - Z_S^{k/p})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

by taking into account that $E_P Z_T^{\alpha} \leqslant 1$ for $0<\alpha\leqslant 1$. Of course, the result is rather crude (one can in (3.7) take $B_p = p^{1/p}(2^{p+1}-4)^{(p-1)/p}$ which gives, in particular, $B_2 = 2\sqrt{2}$), nevertheless this is sufficient for our purposes, that is the application in the course of proving Theorem 4.2 below.

4. SEQUENCES OF BINARY EXPERIMENTS

4.1. In the present section we consider certain applications to sequences of binary experiments

$$(\Omega^n, F^n, P^n, \tilde{P}^n), \quad n = 1,2, \cdots \tag{4.1}$$

with the associated density processes ζ^n and $\tilde{\zeta}^n$ as in (1.1), and the corresponding Hellinger process h^n and processes $k^n(p), p \geqslant 2$ defined as in (1.3) and (1.5).

We remark first that in view of the properties of the distances ρ_p indicated in Subsection 2.2, the limiting (as $n\to\infty$) behaviour of $\rho_p(P^n_{T_n}, \tilde{P}^n_{T_n})$, defined by (1.2) with a sequence of stopping times $T_n, n = 1,2,...$, is controlled under the circumstances

$$h^n_{T_n} \xrightarrow{P^n} 0 \text{ or } h^n_{T_n} \xrightarrow{P^n} +\infty \tag{4.2}$$

in the exactly same way as that of the variational distance $\|P^n_{T_n} - \tilde{P}^n_{T_n}\|$ (see JACOD and SHIRYAEV (1987), Theorem 4.32).

Contrary to (4.2), in the next subsection we consider the situation in which a sequence of the Hellinger processes possesses a certain limit in P^n- probability.

4.2. Let $t \rightsquigarrow C_t$ be a non-decreasing continuous function with $C_0=0$. Let M be a continuous martingale with $M_0=0$ and $<M,M>_t=C_t$, on some stochastic basis $(\Omega,F,\mathbf{F},P)$ (so M is Gaussian).

Let $\mathfrak{D}$ be a dense subset in $\mathbb{R}_+$. Consider the following conditions:

(a) $h_t^n \xrightarrow{P^n} \frac{1}{8} C_t$ for all $t \in \mathfrak{D}$

(that is Condition $[H-D]$ in JACOD and SHIRYAEV (1987), Theorem X.1.12) and

(b) for a certain $p>2$

$$k_t^n(p) \xrightarrow{P^n} 0.$$

Along with the processes h^n and $k^n(p)$, $p \geq 2$, we will associate with (4.1) a new process $I_t^n(a)$ for $a>1$:

$$I^n(a) = 1_{\{1/a<\lambda^n/\tilde{\lambda}^n<a\}^c} |\lambda^n - \tilde{\lambda}^n| * \nu^{\zeta^n, Q^n} \tag{4.3}$$

where $\tilde{\lambda}^n$ and λ^n are defined as in (2.2) and $Q^n=(P^n+\tilde{P}^n)/2$ obviously, and we consider Condition $[L-D]$ in JACOD and SHIRYAEV (1987), Theorem X.1.12:

(c) $I^n(1+\epsilon)_t \xrightarrow{P^n} 0$ for all $t \in D, \epsilon>0$.

Set $Z^n=\tilde{\zeta}^n/\zeta^n$, and consider the following statement:

(i) $Z^n \to Z = e^{M-C/2}$ in low $\mathfrak{L}(P^n)$,

with M and $<M>=C$ defined above.

The following extension of Theorem X.1.12 by JACOD and SHIRYAEV (1987) takes place:

THEOREM 4.1. *The statement* (i) *is equivalent to the following two statements:*
(ii) *Conditions* (a) *and* (c) *hold;*
(iii) *Conditions* (a) *and* (b) *hold.*

PROOF. For (i) $\Leftrightarrow$ (ii) see JACOD and SHIRYAEV (1987), Theorem X.1.12. To show (ii) $\Leftrightarrow$ (iii) denote by $A_a(\lambda,\tilde{\lambda})$ the set the indicator of which is involved in (4.3) (we suppress the index n, as it is superfluous here).

It is easily verified that the validity of the following two statements suffices here:

1) for each ϵ, $0<\epsilon<1$ and $p \geq 2$

$$k(p) \leq \left(\frac{2\epsilon}{1-\epsilon}\right)^{p-2} k(2) + I\left(\frac{1+\epsilon}{1-\epsilon}\right);$$

2) for each $p>2$ and $a>1$ there is a constant $C_{a,p}>0$ such that

$$1_{A_a(\lambda,\tilde{\lambda})} |\lambda^{1/p} - \tilde{\lambda}^{1/p}|^p * \nu^{\zeta,Q} \leq I(a) \leq C_{a,p} 1_{A_a(\lambda,\tilde{\lambda})} |\lambda^{1/p} - \tilde{\lambda}^{1/p}|^p * \nu^{\zeta,Q}$$

Statement 1) follows from the simply verified inequalities

$$(u^{1/p}-1)^p \leq \begin{cases} (u^{1/2}-1)^2 (2\epsilon/(1-\epsilon))^{p-2} & \text{if } 1 \leq u \leq \frac{1+\epsilon}{1-\epsilon}, \\ u-1 & \text{if } u > \frac{1+\epsilon}{1-\epsilon}, \end{cases}$$

and Statement 2) from (2.2) and the fact that the continuous function $|u^{1/p}-1|^p/|u-1|$ vanishes as $u \to 1$ and tends to one as $u \to \infty$.

REMARK 4.1. The relation between $k(p)$ and the Hellinger processes $h(\alpha)$ of order $\alpha\in(0,1)$ indicated at the end of Subsection 2.2, allows one to trace directly the equivalence of (iii) above and (ii) or (iii) in JACOD and SHIRYAEV (1987), Theorem X.1.64.

4.3. Under the circumstances of the previous subsection we have

THEOREM 4.2. *Statement* (i) *implies*

$$\overline{\lim_{n\to\infty}}\, \rho_p^p(P_t^n,\tilde{P}_t^n) \leq K_p C_t^{p/2}$$

with a certain constant K_p.

PROOF. Let $S_n = inf\{s|\ h_s^n \geq C_t+1\}$. Then

$$k_{S_n\wedge t}^n(p) \leq 2h_{S_n\wedge t}^n \leq 2C_t + 3,$$

since $\Delta h\leq 1$, and $\{S_n<t\}\subseteq\{h_t^n\geq C_t+1\}$. Hence $P^n(S_n<t)\to 0$ under (i), and this implies in turn that

$$h_{S_n\wedge t}^n \xrightarrow{P^n} \frac{1}{8}C_t$$

and

$$k_{S_n\wedge t}^n(p) \xrightarrow{P^n} 0.$$

But the sequences $k_{S_n\wedge t}^p(p)$ and $h_{S_n\wedge t}^n$ are bounded and hence under (i)

$$E_{P^n}(h_{S_n\wedge t}^n)^{p/2}\to(\frac{1}{8}C_t)^{p/2}$$

and

$$E_{P^n}(k_{S_n\wedge t}^n)\to 0.$$

This, in view of Theorem 3.2, gives the result.

5. PARAMETRIC FAMILIES OF EXPERIMENTS

5.1. We consider here an application to a parametric family of experiments

$$(\Omega,F,\mathbf{F},\{P^\theta,\theta\in\Theta\},Q)$$

where Θ is a closed subset of the Euclidean space R^d, and Q is a measure dominating the family $\{P^\theta,\theta\in\Theta\}$ of probability measures depending continuously on a parameter θ.

We retain here the assumptions and notations of Introduction (with a general dominating measure Q, however) writing specifically (for $\theta,\theta+u\in\Theta$)

$$\rho_p(P_T^{\theta+u},P_T^\theta) = \{E_Q|\zeta_T(\theta+u)^{1/p}-\zeta_T(\theta)^{1/p}|^p\}^{1/p} \tag{5.2}$$

with $p\geq 2$ and

$$\zeta_T(\theta) = dP_T^\theta/dQ_T. \tag{5.3}$$

Analogously, we define the processes $h(\theta+u,\theta)$ and $k(p;\theta+u,\theta)$ by the formulas (1.3) and (1.5) respectively, with $\zeta=\zeta(\theta)$ and $\zeta=\zeta(\theta+u)$ this time.

5.2. We wish to evaluate the expectation E_Q with respect to the dominating measure Q of the following modulus of continuity (for a certain $p>d$)

$$\omega_p(\delta,L;P_T^\theta,P_T^{\theta+u}) = sup\ |\zeta_T(\theta+u)^{1/p}-\zeta_T(\theta)^{1/p}|^p \tag{5.4}$$

where sup is taken over $\theta,\theta+u\in\Theta$ with $|\theta|\leq L$, $|\theta+u|\leq L$ and $|u|\leq\delta$.

THEOREM 5.1. *Let the following Lipschitz type conditions be satisfied: there is a bounded function B_θ of θ such that for each $\theta,\theta+u\in\Theta$*

$$E^\theta h_T^{p/2}(\theta,\theta+u)\leqslant B_\theta|u|^p,\ E^\theta k_T(p;\theta,\theta+u)\leqslant B_\theta|u|^p \tag{5.5}$$

with the expectation relative to the measure P^θ.

Then for $p>d$

$$E_Q\omega_p(\delta,L;P_T^\theta,P_T^{\theta+u})\leqslant B_0\sup_{|\theta|\leqslant L}B_\theta\ L^{dp}\delta^{p-d}$$

where the constant B_0 depends on d and p only.

PROOF. We apply here Theorem 19 in IBRAGIMOV and HAS'MINSKII (1981), Appendix I. All of its conditions are satisfied: the first one in (7), p. 372 by $E_Q\zeta_T(\theta)\leqslant 1$ and the second one by Theorem 3.2 above which implies

$$\begin{aligned}\rho_p(P_T^{\theta+u},P_T^\theta)&\leqslant C_pE^\theta\{h_T^{p/2}(\theta,\theta+u)\ +\ k_T(p;\theta,\theta+u)\}\\&\leqslant C_pB_\theta|u|^p\end{aligned}$$

in view of (5.2), (5.3) and (5.5).

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