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Conditionally and Strictly Distribution-Free Tests for Randomized Block Designs that are Asymptotically Optimal

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Both the method of ranking after alignment and the Tukey-Quade method of weighted rankings for the analysis of complete blocks are generalized so as to give rise to classes of tests containing a conditionally distribution-free test and strictly distribution-free tests that are asymptotically optimal in the sense that, when the number of blocks tends to infinity, their asymptotic local power reaches the one of the asymptotically minimax test based on block-location-free statistics.

RÉSUMÉ

En généralisant tour à tour la méthode du rangement après alignement et la méthode des rangements pondérés de Tukey-Quade en vue de l'analyse de plans de blòcs complets, il est possible de construire des classes de tests contenant un test conditionellement libre et des tests strictement libres qui sont asymptotiquement optimaux en ce sens que, lorsque le nombre de blocs croît vers l'infini, leur puissance asymptotique locale atteint celle du test asymptotiquement minimax basé sur des statistiques invariantes quant aux effets de bloc.

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1. INTRODUCTION

Consider N observations divided into n blocks of M observations. Let, for each block, $m_j(\geqslant 1)$ be the number of observations under the j-th treatment, $1 \leqslant j \leqslant p(\geqslant 2)$. Also, let $m_0^* = 0$ and $m_j^* = \Sigma_{k=1}^{j} m_j, 1 \leqslant j \leqslant p$. Each observation is described by the model

$$X_{ik} = \mu + \alpha_i + \beta_j + \epsilon_{ik}, \ \ k = m_{j-1}^*, \ldots, m_j^*, \ j = 1, \ldots, p, \ i = 1, \ldots, n,$$

where μ stands for the main effect, the α_i's and the β_j's for block and treatment effects, respectively, and the ϵ_{ik}'s for the residual error components. It is assumed that $(\epsilon_{it}, \ldots, \epsilon_{iM})', 1 \leqslant i \leqslant n$, are independently and identically distributed random vectors having a joint density function $g(x_1, \ldots, x_M)$ which is symmetric in its M arguments. Let $\mathbf{x} = (x_1, \ldots, x_M)$ be a vector-valued variable of $\mathbb{R}^M$, $\mathbf{c} = (c_1, \ldots, c_M)$ a non-null constant vector of $\mathbb{R}^M$ and write $\mathbf{x} + t\mathbf{c}$ for $(x_1 + tc_1, \ldots, x_M + tc_M)$. It is further assumed that

$$\left.\begin{aligned} & g(\mathbf{x}+t\mathbf{c}) \text{ is absolutely continuous in } t \text{ a.e. } (\mathbf{x}), \\ & \frac{d}{dt} g(\mathbf{x}+t\mathbf{c}) = \sum_{k=1}^{M} c_k \dot{g}_k(\mathbf{x}+t\mathbf{c}) \text{ a.e.}(t) \text{ and a.e.}(\mathbf{x}), \\ & \int_{\mathbb{R}^M} \left[\frac{\dot{g}_1(\mathbf{x})}{g(\mathbf{x})}\right]^2 g(\mathbf{x}) d\mathbf{x} < \infty, \end{aligned}\right\} \quad (1.1)$$

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where $\dot{g}_1(\mathbf{x}),...,\dot{g}_M(\mathbf{x})$ denote the partial derivatives of the density function $g(\mathbf{x})$. Note that, if the error components are independent within blocks, i.e., if $g(\mathbf{x})=\Pi_{k=1}^{M} g_0(x_k)$, condition (1.1) is equivalent to the more usual assumption (Hájek and Sidák 1967): $g_0(t)$ is absolutely continuous and has finite Fisher's information. Assuming without loss of generality that $\Sigma_{j=1}^{p} m_j \beta_j = 0$, the hypotheses of interest are expressed as

$$H: \sum_{j=1}^{p} m_j \beta_j^2 = 0 \quad \text{and} \quad K: \sum_{j=1}^{p} m_j \beta_j^2 > 0.$$

Thus, the null hypothesis refers to the homogeneousness of the treatment effects and the alternative hypothesis to their nonhomogeneousness.

The present paper will focus attention on two ranking methods for treating the aforementioned problem. In chronological perspective, FRIEDMAN (1937) introduced the method of n rankings in which observations are ranked separately within each block. Since no between-blocks comparisons are being made, this distribution-free method tends to give rank tests with low efficiency, especially when the number of observations per block is small. HODGES and LEHMANN (1962) noticed that such between-blocks comparisons could be made by first aligning the observations within each block, that is, by substracting from each observation some estimate of the block effect, and then using a combined ranking of all the observations. Their method is known as the ranking after alignment procedure. Since the vector of combined ranks is not distribution-free, even under the null hypothesis, a permutational argument has to be invoked in order for the rank tests to behave like distribution-free tests. Thus, rank tests based on the ranking after alignment procedure are only conditionally distribution-free but they tend to have higher efficiency than those based on the method of n rankings. QUADE (1972, 1979), in an effort to allow comparisons between blocks while still retaining a strictly distribution-free behavior, considered the possibility of weighting the within-block rankings used in the Friedman method according to some stochastic credibility or variability measure of the blocks, an intuitive idea that seems to have been expressed first by TUKEY (1957). The method of n rankings is a particular case of the Tukey-Quade method of weighted rankings since it corresponds to the situation for which the weights are all equal.

Several authors, e.g., PURI and SEN (1971), SCHACH (1979), ROTHE (1983), TARDIF (1980, 1985, 1987), have considered classes of tests based on either one of these methods and have given some asymptotic efficiency results as the number of blocks tends to infinity. In particular, TARDIF (1985) has, for any given joint density function $g(\mathbf{x})$ satisfying (1.1), established the existence of an asymptotically local optimal test for the problem and has shown that, in general, the class of tests based on the ranking after alignment procedure, the alignment being made on the mean, does not contain a member whose asymptotic Pitman-efficiency relative to this asymptotically optimal test is equal to one. Furthermore, TARDIF (1987) has shown that, for $M \geqslant 3$, the asymptotic local power of any rank test based on the method of weighted rankings never reaches the one of the asymptotically optimal test.

The goal of the present paper is to answer the following question: is it possible to modify both methods in such a way that the classes of tests based on them will include asymptotically optimal members? The answer to this question is a definitive yes. On the one hand, it will be seen that the ranking after substitution procedure, a generalization of the ranking after alignment procedure, gives rise to a class of conditionally distribution-free tests broad enough so as to include an asymptotically optimal member. On the other hand, the generalization of the method of weighted rankings will be such that the class of tests induced by it will, surprisingly enough, contain not one but an infinite number of strictly distribution-free tests that are asymptotically optimal.

The exact definition of the asymptotically optimal test is recalled from TARDIF (1985) in Section 2. Sections 3 and 4 are devoted to the ranking after substitution procedure and the generalized method of weighted rankings, respectively.

2. The asymptotically minimax test

Introduce a strictly increasing sequence $\{n_\nu : \nu \geq 1\}$ of number of blocks, let $N_\nu = n_\nu M, \nu \geq 1$, and consider the sequence of composite contiguous alternatives

$$K_\nu : \beta_j = N_\nu^{-1/2} \Delta_j \ , \ j=1,...,p, \nu \geq 1,$$

for which $\mathbf{\Delta} = (\Delta_1,...,\Delta_p)$ is allowed to belong to the ellipsoid $\{\mathbf{\Delta} : \Sigma_{j=1}^p m_j \Delta_j^2 = C^2$ and $\Sigma_{j=1}^p m_j \Delta_j = 0\}$ with C^2 a positive constant. Given any density function $g(\mathbf{x})$ satisfying condition (1.1), Tardif (1985) has established the existence of an asymptotically minimax test for distinguishing between H and the sequence $\{K_\nu\}$. It should be mentioned that, since the block effects $\alpha_1, \ldots, \alpha_{n_\nu}$ act as nuisance parameters in the problem and, as the number of blocks will tend to infinity, there will be an infinite number of nuisance parameters, the asymptotically optimal test should exhibit invariance with respect to these block effects. With this in mind, introduce the mean-aligned observations $Y_{ik} = X_{ik} - M^{-1} \Sigma_{h=1}^M X_{ih}, 1 \leq k \leq M$ and $1 \leq i \leq n_\nu$. Then, the joint density function of $(M-1)$ aligned observations of any given block is, under H, given by

$$f(y_1, \ldots, y_{M-1}) = M \int_{-\infty}^{\infty} g(y_1 + t, \ldots, y_M + t) dt, \tag{2.1}$$

where $y_M = -\Sigma_{k=1}^{M-1} y_k$. Furthermore, define the M functions

$$f'_k(y_1, \ldots, y_{M-1}) = M \int_{-\infty}^{\infty} \dot{g}_k(y_1 + t, \ldots, y_M + t) dt, \quad k = 1,...,M, \tag{2.2}$$

the quantity

$$\mathcal{I}(f) = \int_{\mathbb{R}^{M-1}} \left[\frac{f'_1(\mathbf{y})}{f(\mathbf{y})} \right]^2 f(\mathbf{y}) d\mathbf{y},$$

where $\mathbf{y} = (y_1, \ldots, y_{M-1})$, and introduce the sequence of random variables

$$W_{\nu j} = N_\nu^{-1/2} \sum_{i=1}^{n_\nu} \sum_{k=m_{j-1}^*+1}^{m_j^*} \left\{ -\frac{f'_k}{f}(Y_{i1}, \ldots, Y_{i,M-1}) \right\}, \ j = 1, \ldots, p \text{ and } \nu \geq 1.$$

It was shown in Tardif (1985) that, among the class of tests based on mean-aligned observations, which amounts to say among the class of tests based on block-location-free statistics, the one based on $\{\mathcal{I}(f)/(M-1)\}^{-1} \Sigma_{j=1}^p m_j^{-1} W_{\nu j}^2$ is asymptotically maximin most powerful, or minimax, for testing H against $\{K_\nu\}$. It was also shown that this statistic is, under $\{K_\nu\}$, asymptotically distributed as a chi-squared variable with $(p-1)$ degrees of freedom and noncentrality parameter

$$\delta_{opt}^2 = C^2 \frac{\mathcal{I}(f)}{M-1}. \tag{2.3}$$

Any rank test, either conditionally or strictly distribution-free, will said to be asymptotically optimal if and only if the asymptotic distribution of its test statistic is a chi-square with $(p-1)$ degrees of freedom and noncentrality parameter equal to (2.3) since then its asymptotic Pitman-efficiency relative to the asymptotically minimax test will be equal to one.

3. The method of ranking after substitution

With the ranking after alignment procedure, the observations are made comparable by aligning them, that is, by removing from them the influence of the block effects so a combined ranking can be envisaged. The method of ranking after substitution (MRS) is inspired by a similar principle. Introduce M real functions $e_1(x_1, \ldots, x_M), \ldots, e_M(x_1, \ldots, x_M)$ satisfying for all $(x_1, \ldots, x_M) \in \mathbb{R}^M$ and for $1 \leq k \leq M$:

$$\left.\begin{array}{l} \text{(a) } e_k(x_1 + c, \ldots, x_M + c) = e_k(x_1, \ldots, x_M) \text{ for every } c \in \mathbb{R}, \\ \text{(b) } e_k(x_{r_1}, \ldots, x_{r_M}) = e_{r_k}(x_1, \ldots, x_M) \text{ for every } (r_1, \ldots, r_M) \in \mathfrak{R}, \end{array}\right\} \tag{3.1}$$

where $\mathfrak{R}$ is the group of all permutations of the first M integers, and define the random variables $E_{ik}=e_k(X_{i1},\ldots,X_{iM}), 1\leqslant k\leqslant M$ and $1\leqslant i\leqslant n_\nu$. Now, consider the observations of a given block, the i-th, say, let R_{ik} be the within-block rank of X_{ik} among $X_{i1},\ldots,X_{iM}$ and designate the vector of within-block ordered observations by $\mathbf{X}_{i(\cdot)}=(X_{i(1)},\ldots,X_{i(M)})$. Then, for $1\leqslant k\leqslant M$:

$$E_{ik} = e_k(X_{i1},\ldots,X_{iM}) = e_k(X_{i(R_{i1})},\ldots,X_{i(R_{iM})}) = e_{R_{ik}}(\mathbf{X}_{i(\cdot)}). \tag{3.2}$$

Consequently, if $\mathbf{X}_{i(.)}$ is fixed, the random variables $E_{i1},\ldots,E_{iM}$ are solely functions of the within-block ranks of $X_{i1},\ldots,X_{iM}$, respectively. The MRS thus recommends the use of the random variables $E_{ik}, 1\leqslant k\leqslant M$ and $1\leqslant i\leqslant n_\nu$, as substitutes for the original observations $X_{ik}, 1\leqslant k\leqslant M$ and $1\leqslant i\leqslant n_\nu$.

REMARK 3.1. If the substitution functions are required to verify the additional property:

$$\left.\begin{array}{l} e_1(\mathbf{x}_{(\cdot)})\leqslant \cdots \leqslant e_M(\mathbf{x}_{(\cdot)}) \text{ for every } \mathbf{x}_{(\cdot)} = (x_{(1)},\ldots,x_{(M)})\in\mathbb{R}^M \\ \text{such that } x_{(1)}\leqslant \cdots \leqslant x_{(M)} \text{ holds,} \end{array}\right\} \tag{3.3}$$

then the vector of the within-block ranks of $E_{i1},\ldots,E_{iM}$ coincides with the one of $X_{i1},\ldots,X_{iM}$. To see this, let Q_{ik} be the rank of E_{ik} among $E_{i1},\ldots,E_{iM}, 1\leqslant k\leqslant M$. In view of (3.2), $E_{i(1)}\leqslant \cdots \leqslant E_{i(M)}$ are the order statistics of $e_{R_{i1}}(\mathbf{X}_{i(\cdot)}),\ldots,e_{R_{iM}}(\mathbf{X}_{i(\cdot)})$ or, equivalently, of $e_1(\mathbf{X}_{i(\cdot)}),\ldots,e_M(\mathbf{X}_{i(\cdot)})$. Property (3.3) entails however that $e_1(\mathbf{X}_{i(\cdot)})\leqslant \cdots \leqslant e_M(\mathbf{X}_{i(\cdot)})$ so $E_{i(k)}=e_k(\mathbf{X}_{i(\cdot)}), 1\leqslant k\leqslant M$. Consequently,

$$E_{ik} = E_{i(Q_{ik})} = e_{Q_{ik}}(\mathbf{X}_{i(\cdot)}),\ k=1,\ldots,M. \tag{3.4}$$

It then follows from (3.2) and (3.4) that $Q_{ik}=R_{ik}, 1\leqslant k\leqslant M$.

In view of (3.1.a), the substitutes $E_{ik}, 1\leqslant k\leqslant M$ and $1\leqslant i\leqslant n_\nu$, are block-location-free. Moreover, in view of (3.1.b) and since the original observations are exchangeable within blocks under the null hypothesis, it holds, for any $1\leqslant i\leqslant n_\nu$ and any $(r_1,\ldots,r_M)\in\mathfrak{R}$, that:

$$\begin{aligned}(E_{ir_1},\ldots,E_{ir_M}) &= (e_{r_1}(X_{i1},\ldots,X_{iM}),\ldots,e_{r_M}(X_{i1},\ldots,X_{iM}))\\ &= (e_1(X_{ir_1},\ldots,X_{ir_M}),\ldots,e_M(X_{ir_1},\ldots,X_{ir_M}))\\ &\overset{d}{=}(e_1(X_{i1},\ldots,X_{iM}),\ldots,e_M(X_{i1},\ldots,X_{iM}))\\ &= (E_{i1},\ldots,E_{iM}),\end{aligned}$$

where $\overset{d}{=}$ denotes an equality in distribution, so the substitutes are also exchangeable within blocks under H. Consequently, they are comparable and a combined ranking of them makes sense. Let $R_{N_\nu;ik}$ be the rank of E_{ik} among the N_ν substitutes and consider a set of scores $\{a_{N_\nu}(1),\ldots,a_{N_\nu}(N_\nu)\}$. For $\nu\geqslant 1$, define

$$\begin{aligned} a_{N_\nu}(R_{N_\nu;i\cdot}) &= M^{-1}\sum_{k=1}^{M} a_{N_\nu}(R_{N_\nu;ik}) \ ,i=1,\ldots,n_\nu,\\ \tau^2_{N_\nu} &= \{(M-1)n_\nu\}^{-1}\sum_{i=1}^{n_\nu}\sum_{k=1}^{M}\Big[a_{N_\nu}(R_{N_\nu;ik}) - a_{N_\nu}(R_{N_\nu;i\cdot})\Big]^2,\\ \xi_{N_\nu j} &= (m_j n_\nu)^{-1/2}\sum_{i=1}^{n_\nu}\sum_{k=m_{j-1}+1}^{m_j}\Big[a_{N_\nu}(R_{N_\nu;ik}) - a_{N_\nu}(R_{N_\nu;i\cdot})\Big],\ j=1,\ldots,p,\\ Q_{N_\nu} &= \tau^{-2}_{N_\nu}\sum_{j=1}^{p}\xi^2_{N_\nu j}.\end{aligned}$$

The MRS suggests the use of the quadratic form Q_{N_ν} as a test statistic. This test statistic is of the same form as the one associated with the ranking after alignment procedure (see TARDIF (1980)). In fact, the MRS constitutes a generalization of the former. The reason is as follows. Any ranking-after-alignment test statistic is a function of the combined ranks of the aligned observations $X_{ik}-a(X_{i1},\ldots,X_{iM}), 1\leqslant k\leqslant M$ and $1\leqslant i\leqslant n_\nu$, where $a(x_1,\ldots,x_M)$ is some real translation equivariant symmetric function. But the functions

$$e_k(x_1,\ldots,x_M) = x_k - a(x_1,\ldots,x_M),\ k=1,\ldots,M,$$

are easily seen to form a system of substitution functions satisfying (3.1). Moreover, property (3.3) is also seen to be verified. Consequently, aligned observations are just particular choices of substitutes.

REMARK 3.2. An interesting system of substitution functions can be defined as follows. Consider the functions $f(\mathbf{y}), f'_1(\mathbf{y}),\ldots,f'_M(\mathbf{y})$ introduced in (2.1) and (2.2), respectively, and put

$$e_k(x_1,\ldots,x_M) = -\frac{f'_k}{f}(x_1-\bar{x},\ldots,x_{M-1}-\bar{x}),\ k=1,\ldots,M, \tag{3.4}$$

where $\bar{x}=M^{-1}\Sigma_{k=1}^{M}x_k$. On account of Lemma 4.1 of TARDIF (1980), it is easily seen that condition (3.1) is satisfied. Furthermore, if the partial derivatives of the density function $g(\mathbf{x})$ verify the property:

$$\left.\begin{array}{l} -\dot{g}_1(\mathbf{x}_{(\cdot)})\leqslant\cdots\leqslant-\dot{g}_M(\mathbf{x}_{(\cdot)}) \text{ for every } \mathbf{x}_{(\cdot)} = (x_{(1)},\ldots,x_{(M)})\in\mathbb{R}^M \\ \text{such that } x_{(1)}\leqslant\cdots\leqslant x_{(M)} \text{ holds,} \end{array}\right\} \tag{3.5}$$

then property (3.3) is also verified. In particular, if $g(\mathbf{x})$ is the joint density function of M independent random variables, that is, $g(\mathbf{x})=\Pi_{k=1}^{M}g_0(x_k)$, then property (3.5) will hold if and only if $g_0(t)$ is a strongly unimodal density function (see TARDIF (1987)). It may also be noted that the use of the substitution functions (3.4) for two particular choices of the density function $g(\mathbf{x})$ will produce substituted-rank tests that are in fact aligned-rank tests. The first case arises when $g(\mathbf{x})$ is the density function of a multivariate normal random vector with a covariance matrix having the equal-variance, equal-correlation pattern. It can be deduced from Section 4 of TARDIF (1987) that the substitution functions (3.4) are then proportional to $x_1-\bar{x},\ldots,x_M-\bar{x}$, respectively. Consequently, the corresponding substituted-rank test coincides with the mean-aligned rank test. The second case arises when $g(\mathbf{x})$ is the density function of M independent random variables having the extreme-value distribution, that is, $g(\mathbf{x})=\Pi_{k=1}^{M}exp\{x_k-exp(x_k)\}$. Once again, it can be deduced from Section 4 of TARDIF (1987) that the substitution functions (3.4) then become

$$e_k(x_1,\ldots,x_M) = \frac{M\,exp(x_k)}{\Sigma_{h=1}^{M}exp(x_h)} - 1,\ k=1,\ldots,M.$$

Now, since $log(x+1)$ is a nondecreasing function of x, the combined ranking of the substitutes $E_{ik}, 1\leqslant k\leqslant M$ and $1\leqslant i\leqslant n_\nu$, and the combined ranking of the random variables $log(E_{ik}+1)=X_{ik}-log\{M^{-1}\Sigma_{h=1}^{M}exp(X_{ih})\}, 1\leqslant k\leqslant M$ and $1\leqslant i\leqslant n_\nu$, lead to identical vectors of ranks. Therefore, the corresponding substituted-rank test is equivalent to the aligned-rank test when the alignment is made on the function $a(x_1,\ldots,x_M)=log\{M^{-1}\Sigma_{k=1}^{M}exp(x_k)\}$.

Rank tests based on the MRS are to be performed conditionally, given the configuration, exactly in the same way as the rank tests based on the ranking after alignment method. For a precise definition and a full discussion on the configuration and the corresponding conditional permutation distribution, see HODGES and LEHMANN (1962) or PURI and SEN (1971). The asymptotic (unconditional) distribution of Q_{N_ν} under the null hypothesis as well as under the sequence of alternatives $\{K_\nu\}$ will now be considered. For that purpose, some additional conditions are required. Let $G_E(x)$ denote the common c.d.f. of the substitutes under the null hypothesis and, to simplify proofs, suppose that

$G_E(x)$ is a continuous c.d.f. (3.6)

(this requirement rules out, for instance, substitutes that are median-aligned observations when M is odd since these observations are, under H, equal to zero with probability $1/M$). Suppose also that the sequence of scores $\{(a_{N_\nu}(1),...,a_{N_\nu}(N_\nu)):\nu\geqslant 1\}$ is generated by a square-integrable function φ on $(0,1)$ in the sense that

$$\lim_{\nu\to\infty}\int_0^1\{a_{N_\nu}(1+[uN_\nu]) - \varphi(u)\}^2du = 0. \tag{3.7}$$

Assume further that

$$\left.\begin{array}{l}\varphi \text{ is not constant and such that}\\ P[\varphi\{G_E(E_{11})\} = \varphi\{G_E(E_{12})\}|H]<1.\end{array}\right\} \tag{3.8}$$

Finally, define

$$H_k = -\frac{f_k}{f}(Y_{11},\ldots,Y_{1,M-1}),\ k=1,...,M,$$

and

$$\tau^2 = \int_0^1\varphi^2(u)du - \mathcal{E}(\varphi\{G_E(E_{11})\}\varphi\{G_E(E_{12})\}),$$

where $\mathcal{E}$ stands for an expectation taken under H. The fulfillment of condition (3.8) ensures that $\tau^2>0$.

Theorem 3.1. *Under assumptions* (1.1), (3.1), (3.6), (3.7) *and* (3.8), Q_{N_ν} *has, under H, asymptotically a central chi-squared distribution with* $(p-1)$ *degrees of freedom and has, under* $\{K_\nu\}$, *asymptotically a noncentral chi-squared distribution with the same number of degrees of freedom and with noncentrality parameter*

$$\delta^2_{MRS} = C^2\frac{M[\mathcal{E}(\varphi\{G_E(E_{11})\}H_1)]^2}{(M-1)^2\tau^2}.$$

Proof. Lemma 3.1 of Tardif (1980) can be invoked to show that, under $H, \tau^2_{N_\nu}$ converges in probability to τ^2. Furthermore, Theorem 3.1 of Tardif (1980) can be applied without modification to get the asymptotic distribution of Q_{N_ν} under the null hypothesis. Finally, straightforward adaptations to the present situation of parts of the proof of Theorem 5.1 of Tardif (1980) show that Q_{N_ν} is asymptotically a $\chi^2_{p-1}(\delta^2_{MRS})$ under $\{K_\nu\}$. Q.E.D.

Remark 3.3. There was an imbalance between Theorems 3.1 and 5.1 of Tardif (1980) in the sense that the former is valid for any alignment while the latter is valid for alignment on the block mean only. Theorem 3.1 now compensates for this imbalance since its validity holds, in particular, for any alignment function.

There only remains to show that the class of rank tests associated with the MRS contains an asymptotic optimal member. To see this, define substitutes via the system of substitution functions introduced in Remark 3.2, that is, take $E_{1k}=H_k, 1\leqslant k\leqslant M$, and let $\varphi(u)=G_E^{-1}(u), 0<u<1$. Then, according to Lemma 4.1 of Tardif (1980), it is seen that

$$\mathcal{E}(\varphi\{G_E(E_{11})\}H_1) = \mathcal{E}(H_1^2) = \mathcal{I}(f)$$

and

$$\tau^2 = \mathcal{E}(H_1^2) - \mathcal{E}(H_1H_2) = \frac{M}{M-1}\mathcal{I}(f)$$

so δ^2_{MRS} becomes $C^2\mathcal{I}(f)/(M-1)=\delta^2_{opt}$. In view of Remark 3.2, the conditionally distribution-free test which is asymptotically optimal turns out to be an aligned-rank test when the observations have either the normal or the extreme-value distribution. It should be noted however that, under normality, the asymptotically minimax test is actually the asymptotic version of the classical variance-ratio test. Moreover, SEN (1968) had already established that the mean-aligned rank test for which normal scores are used has, under normality, an asymptotic Pitman-efficiency relative to the variance-ratio test equal to one.

4. THE GENERALIZED METHOD OF WEIGHTED RANKINGS

The method of weighted rankings (MWR) has been described by QUADE (1979), ROTHE (1983) and TARDIF (1987). The generalized MWR is based on the following simple idea. First, the total number of blocks is partitioned into $M-1$ subgroups of blocks. Then, on each of these subgroups, different sets of sums of scores for each treatment are computed as advocated by the MWR. Finally, the different partial sums of scores for each treatment are added together to form total sums of scores for each treatment and a quadratic form of these is used as a test statistic. More precisely, let, for each $\nu\geq 1$ and $1\leq k\leq M-1, n_{\nu k}$ be the number of blocks contained in the kth subgroup of blocks, so $\Sigma_{k=1}^{M-1} n_{\nu k}=n_\nu$, and put $n^*_{\nu 0}=0$ and $n^*_{\nu k}=\Sigma_{h=1}^{k} n_{\nu h}$. For each subgroup of blocks, that is, for each $1\leq k\leq M-1$, introduce a set of within-block scores $\mathbf{b}_k=(b_{k1},\ldots,b_{kM})'$ satisfying $\sigma_k^2=(M-1)^{-1}\Sigma_{h=1}^{M}(b_{kh}-\bar{b}_k)^2>0$, where $\bar{b}_k=M^{-1}\Sigma_{h=1}^{M} b_{kh}$, let R_{ih} denote the within-block rank of X_{ih} among $X_{i1},\ldots,X_{iM}, 1\leq h\leq M$ and $n^*_{\nu,k-1}+1\leq i\leq n^*_{\nu k}$, and define

$$B_{ikj} = m_j^{-1/2}\sum_{h=m^*_{j-1}+1}^{m^*_j}[b_{kR_{ih}}-\bar{b}_k], i=n^*_{\nu,k-1}+1,\ldots,n^*_{\nu k}, k=1,\ldots,M-1, j=1,\ldots,p.$$

Next, for each $1\leq k\leq M-1$, introduce a variability measure $d_k(x_1,\ldots,x_M)$, that is, a real-valued function satisfying, for all $(x_1,\ldots,x_M)\in\mathbb{R}^M$:

$$\left.\begin{array}{l}\text{(a) } d_k(x_1+c,\ldots,x_M+c) = d_k(x_1,\ldots,x_M) \text{ for every } c\in\mathbb{R},\\ \text{(b) } d_k(x_{r_1},\ldots,x_{r_M}) = d_k(x_1,\ldots,x_M) \text{ for every } (r_1,\ldots,r_M)\in\mathfrak{R},\end{array}\right\} \quad (4.1)$$

define $D_{ik}=d_k(X_{i1},\ldots,X_{iM})$ as the observed variability of the ith block, $n^*_{\nu,k-1}+1\leq i\leq n^*_{\nu k}$, and let $Q_{n_\nu ik}$ be the rank of D_{ik} among the set $\{D_{i'k}: n^*_{\nu,k-1}+1\leq i'\leq n^*_{\nu k}\}$. Furthermore, introduce, for each $1\leq k\leq M-1$, a sequence of between-blocks scores $\{(a_{n_{\nu k}}(1),\ldots,a_{n_{\nu k}}(n_{\nu k})):\nu\geq 1\}$. Then

$$S_{n_\nu kj} = \sum_{i=n_{\nu,k-1}+1}^{n^*_{\nu k}} a_{n_{\nu k}}(Q_{n_\nu ik})B_{ikj}, \; k=1,\ldots,M-1, j=1,\ldots,p,$$

are the partial sums of scores for each treatment and $S_{n_\nu j}=\Sigma_{k=1}^{M-1}S_{n_\nu kj}, 1\leq j\leq p$, are the total sums of scores for each treatment. The test statistic is finally defined as the quadratic form

$$Q_{n_\nu} = \left\{\sum_{k=1}^{M-1}\sum_{i=1}^{n_{\nu k}} a^2_{n_{\nu k}}(i)\sigma_k^2\right\}^{-1}\sum_{j=1}^{p} S^2_{n_\nu j}. \quad (4.2)$$

Because the observations are independent between blocks and are, under H, exchangeable within blocks, the vectors of within-block ranks $(R_{i1},\ldots,R_{iM}), 1\leq i\leq n_\nu$, are, under H, independently and uniformly distributed over the group of permutations of the first M integers and, for $1\leq k\leq M-1$, the vectors of between-blocks ranks $(Q_{n_\nu, n^*_{\nu,k-1}+1,k},\ldots,Q_{n_\nu n^*_{\nu k}k})$ are mutually independent and uniformly distributed over the group of permutations of the first $n_{\nu k}$ integers, respectively. Moreover, the observed variabilities are solely functions of the within-block order statistics on account of (4.1.b) and hence are, under H, independent of the vectors of within-block ranks. This entails that the vectors of between-blocks ranks and of within-block ranks are, under H, mutually independent. Consequently, a test based on Q_{n_ν} is strictly distribution-free.

The asymptotic distribution of the statistic (4.2) under the null hypothesis and under the sequence of contiguous alternatives $\{K_\nu\}$ is provided in the next theorem under the following additional conditions. Assume that

$$\lim_{\nu\to\infty}\frac{n_{\nu k}}{n_\nu} = \eta_k, k=1,\dots,M-1, \text{ as} \lim_{\nu\to\infty}\min\{n_{\nu 1},\dots,n_{\nu,M-1}\} = \infty, \tag{4.3}$$

where $\eta_1,\dots,\eta_{M-1}$ are real numbers lying on the unit interval, and that the sequences of between-blocks scores are generated by nonconstant square-integrable functions $\varphi_1(u),\dots,\varphi_{M-1}(u), 0<u<1$, that is:

$$\lim_{\nu\to\infty}\int_0^1\{a_{n_\nu k}(1+[un_{\nu k}]) - \varphi_k(u)\}^2 du = 0, k=1,\dots,M-1. \tag{4.4}$$

Also, designate the order statistics of the mean-aligned observations $Y_{11},\dots,Y_{1M}$ by $Y_{1(1)}\leqslant\cdots\leqslant Y_{1(M)}$, define

$$V_h = -\frac{f_h'}{f}(Y_{1(1)}\dots,Y_{1(M-1)}),\ h=1,\dots,M,$$

$$D_k = d_k(X_{11},\dots,X_{1M}),\ k=1,\dots,M-1,$$

and let $G_k(x)$ denote the c.d.f. of D_k under the null hypothesis, $1\leqslant k\leqslant M-1$.

THEOREM 4.1. *Under assumptions* (1.1), (4.1), (4.3) *and* (4.4), Q_{n_ν} *has, under H, asymptotically a central chi-squared distribution with* $(p-1)$ *degrees of freedom and has, under* $\{K_\nu\}$, *asymptotically a noncentral chi-squared distribution with the same number of degrees of freedom and with noncentrality parameter*

$$\delta^2_{MWR} = C^2\frac{\left[\sum_{k=1}^{M-1}\eta_k^{1/2}\sum_{h=1}^{M}b_{kh}\mathcal{E}(\varphi_k\{G_k(D_k)\}V_h)\right]^2}{M(M-1)^2\sum_{k=1}^{M-1}\eta_k\int_0^1\varphi_k^2(u)du\sigma_k^2}, \tag{4.5}$$

where $\mathcal{E}$ *stands for an expectation taken under H.*

PROOF. Introduce the random variables $\zeta_{n,j}=\sum_{k=1}^{M-1}\gamma_{n_\nu k}\zeta_{n_\nu kj}, 1\leqslant j\leqslant p$, where, for $1\leqslant k\leqslant M-1$ and $1\leqslant j\leqslant p$:

$$\gamma_{n_\nu k} = \left\{\frac{\sum_{i=1}^{n_{\nu k}}a^2_{n_\nu k}(i)\sigma_k^2}{\sum_{h=1}^{M-1}\sum_{i=1}^{n_{\nu h}}a^2_{n_\nu h}(i)\sigma_h^2}\right\}^{1/2}$$

and

$$\zeta_{n_\nu kj} = \left\{\frac{1}{n_{\nu k}}\sum_{i=1}^{n_{\nu k}}a^2_{n_\nu k}(i)\sigma_k^2\right\}^{-1/2}\frac{1}{\sqrt{n_{\nu k}}}S_{n_\nu kj}.$$

Evidently, $Q_{n_\nu}=\sum_{j=1}^{p}\zeta^2_{n,j}$. Furthermore, define

$$\gamma_k = \left\{\frac{\eta_k\int_0^1\varphi_k^2(u)du\sigma_k^2}{\sum_{h=1}^{M-1}\eta_h\int_0^1\varphi_h^2(u)du\sigma_h^2}\right\}^{1/2}, k=1,\dots,M-1.$$

It follows easily from (4.3) and (4.4) that $\lim_{\nu\to\infty}\gamma_{n_\nu k}=\gamma_k, 1\leq k\leq M-1$. Now, define $\kappa_j=(m_j/M)^{1/2}, 1\leq j\leq p$, consider the $p\times p$ matrix K with entries $-\kappa_j\kappa_{j'}$ for $j\neq j'$ and $1-\kappa_j^2$ for $j=j', 1\leq j$ and $j'\leq p$, let $b_{kh}^*=(b_{kh}-\bar{b}_k)/\{(M-1)^{1/2}\sigma_k\}, 1\leq k\leq M-1$ and $1\leq h\leq M$, and $\varphi_k^*(u)=\{\int_0^1\varphi_k^2(v)dv\}^{-1/2}\varphi_k(u), 0<u<1$ and $1\leq k\leq M-1$. Theorems 2.1 and 2.2 of TARDIF (1987) entail that, for $1\leq k\leq M-1$, the vector $(\zeta_{n_\nu k1},\ldots,\zeta_{n_\nu kp})'$ has, under both H and $\{K_\nu\}$, asymptotically a multivariate normal distribution with covariance matrix equal to K and with mean vector equal to $(0,\ldots,0)'$ under H but equal to $(\kappa_1\Delta_1,\ldots,\kappa_p\Delta_p)'\epsilon_k$ under $\{K_\nu\}$, where

$$\epsilon_k = (M-1)^{-1/2}\sum_{h=1}^{M} b_{kh}^*\mathcal{E}(\varphi_k^*\{G_k(D_k)\}V_h).$$

Since these vectors are mutually independent and since $\Sigma_{k=1}^{M-1}\gamma_k^2=1$, Slutsky's theorem entails that

$$(\zeta_{n,1},\ldots,\zeta_{n,p})' = \sum_{k=1}^{M-1}\gamma_{n_\nu k}(\zeta_{n_\nu k1},\ldots,\zeta_{n_\nu kp})'$$

is asymptotically a multivariate normal with covariance matrix equal to K and with mean vector equal to $(0,\ldots,0)'$ under H but equal to $(\kappa_1\Delta_1,\ldots,\kappa_p\Delta_p)'\Sigma_{k=1}^{M-1}\gamma_k\epsilon_k$ under $\{K_\nu\}$. Hence, Q_{n_ν} is asymptotically distributed as a χ_{p-1}^2 under H and as a $\chi_{p-1}^2(\delta^2)$ under $\{K_\nu\}$, where

$$\delta^2 = \sum_{j=1}^{p}\kappa_j^2\Delta_j^2\left[\sum_{k=1}^{M-1}\gamma_k\epsilon_k\right]^2,$$

in accordance with Lemma I.4.1 of HÁJEK and SIDÁK (1967). Finally, straightforward computations lead to $\delta^2=\delta_{MWR}^2$. Q.E.D.

It may be noted that, if $\eta_2=\cdots=\eta_{M-1}=0$, which entails $\eta_1=1$, the noncentrality parameter (4.5) reduces to the one given in Theorem 2.2 of TARDIF (1987). Thus, Theorem 4.1 may be viewed as a generalization of the latter.

REMARK 4.1. Using an argument similar to the one of Theorem V.2.2 of HÁJEK and SIDÁK (1967), it is noticed that, to establish the asymptotic distribution of Q_{n_ν} under the null hypothesis, the part of condition (4.3) requiring the ratios $n_{\nu k}/n_\nu$ to converge to finite numbers $\eta_k, 1\leq k\leq M-1$, can be dispensed with. It is preferable however to impose such a condition for establishing the asymptotic distribution of Q_{n_ν} under the sequence $\{K_\nu\}$ in order that δ_{MWR}^2 is a constant.

It will now be seen that the class of distribution-free tests induced by the generalized MWR contains asymptotically optimal members. Let the within-block scores be the columns of any $M\times(M-1)$ Helmert matrix, or matrix of orthonormal contrasts, $B^*=[\mathbf{b}_1\cdots\mathbf{b}_{M-1}]$, where $\mathbf{b}_k=(b_{k1},\ldots,b_{kM})', k=1,\ldots,M-1$, and assume that $\eta_1>0,\ldots,\eta_{M-1}>0$. Now, define the optimal variability measures as

$$d_k(x_1,\ldots,x_M) = \sum_{h=1}^{M} b_{kh}\{-\frac{f'_h}{f}(x_{(1)}-\bar{x},\ldots,x_{(M-1)}-\bar{x})\}, k=1,\ldots,M-1, \quad (4.6)$$

where $x_{(1)}\leq\cdots\leq x_{(M)}$ is the enumeration of $x_1,\ldots,x_M$ in ascending order and where $\bar{x}=M^{-1}\Sigma_{h=1}^{M}x_h$. Note that (4.6) entails $D_k=\Sigma_{h=1}^{M}b_{kh}V_h, 1\leq k\leq M-1$. Moreover, define the optimal score-generating functions as $\varphi_k(u)=\eta_k^{-1/2}G_k^{-1}(u)$, where $G_k^{-1}(u)$ is the quantile function of D_k under $H, 1\leq k\leq M-1$. Then, the noncentrality parameter (4.5) becomes

$$\delta_{MWR}^2 = C^2\frac{\sum_{k=1}^{M-1}\mathcal{E}(D_k^2)}{M(M-1)}.$$

On the other hand, introduce the $M\times M$ matrix Σ with entries $\sigma_{kh}=\mathcal{E}(V_kV_h), 1\leq k$ and $h\leq M$. TARDIF (1987) has noted that Σ is, in general, of rank $(M-1)$, its smallest characteristic root being 0 and $\mathbf{1}$, the $M\times 1$ vector of ones, being the associated characteristic vector. Since $\mathbf{1}'\Sigma\mathbf{1}=0$ and since the

$M \times M$ matrix $B = [B^* M^{-1/2}\mathbf{1}]$ is orthonormal, it follows that

$$\sum_{k=1}^{M-1} \mathcal{E}(D_k^2) = \sum_{k=1}^{M-1} \mathbf{b}_k{}'\Sigma\mathbf{b}_k = tr(B'\Sigma B) = tr(\Sigma),$$

where $tr(A)$ denotes the trace of the matrix A. Finally, it was noticed in TARDIF (1987) that $tr(\Sigma) = M\mathcal{I}(f)$ so δ^2_{MWR} is equal to $C^2\mathcal{I}(f)/(M-1) = \delta^2_{opt}$. The latter result holds for any value of the constants $\eta_1 > 0, \ldots, \eta_{M-1} > 0$, provided, of course, $\Sigma_{k=1}^{M-1}\eta_k = 1$, and any choice of Helmert matrix B^*. Consequently, the class of tests based on the generalized MWR contains an infinite number of optimal members. In other words, no matter how the partitioning of blocks is made and no matter which within-block scores are used, asymptotic optimality can be reached as long as the variability measures and the score-generating functions are defined accordingly.

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