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BOOLEAN ELEMENTS IN COMBINATORIAL OPTIMIZATION - A SURVEY

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BOOLEAN ELEMENTS IN COMBINATORIAL OPTIMIZATION - A SURVEY *)

by

Peter L. Hammer **)

ABSTRACT

This paper surveys several recent developments in the use of Boolean methods in 0-1 programming. After a brief introductory section some elements of Boolean algebra are presented. The next section deals with transformations of linear or nonlinear constraints in 0-1 programming to Boolean equations and these results are used in the following two sections for developing algorithms to solve 0-1 problems (with special emphasis on the linear and on the quadratic cases, as well as on the special case of knapsack problems) and for finding equivalent formulations of 0-1 problems (e.g. for proving the fact that "almost" every linear or nonlinear 0-1 programme can be transformed to an equivalent covering problem in the original variables). A further section characterizes packing problems which are equivalent to knapsack problems. Different possibilities of coefficient changes for a given linear inequality in 0-1 variables are exploited in the next section. The following section analyzes some connections between the Boolean and the geometric representation of certain polytopes in the unit cube and establishes a one-to-one correspondence between certain prime implicants of the problem and certain facets of the polytope. The last section deals with n-person characteristic function games, examines different value concepts (selections, core elements, Shapley value) as linear approximations of the nonlinear psuedo-Boolean function which represents the game, and establishes connections between these concepts.

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INTRODUCTION

The possibility of using Boolean elements in the formulation and interpretation of combinatorial optimization problems has been first pointed out by R. FORTET [12], [13]. This approach was continued by P. CAMION [5], R. FAURE & Y. MALGRANGE [11], P.L. HAMMER (Ivanescu), I. ROSENBERG & S. RUDEANU [29]. A monograph [31] on this subject has appeared in 1968, and since then numerous publications have been devoted both to theoretical and to practical (algorithmic) aspects of this topic. RUDEANU's recent monograph [44] is devoted to the problems of Boolean equations.

Most of the generally available algorithms for the solution of discrete optimization problems are based either on implicit enumeration, or on linear algebra. The use of linear algebra is motivated by the excellent results it yields in the solution of (continuous) linear programming problmes, and by the possibility of "relaxing" a typical discrete condition of the form $x \in \{0,1\}$ to its continuous counterpart $0 \leq x \leq 1$. However, in this relaxation one risks to lose essential features of the original discrete problem. (Consider for example the system $2x - 6y \geq -5$, $2x + 6y \geq 1$, with $x, y \in \{0,1\}$; this system obviously implies $x = 1$. If we relax $x, y \in \{0,1\}$ to $0 \leq x, y \leq 1$ and examine *all* the possible surrogates of the above two inequalities, i.e. all inequalities of the form $(2+2\lambda)x + (-6+6\lambda)y \geq -5 + \lambda$, we see that they have the following 0-1 solutions: (0,0), (1,0), (1,1) for $0 \leq \lambda \leq 1/5$, (0,0), (0,1), (1,0), (1,1) for $1/5 \leq \lambda \leq 5$, and (0,1), (1,0), (1,1) for $\lambda \geq 5$. In other words, there is no surrogate of our problem implying $x=1$). On the other hand, the degree of implicitness of an enumeration-type algorithm depends heavily on the art of using it. The interaction of constraints being usually hard to realize (unless it is strong enough to be detected in the continuous relaxation of the problem) is bypassed and taken care of only at later steps when sufficient variables have been fixed to arrive at conclusions from one of the particular constraints of the problem (e.g. how "implicit" is the enumeration which tells us that in every solution of the above problem $x = 1$, while y is arbitrary?). The difficulties arising in connection with discrete nonlinear problems are even greater.

The necessity of complementing rather than replacing the presently utilized methods with other ones seems obvious and Boolean algebra appears to be likely candidate for this task. In our above discussed example, it would tell that the first inequality is equivalent to $\bar{x}y = 0$, the second to $\bar{x}\bar{y} = 0$, and the system to $\bar{x}y \vee \bar{x}\bar{y}(=\bar{x}) = 0$, i.e. to $x = 1$).

On the other hand, the role of a Boolean viewpoint in combinatorial optimization does not reduce to that of assiting the computations. Boolean procedures can be used to transform problems to simpler ones and to get a better insight into their structure. Irrelevant elements can be disposed of (in the above example the variable y was irrelevant, since our problem did not depend on it), inessential data simplified (e.g. the inequality $2x + 6y \geq 1$ can be reduced to $x + y \geq 1$). Further, some familiar problems can be given new and possibly advantageous formulations (e.g. see [48] for a new formulation of the plant-location problem). Moreover one can expect connections to be established between apparently different questions and structural results to be obtained (e.g. "almost" every 0-1 programming problem can be reduced to a covering problem in the original variables, there is a strong connection between prime implicants of threshold functions and facets of the polytope of 0-1 solutions of knapsack problems, different concepts of value in n-person characteristic function games can be viewed as linear approxamations of nonlinear pseudo-Boolean functions, etc.)

The aim of this survey is not to present a comprehensive bibliography of all pertinent developments, but rather to discuss a relatively small (and subjective) selection of possibly useful ideas which have been reported in the literature of the last few years.

1. ELEMENTS OF BOOLEAN ALGEBRA

Let $B = \{0,1\}$. For $x \in B$ we shall denote $\bar{x} = 1 - x$ its *complement* or *negation*. We shall also write frequently $x^\alpha = x$ if $\alpha = 1$, and $x^\alpha = \bar{x}$ if $\alpha = 0$. This notation can cause no confusion, because the regular powers of $x \in B$ being all equal to x (idempotency of multiplication) we shall never use them.

For any $x, y \in B$, we shall define their *union* $x \vee y$ by $x \vee y =$ $= x + y - xy$.

Some of the most commonly utilized properties of the above defined operations are the following: $x \vee y = y \vee x$ (commutativity), $x \vee (y \vee z) =$ $= (x \vee y) \vee z$ (associativity), $x \vee x = x$ (idempotency), $x \vee y = 0$ if and only if $x = y = 0$, $x \vee 0 = x$, $x \vee 1 = 1$, $x \vee \bar{x} = 1$, $x \vee yz = (x \vee y)(x \vee z)$ and $x(y \vee z) = xy \vee xz$ (distributivities), $x \vee xy = x$ (absorption), $x \vee \bar{x}y = x \vee y$, $\overline{xy} = \bar{x} \vee \bar{y}$ and $\overline{x \vee y} = \bar{x} \cdot \bar{y}$ (De Morgan's Laws), $\bar{\bar{x}} = x$ (double negation), $x \leq y$ if and only if $xy = x$, $x \leq y$ if and only if $x\bar{y} = 0$, $x = y$ if and only if $x\bar{y} \vee \bar{x}y = 0$.

A function $f(x_1, \ldots, x_n)$ whose variables and values belong to B, will be called a *Boolean function*. Examples of such functions are $x \vee yz$, $x \vee yz \vee \bar{x}\bar{z}$, $(x \vee \bar{y})(\overline{y \vee xz})$, etc. The algebraic expression of a Boolean function is not unique, e.g. the expressions $x \vee y \vee \bar{z}$ and $x \vee yz \vee \bar{x}\bar{z}$ define the same function (this can be seen either by giving to x, y, z all 2^3 possible combinations of values, or noticing that $x \vee yz \vee \bar{y}\bar{z} = x \vee yz \vee \bar{z} =$ $= x \vee y \vee \bar{z}$.

A variable x, or its negation $\bar{x}$ will be called a *literal* X. A finite product of literals will be called an *elementary conjunction* $C = \prod_{j \in S} x_j^{\alpha_j}$ by convention, we shall consider sometimes also the constant $\underline{1}$ as being an elementary conjunction (with $S = \emptyset$). A finite union of elementary conjunctions $E = C_1 \vee C_2 \vee \ldots \vee C_m$ will be called a *disjunctive form*. It can be shown easily that every Boolean function can be expressed in a disjunctive form.

We shall say that an elementary conjunction C is *contained* in the elementary conjunction C' if every literal appearing as a factor in C is also a factor of C'. e.g. $x\bar{y}$ is contained in $x\bar{y}zu$, also in $x\bar{y}$, but is not contained in xz or in xyz.

An elementary conjunction I is said to be an *implicant* of the Boolean function $f(x_1,\ldots,x_n)$, if $I = 1$ implies $f(x_1,\ldots,x_n) = 1$. For example, $x\bar{y}$ is an implicant of $x\bar{y} \vee \bar{y}z(x\vee\bar{z})$. Also, $x\bar{y}$ is an implicant of $xz \vee \bar{y}\bar{z}$ (indeed, if $x\bar{y} = 1$, then $x = 1$, $y = 0$, and hence $xz \vee \bar{y}\bar{z}$ becomes $z \vee \bar{z}$ which is equal to 1).

An implicant P of a Boolean funciton $f(x_1,\ldots,x_n)$ is said to be a *prime implicant* if there is no other implicant P' of f contained in P. For example, $x\bar{y}$ is a prime implicant of $f = xz \vee \bar{y}\bar{z}$, but $x\bar{y}z$ is a non-prime implicant of f. If all the prime implicants of a Boolean function f are $P_1,\ldots,P_t$, then it is easy to see that $f = P_1 \vee\ldots\vee P_t$.

We shall see later that the knowledge of the prime implicants of a given Boolean function is extremely useful. A way of finding all the prime implicants is offered by the so-called *consensus* method.

Given two elementary conjunctions C and C', such that there is precisely one variable (x_0) appearing unnegated (x_0) in one of them, and negated $(\bar{x}_0)$ in the other, then the elementary conjunction obtained from the juxtaposition CC' of C and C' after deleting x_0, $\bar{x}_0$ and repeated literals, will be called the *consensus* of C and C'. For example, let $C = x\bar{y}\bar{z}u$ and $C = \bar{y}zu\bar{w}$; then their consensus is $C'' = x\bar{y}u\bar{w}$.

The *consensus method* consists in applying as many times as possible the following two operations to a disjunctive form of a Boolean function:

(i) eliminate any elementary conjunction which contains another one;
(ii) add as a new elementary conjunction the consensus of two elementary conjunctions, provided this consensus does not include any of the listed undeleted elementary conjunctions.

All the different expressions obtained along this process represent the same Boolean function, and the elementary conjunctions appearing in the final form at the end of this (finite, but long) process are exactly the prime implicants of the given functions.

It is likely that in practical problems finding all the prime implicants of a Boolean function might require an excessive amount of computation.

Therefore, in the more practical procedures described in Section III, we shall work with implicants which are not necessarily prime, but which allow an efficient solution of many 0-1 programs. A particular way of finding them is described in [20], and numerous other alternatives are easy to describe.

II. THE RESOLVENT

Let $S \subseteq B^n$ be the set of solutions of the system Σ of pseudo-Boolean inequalities $f(X) \le 0$ $(i=1,\dots,m)$ and let $\rho(X)$ be a Boolean function which takes the value 0 iff $X \in S$. The function ρ will be called the *resolvent* of Σ, and also the resolvent of S.

Let us consider the linear inequality

$$(1) \qquad \sum_{j=1}^{n} a_j x_j \le a_0$$

and let $\mathcal{l}$ be the family of all *minimal covers* of (1), i.e. the family of all the minimal sets $C \subseteq \{1,\dots,n\}$ with the property

$$\sum_{j\in C} |a_j| > a_0 - \sum_{j=1}^{n} \min(0,a_j).$$

It can be seen ([19]) that the function

$$(2) \qquad \phi(X) = \bigvee_{C\in\mathcal{l}} \prod_{j\in C} x_j^{\alpha_j}$$

(where $\alpha_j=1$ if $a_j\ge 0$ and $\alpha_j=0$ if $a_j<0$) is the resolvent of (1).

It has been shown in [29] (see also [31]) that every pseudo-Boolean function f(X) has a polynomial expression, which is linear in each variable. Hence, every pseudo-Boolean inequality can be written in the form

$$(3) \qquad \sum_{h=1}^{\mathcal{l}} b_h y_h \le b_0$$

where

$$(4) \qquad y_h = \prod_{j \in H_h} x_j \qquad (h=1,\ldots,k)$$

are themselves taking only the values 0 and 1. If $\psi(Y)$ is the resolvent of (3) (viewed as a linear inequality in the y_h's), then it is easy to see that the resolvent $\phi(X)$ of (3) (viewed as an inequality in the x_j's) can be obtained from $\psi(Y)$ by simply substituting (4) into it.

Further, if $\phi_i(X)$ are the resolvents of the pseudo-Boolean inequalities $f_i(X) \le 0$ $(i=1,\ldots,m)$ then $\phi(X) = \vee_{i=1}^{m} \phi_i(X)$ will be the resolvent of the system Σ.

Consider for example the system consisting of $x_j \in B (j=1,\ldots,6)$ and

$$(5\text{-}1) \qquad 5x_1 - 4x_2 - 2x_3 - x_4 - 4x_5 + 3x_6 \le -2$$

$$(5\text{-}2) \qquad -5x_2 + 6x_2x_6 - 8x_1x_3x_4 - 4x_2x_4 \le -7$$

or

$$(5\text{-}1)' \qquad 5x_1 + 4\bar{x}_2 + 2\bar{x}_3 + \bar{x}_4 + 4\bar{x}_5 + 3x_6 \le 9$$

$$(5\text{-}2)' \qquad 5\bar{x}_2 + 6x_2x_6 + 8\overline{x_1x_3x_4} + 4\overline{x_2x_4} \le 10$$

The resolvents of these inequalities are, respectively.

$$\begin{aligned}(6\text{-}1) \qquad \phi_1 = {} & x_1\bar{x}_2\bar{x}_3 \vee x_1\bar{x}_2\bar{x}_4 \vee x_1\bar{x}_2\bar{x}_5 \vee x_1\bar{x}_2x_6 \vee x_1\bar{x}_3\bar{x}_5 \vee \\ & x_1\bar{x}_4\bar{x}_5 \vee x_1\bar{x}_5x_6 \vee x_1\bar{x}_3x_6 \vee \bar{x}_2\bar{x}_3\bar{x}_5 \vee \bar{x}_2\bar{x}_5x_6 \vee \\ & \bar{x}_2\bar{x}_3\bar{x}_4x_6 \vee \bar{x}_3\bar{x}_4\bar{x}_5x_6,\end{aligned}$$

$$(6\text{-}2) \qquad \phi_2 = \bar{x}_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee \bar{x}_1x_6 \vee \bar{x}_3x_6 \vee \bar{x}_4,$$

while the resolvent of the system (5-1) - (5-2) is

$$\begin{aligned}(7) \qquad \phi = \phi_1 \vee \phi_2 = {} & \bar{x}_1\bar{x}_2 \vee \bar{x}_1x_6 \vee x_1\bar{x}_3\bar{x}_5 \vee \bar{x}_2\bar{x}_3 \vee \bar{x}_2\bar{x}_5 \vee \\ & \bar{x}_2x_6 \vee \bar{x}_3x_6 \vee \bar{x}_4 \vee \bar{x}_5x_6,\end{aligned}$$

(showing in particular that in every solution of (5-1) - (5-2), $x_4=1$).

III. ALGORITHMS *)

Due to the fact that the resolvent of a system of inequalities might involve an excessive number of (prime) implicants, practical algorithms based on the ideas outlined in the previous section can utilize only partially the information contained in it. Spielberg's minimal preferred inequalities method [44] belongs essentially to this class. Another example, APOSS (A Partial Order in the Solution Space), an algorithm given in [32] for solving linear 0-1 programs, utilizes only those minimal covers of the individual constraints which involve at most 3 elements. The corresponding implicants are combined to produce more implicants of lengths 1, 2 and 3. To every implicant of length 2 an order relation between variables is naturally associated ($x\bar{y}=0$ means $x\leq y$, $xy=0$ means $x\leq\bar{y}$, $\bar{x}\bar{y}=0$ means $\bar{x}\leq y$). If two binary relations involving the same pair of variables can be detected, then one of the variables can be eliminated ($xy=x\bar{y}=0$ implies $x=0$, $\bar{x}y=\bar{x}\bar{y}=0$ implies $x=1$, $xy=\bar{x}\bar{y}=0$ implies $x=\bar{y}$, $x\bar{y}=\bar{x}y=0$ implies $x=y$). When all these informations are exhausted, the same binary relations are reused as cuts in the associated linear program, and finally, if no further use of the binary relations is apparent, a branching technique is applied.

Consider for example a problem involving the constraints

$$8x_1 + 7x_2 + 5x_3 + 4x_4 + 2x_5 + 2x_6 \leq 14$$

$$4x_1 + 2x_2 + 6x_3 + 3x_4 + x_5 + 5x_6 \geq 12$$

The minimal covers of lenghts not exceeding 3 give rise to the "partial resolvents"

$$\psi_1 = x_1x_2 \vee x_1x_3 \vee x_1x_4 \vee x_2x_3x_4$$

$$\psi_2 = \bar{x}_1\bar{x}_3 \vee \bar{x}_2\bar{x}_3\bar{x}_4 \vee \bar{x}_2\bar{x}_4\bar{x}_6 \vee \bar{x}_3\bar{x}_4\bar{x}_5 \vee \bar{x}_3\bar{x}_6$$

From $x_1x_3 = \bar{x}_1\bar{x}_3 = 0$ it follows that $x_3 = \bar{x}_1$. Substituting we get

$$\psi_1' = x_1x_2 \vee x_1x_4 \vee x_2x_4$$

$$\psi_2' = x_1\bar{x}_2\bar{x}_4 \vee \bar{x}_2\bar{x}_4\bar{x}_6 \vee x_1\bar{x}_4\bar{x}_5 \vee x_1\bar{x}_6,$$

---

*) A survey on Boolean-based algorithms is given in [20].

hence $\psi' = \psi'_1 \vee \psi'_2 = x_1 \vee x_2x_4 \vee \bar{x}_2\bar{x}_4\bar{x}_6$, implying in particular $x_1 = 0$, and hence $x_3 = 1$. Substituting $x_1 = 0$, $x_3 = 1$ into our original system, we get $7x_2 + 4x_4 + 2x_5 + 2x_6 \le 9$, $2x_2 + 3x_4 + x_5 + 5x_6 \ge 6$, the partial resolvents of which are

$$\psi''_1 = x_2x_4 \vee x_2x_5x_6$$
$$\psi''_2 = \bar{x}_2\bar{x}_6 \vee \bar{x}_4\bar{x}_6 \vee \bar{x}_5\bar{x}_6 \vee \bar{x}_2\bar{x}_4\bar{x}_5;$$

hence $\psi'' = \psi''_1 \vee \psi''_2 = \bar{x}_6 \vee x_2x_4 \vee x_2x_5 \vee \bar{x}_2\bar{x}_4\bar{x}_5$ showing that $x_6 = 1$ in every feasible solution.

This algorithm has been coded on a CDC-6600 and a few hundred test problems involving up to 200 variables have been solved; the execution times (varying from .35 up to 65 sec.) compare favourably with those given by other methods.

The special case of quadratic 0-1 programs has been examined in [21], [30]. Consider a quadratic function in 0-1 variables

$$f = \sum_{j=1}^{n} c_j x_j + \sum_{\substack{i,j=1 \\ i<j}} d_{ij} x_i x_j ,$$

and let us put

$$\Delta_j = c_j + \sum_{i=1}^{j-1} d_{ij} x_i + \sum_{i=j+1}^{n} d_{ji} x_i \qquad (j=1,\dots,n)$$

$$\Delta_{jk} = \Delta_j - \Delta_k - d_{jk} x_k + d_{kj} x_j \qquad (j,k=1,\dots,n;j<k).$$

It is easy to see that in every minimizing point of f, $\Delta_j > 0$ ($\Delta_j<0$) implies $x_j = 0$ ($x_j=1$), while $\Delta_{jk} > 0$ ($\Delta_{jk}<0$) implies $x_j \le x_k$ ($x_j \ge x_k$). These relations can be exploited exactly as in the linear case to obtain information about variables with fixed values and about equal or complementary variables. If for example,

$$f = -x_1 + 3x_2 + x_1x_4 - 3x_1x_3 + 2x_2x_4 + 3x_3x_4 - 4x_2x_3$$

then from $\Delta_1 = -1 - 3x_3 - x_4$ we get $x_4 = 0 \rightarrow \Delta_1 < 0 \rightarrow x_1 = 1$,
and from $\Delta_4 = x_1 + 2x_2 + 3x_3$ we get $x_1 = 1 \rightarrow \Delta_4 > 0 \rightarrow x_4 = 0$,
i.e. $\bar{x}_1\bar{x}_4 = x_1x_4 = 0$, or $x_4 = \bar{x}_1$. Replacing now x_4 by $1 - x_1$ in f gives

$$f' = -x_1 + 5x_2 + 3x_3 - 2x_1x_2 - 6x_1x_3 - 4x_2x_3;$$

now $\Delta_1 < 0$, and hence $x_1 = 1$; f' becomes

$$f'' = -1 + 3x_2 - 3x_3 - 4x_2x_3,$$

where $\Delta_3 < 0$, showing that $x_3 = 1$; finally, f'' becomes

$$f''' = -4 - x_2,$$

showing that $x_2 = 1$, and the minimum (-5) is obtained in (1,1,1,0).

Another device which gives some insight into the problem is the examination of a "penalty relaxation inequality". This inequality has the form $\ell(x) \leq b^*$, where $\ell(x)$ is a linear lower bound of the quadratic function f(x), and b^* is an upper bound of the minimum of f(n); the rôle of b^* can be played by the value of f(x) in an arbitrary 0-1 point, while the construction of $\ell(x)$ (see [21]) is based on Hansen's additive penalties [34]. Such an $\ell(x)$ for our function is $-6 + \frac{5}{2}\bar{x}_1 + \bar{x}_2 + \frac{7}{2}\bar{x}_3$, and if we take as b^* the value f(1 0 1 0) = -4, we find that $\frac{5}{2}\bar{x}_1 + \bar{x}_2 + \frac{7}{2}\bar{x}_3 \leq 2$, i.e. $x_1 = x_3 = 1$.

Of course, the examination of the Δ_j's, Δ_{ij}'s and of the penalty-relaxation inequalities does not usually solve the entire problem, but can give valuable information when coupled with branch-and-bound type method. Since every quadratic 0-1 problem can be brought to a form where the quadratic form is positive (negative) definite (see [30]), there are possibilities of "bounding" by the use of continuous quadratic programming.

A special case of quadratic 0-1 programming has been studied in [15]. The question of maximizing a quadratic function with a single linear constraint ("quadratic knapsack problem") arose in connection with a location problem for airports in Italy, and the method suggested in [15] for its solution consists in determining linear upper bounds of the objective function and solving a sequence of associated (linear) knapsack problems.

A question which arises frequently in applications is that of minimizing an unconstrained polynomial in 0-1 variables. A method of successive elimination of variables has been given in [29] (see also [31]) for its solution. Branch-and-bound methods for the same problem have been devised in [3], [25], [33], [47], [51]; the main characteristic of these methods is the fact that branching is not performed according to single variables, but according to the 0-1 values of the nonlinear terms appearing in the polynomial. A variant of these procedures (see [25]) has been programmed on an IBM 360/50; problems with 10 - 30 variables, involving 10 - 50 nonlinear terms required between 0.48 and 239 seconds of execution time (including input-output time).

An efficient method for minimizing quotients of linear functions in 0-1 variables has been given by M. FLORIAN & P. ROBILLARD [41], [42]. (see also [31]).

Another question which has been examined was that of constraint pairing and its application to knapsack problems. Single linear constraints can be used in a straightforward way for deriving bounds on the variables of discrete optimization problems from the examination of all the surrogate constraints associated to pairs of constraints. Different surrogates might be helpful in fixing the values (or at least improving the bounds) of different variables; it might of course happen that no surrogate constraint fixes a variable, although the system does. It was however shown in [24] that if any variables can be fixed (or its bounds improved) by using arbitrary surrogates, then the same conclusion can also be obtained from the examination of n + 2 "special" surrogates (n of which correspond to those multipliers for which the coefficient of one of the variables in this surrogate is 0). R. DEMBO [9] shows that many of the conclusions so obtainable, are also available from the "best" surrogate. A. CHARNES, D. GRANOT & F. GRANOT [6] show how to extend these ideas to the case of more than two constraints.

An efficient application of this approach to knapsack problems [10]

$$
(\mathrm{KP})\begin{cases}\text{maximize} & \sum_{j=1}^{n} a_j x_j \\ \text{subject to} & \sum_{j=1}^{n} b_j x_j \le b_0 \\ & x_j \in \{0,1\}, \quad j=1,\ldots,n.\end{cases}
$$

associates to (KP) a pair of constraints and derives conclusions from the resulting system.

Let us assume that $a_1/b_1 \ge \ldots \ge a_n/b_n$. Let $\Xi = (\xi_1,\ldots,\xi_n)$ be the optimal solution of (KP'), obtained from (KP) by replacing all the constraints $x_j \in \{0,1\}$ by $0 \le x_j \le 1$ $(j=1,\ldots,n)$. If Ξ is not an integer vector, then we have $\xi_j = 1$ $(j=1,\ldots,t)$, $\xi_j = 0$ $(j=t+2,\ldots,n)$, and $0 < \xi_{t+1} < 1$. A very good (frequently optimal) solution is obtained by fixing $x_j^* = 1$ $(j=1,\ldots,t)$, $x_{t+1}^* = 0$, and re-solving a new KP' for b_0 replaced by $b_0 - \sum_{j=1}^{t} b_j$, etc., until arriving to a problem with x_j fixed for $j=1,\ldots,t^*$, and such that all the b_j's $(j=t^*+1,\ldots,n)$ are larger than the remaining b_0. Then the already fixed values x_j^* $(j=1,\ldots,t^*)$ together with $x_j^* = 0$ $(j=t^*+1,\ldots,n)$ form a good initial solution: let $a^* = \sum_{j=1}^{n} a_j x_j^*$. If the data are integer, than any better solution will have $a^* + 1$ as a lower bound. An upper bound to it is $\hat{a} = \sum_{j=1}^{t} a_j + [a_{t+1}\xi_{t+1}]$, (where $[\alpha]$ means the integer part of α). Hence if X^* is not an optimal solution, then any better solution satisfies

$$
\sum_{j=1}^{n} a_j x_j = a^* + p + 1,
$$

where p is a nonnegative integer not exceeding $\hat{a} - a^* - 1$. Pairing this with the constraint

$$
\sum_{j=1}^{n} b_j x_j + s = b_0 \qquad (s \ge 0)
$$

usually supplies enough information to fix at least some of the variables. These informations can be supplemented by those given by the binary and ternary relations among the variables.

Consider for example the 0-1 knapsack problem of maximizing

$$15x_1 + 16x_2 + 13x_3 + 9x_4 + 17x_5 + 11x_6$$

subject to

$$9x_1 + 10x_2 + 11x_3 + 8x_4 + 16x_5 + 11x_6 \le 29.$$

Here $\Xi = (1,1,\frac{10}{11},0,0,0)$, $X^* = (1,1,0,1,0,0)$, $\hat{a} = 42$, $a^* = 40$. Hence, if X^* is not optimal, then any optimal solution satisfies

$$15x_1 + 16x_2 + 13x_3 + 9x_4 + 17x_5 + 11x_6 = 41 + p$$

$$(0 \le p \le 1)$$

$$9x_1 + 10x_2 + 11x_3 + 8x_4 + 16x_5 + 11x_6 + s = 20$$

$$(s \ge 0)$$

Multiplying the first equation by 11, the second one by 13 and subtracting, we get

$$48x_1 + 46x_2 - 5x_4 - 21x_5 - 22x_6 - 13s = 74 + 11p$$

or

$$48x_1 + 46x_2 + 5\bar{x}_4 + 21\bar{x}_5 + 22\bar{x}_6 + 11\bar{p} = 133 + 13s,$$

implying $x_1 = 1$, $x_2 = 1$, $x_5 = 0$, $x_6 = 0$, and the last relation reduces to $5\bar{x}_4 + 11\bar{p} = -4 + 13s$, which obviously has no nonnegative integer solutions, showing that X^* was the optimal solution of our problem.

Experiments carried out with this idea show that it is extremely useful for fixing variables in 0-1 knapsack problems. In experiments carried out on an IBM 370/145 it turned out that in randomly generated problems involving 50 - 10,000 variables, the average number of fixed variables was between 74% and 93% of the total number of variables, while the computing time was less then one second.

IV. EQUIVALENT FORMS OF 0-1 PROGRAMS

Let us rewrite the resolvent $\phi(X)$ of a system Σ of linear or nonlinear inequalities, in the form

$$\phi(X) = \bigvee_{t=1}^{T} \Big(\prod_{j\in U_t} x_j\Big)\Big(\prod_{j\in V_t} \bar{x}_j\Big) \tag{8}$$

where U_t, V_t $(t=1,\ldots,T)$ are disjoint subsets of $\{1,\ldots,n\}$. Then, it is easy to see that $\phi(X) = 0$ iff X is a solution of the following *generalized covering problem*

$$\sum_{j\in V_t} x_j - \sum_{j\ U_t} x_j \geq 1 - |U_t| \qquad (t=1,\ldots,T) \tag{9}$$

Hence ([19])* every linear or nonlinear 0-1 programming problem is *strongly equivalent* to (i.e. has the same set of feasible solutions as) a generalized covering problem.

Consider now the problem (PI) of minimizing a pseudo-Boolean function $f(X)$ subject to Σ. Assume that f_0 is *strictly monotonic,* i.e. changing any 1 of any $X \in B^n$ to a 0 strictly decreases the value of f_0. This assumption holds for example for all linear f_0's having only positive coefficients. Specializing (8) to the case where $(\Pi_{j\in U_t} x_j)$ $(\Pi_{j\in V_t} \bar{x}_j)$ $(t=1,\ldots,T)$ are the prime implicants of $\phi(X)$, and assuming that $U_t = \emptyset$ for $t=1,\ldots,T_0$ and $U_t \neq \emptyset$ for $t = T_0 + 1,\ldots,T$, it can be shown ([22]) that (PI) is *equivalent* to (i.e. has the same optimal solutions as) the problem (PII) of minimizing $f_0(X)$ subject to $\phi'(X) = 0$, where

$$\phi'(X) = \bigvee_{t=1}^{T_0} \prod_{j\in V_t} \bar{x}_j \ , \tag{10}$$

i.e. to the *covering problem:* minimize $f_0(X)$ subject to

$$\sum_{j\in V_t} x_j \geq 1 \qquad (t=1,\ldots,T_0).$$

---

* This remark appears in a somewhat stronger form for the special case of a single linear pseudo-Boolean inequality in [2].

This equivalence holds because every feasible solution of PI is a feasible solution of the covering problem, while a feasible solution of the covering problem cannot be optimal unless it is feasible for PI too. For example minimizing

$$\sum_{j=1}^{6} c_j x_j \qquad (c_j>0, j=1,\dots,6)$$

subject to (5-1) - (5-2) is strongly equivalent to the generalized covering problem: minimize $\sum_{j=1}^{6} c_j x_j$ subject to $x_4 = 1$ and to $x_1 + x_2 \geq 1$, $x_1 - x_6 \geq 0$, $-x_1 + x_3 + x_5 \geq 1$, $x_2 + x_3 \geq 1$, $x_2 + x_5 \geq 1$, $x_2 - x_6 \geq 0$, $x_3 - x_6 \geq 0$, $x_5 - x_6 \geq 0$ and is equivalent to the covering problem: minimize $\sum_{j=1}^{6} c_j x_j$ subject to $x_4 = 1$ and to $x_1 + x_2 \geq 1$, $x_2 + x_3 \geq 1$, $x_2 + x_5 \geq 1$ (x_6 does not appear in any of the constraints, hence $x_6 = 0$ in any optimal solution).

Numerous equivalences between different forms of 0-1 programs have been described in [28].

V. PACKING AND KNAPSACK PROBLEMS

By a *packing problem* we shall mean a set of linear inequalities in 0-1 variables of the form $x_i + x_j \leq 1$ ($(i,j) \in \Gamma$). A linear inequality $\sum_{j=1}^{n} a_j x_j \leq a_0$ ($a_j \geq 0, j=0,1,\dots,n$) is equivalent to a packing problem iff all its minimal covers contain exactly two elements.

The converse problem, of characterizing those packing problems which are equivalent to a single linear inequality, has been examined in [7]. It has been shown that the following two characterizations follow from the theory of threshold functions.

I. PP is not 0-1 equivalent to a single linear inequality iff it is possible to find 4 distinct indices h, i, j, k such that

$$(h,i) \in \Gamma,\ (h,j) \notin \Gamma,\ (h,k) \notin \Gamma,\ (i,j) \notin \Gamma,$$
$$(i,k) \notin \Gamma,\ (j,k) \in \Gamma,$$

or such that

$$(h,i) \in \Gamma,\ (h,j) \notin \Gamma,\ (h,k) \notin \Gamma,\ i,j) \in \Gamma,$$
$$(i,k) \notin \Gamma,\ (j,k) \in \Gamma,$$

or such that

$$(h,i) \in \Gamma,\ (h,j) \notin \Gamma,\ (h,k) \in \Gamma,\ (i,j) \in \Gamma,$$
$$(i,k) \notin \Gamma,\ (j,k) \in \Gamma.$$

II. PP is 0 - 1 equivalent to a single linear inequality iff there exists a partitioning of $\{1,\ldots,n\}$ into two subsets N' and N'' and a permutation $(j_1,\ldots,j_r)$ of the elements of N'' such that

i) $\forall\, i,j \in N',\quad (i,j) \in \Gamma$

and

ii) $\forall\, i,j \in N'',\quad (i,j) \notin \Gamma$

iii) $\forall\, i,j_t \in N'',\ (s<t),\ \forall i \in N',\quad (i,j_t) \in \Gamma$

implies

$$(i,j_s) \in \Gamma.$$

An efficient algorithm was also presented in [7] for finding such a 0-1 equivalent single linear inequality, if any, or otherwise to find a "small" system of linear inequalities equivalent to the given PP. Peled studies in a recent paper the more general question of reducing the number of linear constraints in an arbitrary system of inequalities involving only 0-1 variables.

VI. COEFFICIENT TRANSFORMATION

It is obvious that different linear inequalities may have the same 0-1 solutions, and it might be useful to be able to transform a given inequality to an equivalent one which has a "better" form. For example

$x + y \leq 1$ seems to be a better form than $173x + 89y \leq 244.5$, but obviously the two inequalities have the same 0-1 solutions. This problem is studied in [4] and it is shown that the "optimal" coefficients (according to a large variety of criteria) can be determined by solving an associated linear program.

Let us consider a linear inequality

$$(11) \qquad \sum_{j=1}^{n} a_j x_j \leq a_0,$$

where $a_1 \geq \ldots \geq a_n \geq 0$. A minimal cover $R \subseteq \{1,\ldots,n\}$ such that $\sum_{j\in R} a_j - a_r + a_{r'} \leq a_0$ holds for any $r \in R$, $r' \notin R$, $r < r'$, is called a *roof* of (11). Similarly a set $C \subseteq \{1,\ldots,n\}$, maximal with the property that $\sum_{j\in C} a_j \leq a_0$, and such that $\sum_{j\in C} a_j - a_c + a_{c'} > a_0$ holds for any $c \in C$, $c' \notin C$, $c > c'$, is called a *ceiling* of (11). It is shown that every inequality

$$(12) \qquad \sum_{j=1}^{n} b_j x_j \leq b_0$$

0-1 equivalent to (11) and such that $b_1 \geq \ldots \geq b_n \geq 0$, is proportional to a solution of the system

$$\sum_{j\in R} b_j \geq b_0 + 1 \qquad \text{(for all roofs } R \text{ of (12))}$$

$$\sum_{j\in C} b_j \leq b_0 \qquad \text{(for all ceilings } C \text{ of (12))}$$

$$b_1 \geq \ldots \geq b_n \geq 0.$$

For example all the inequalities 0-1 equivalent to

$$7x_1 + 5x_2 + 3x_3 + 3x_4 + x_5 \leq 10$$

and having the coefficients ordered in the same way, are characterized by the system

$$b_1 + b_2 \geq b_0 + 1, \qquad b_1 + b_3 \leq b_0,$$

$$b_1 + b_4 + b_5 \geq b_0 + 1, \quad b_2 + b_3 + b_5 \leq b_0,$$

$$b_2 + b_3 + b_4 \geq b_0 + 1,$$

$$b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \geq 0.$$

If the criterion is to minimize b_0, the optimal solution is (4,3,2,2,1;6), i.e. the inequality

$$4x_1 + 3x_2 + 2x_3 + 2x_4 + x_5 \leq 6.$$

Numerous problems of similar nature have been studied in threshold logic (e.g. see [35], [36]). The usefulness of such transformations for increasing the efficiency of branch-and-bound methods is pointed out in [49].

VII. POLYTOPES IN THE UNIT CUBE

The convex hull $\hat{S}$ of a set S of vertices of the unit cube can be characterized by its facets. The set S is characterized by a Boolean Function $\delta_S(X)$ equal to 0 for $X \in S$ and to 1 elsewhere. The question of relating the Boolean and the geometric structures of a system of inequalities in 0-1 variables arises naturally. M.A. POLLATSCHEK [40] seems to have been the first to examine such questions. M.W. PADBERG [39] has given a procedure for producing facets of $\hat{S}$. A systematic investigation of this topic has been attempted in [23]. Some of the results of [23] overlap with those of [1], [17], [18], [37], [39], [50].

It was noticed in section IV that every 0-1 programming problem with a strictly monotone objective function can be reduced to a covering problem. Therefore in this section we shall mainly deal with facets of covering problems. For notational convenience we shall put $y_j = \bar{x}_j (j=1,\ldots,n)$; thus the constraints $\sum_{j \in T_i} x_j \geq 1$ $(i=1,\ldots,h)$ of the given covering problem become

$$(13) \qquad \sum_{j\in T_i} y_j \leq t_i - 1 \qquad (i=1,\ldots,h)$$

where $t_i = |T_i|$. Let S be the set of 0-1 solutions of (13), and $\hat{S}$ its convex hull. We shall assume that $\hat{S}$ is n-dimensional. It can be seen easily that $-y_j \leq 0$ is a facet of $\hat{S}$ for all $j=1,\ldots,n$, but $y_j \leq 1$ is a facet of $\hat{S}$ iff $t_1 = 2$ implies that $j \notin T_i$.

It has been shown in [23] that the constraint [13] is a facet of $\hat{S}$ iff for any $k \notin T_j$, the intersection of all those T_j which are contained in $\{k\} \cup T_i$ is nonempty. Further, if (13) is not a facet of $\hat{S}$, a procedure was given for strengthening it to a facet by changing the coefficients of the variables $y_k (k\notin T_i)$ from 0 to certain positive values. The procedure becomes particularly efficient for an apparently special class of covering problems, the so-called *regular covering problems*, i.e. those covering problems where the feasibility of any point $(y_1^*,\ldots,y_n^*)$ (where $y_{j_1}^*=\ldots=y_{j_s}^*=1$, the other components are 0) implies the feasibility of any point $(y_1^{**},\ldots,y_n^{**})$ having the same number of 1 components ($y_{\ell_1}^{**}=\ldots=y_{\ell_s}^{**}=1$, the other components are 0) when $j_1 \leq \ell_1,\ldots,j_s \leq \ell_s$. However (see [22]), a very wide class of covering problems can be brought to such a form.

The extension procedure becomes extremely simple for the case of regular covering problems and it can be shown that there is a 1-1 correspondence between those factors of $\hat{S}$ which have only 0-1 coefficients and those sets T_i which have the following two properties:
(i) if $u_i = \min\{j\,|\,j\in T_i\}$, $w_i = \min\{j\,|\,j\notin T_i,\ j > u_i\}$ (if any), and if $P_i \in B^n$ is the point whose 1-components are all the elements of the set

$$\begin{cases} (\{w_i\} \cup T_i) - \{u_i\} & \text{(if } w_i \text{ is defined)} \\ T_i - \{u_i\} & \text{(otherwise)}, \end{cases}$$

then $P_i \in S$; (ii) if $v_i = \min\{j\,|\,j\in T_i,\ j \neq u_i\}$ and $R_i \in S$ is the point whose

1-components are all the elements of the set $(\{1\} \cup T_i) - \{u_i, v_i\}$, then $R_i \in S$.

The most common case of a regular covering problem corresponds to knapsack problems, when the T_i's are its minimal covers. A list of all the facets of all the knapsack problems with at most 5 variables is given in [23].

VIII. PSEUDO-BOOLEAN FUNCTIONS AND GAME THEORY.*

A *characteristic function game* (N,W) is a set of "players" $N = \{1,2,\ldots,n\}$ and a real-valued function $W: 2^N \to R$ (called the *characteristic function*), defined for all subsets T of N. If T is a "coalition", then W(T) is the "payoff" it can secure. It is clear that 2^N is mapped in a 1-1 way onto B^n by mapping a subset T of N to its *characteristic vector* X, defined by $x_k = 1$ for $k \in T$ and $x_k = 0$ for $k \notin T$. Hence as remarked by Owen ([38]) a characteristic function game is actually the same as a pseudo-Boolean function.

It is well known [31] that every pseudo-Boolean function f in n variables has a unique polynomial expression of the form

$$f(x) = \sum_{T \subseteq N} [a_T \prod_{k \in T} x_k],$$

called its *canonical form.* The corresponding characteristic function game (N,W) then satisfies

$$W(T) = \sum_{S \subseteq T} a_S, \quad T \subseteq N.$$

Shapley ([45]) has shown that this relation gives

$$a_T = \sum_{S \subseteq T} (-1)^{t-s} W(S), \quad T \subseteq N,$$

---

* See [26].

where t and s are the cardinalities of T and S, respectively. Thus the a_T's can be found from the function f(x).

As an example let us consider the 3-person characteristic function game defined by the following table:

T	∅	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
W(T)	0	0	0	0	3	2	2	4

The corresponding pseudo-Boolean function on B^3 is $f(X) = 3x_1x_2\bar{x}_3 + 2x_1\bar{x}_2x_3 + 2\bar{x}_1x_2x_3 + 4x_1x_2x_3$. By replacing each $\bar{x}_j$ by $1 - x_j$ and simplifying, we obtain the canonical expression:

(14) $$f(X) = 3x_1x_2 + 2x_1x_3 + 2x_2x_3 - 3x_1x_2x_3.$$

A game (N,W) is said to be *superadditive* if for any disjoint sets S,T of N, we have $W(S) + W(T) \le W(S \cup T)$ (i.e. it always "pays" to form a larger coalition). It can be easily seen that the game of the example in section 1 is superadditive. The following result holds:

Let f be the pseudo-Boolean function corresponding to the game W. Then the following are equivalent: (a) W is superadditive, (b) $X + Y \le 1$ implies $f(X) + f(Y) \le f(X+Y)$ for all $X, Y \in B^n$, (c) $XY = 0$ implies $f(X) + f(Y) \le f(X+Y)$ for all $X, Y \in B^n$, (d) $f(XY) + f(X\bar{Y}) \le f(X)$ for all $X, Y \in B^n$ (here $\bar{Y} = \underline{1} - Y$, $\underline{1} = (1,\ldots,1)$).

The goal of n=person characteristic function game theory is to find a "solution", i.e. a value for each player based upon the coalitions he may join. If a game (N,W) satisfies $W(\emptyset) = 0$ then, as SHAPLEY [45] mentions, we may regard a solution as an inessential game (N,Z) which "approximates" (N,W) by some method and which assigns a value Z({j}) to each player j. In this paper we discuss a few specific solutions in terms of pseudo-Boolean functions. Since such a function defines a game we can speak about the core and the Shapley value of a function. Throughout this section let f be a pseudo-Boolean function with f(0) = 0. A *core element* of f is a linear pseudo-Boolean function $h(X)$ satisfying $h(X) \geq f(X)$ for all $X \in B^n$ and $h(1) = f(1)$. We shall also say that the vector h of coefficients of $h(X)$ is a *core element* of f. The polyhedron of all the core elements of f is called the *core* of f. It may be empty for some f. In this selection we construct another polyhedron (the *selectope*) and show that it contains the core of f.

Consider the canonical form of the pseudo-Boolean function f. We denote $T^+ = \{T \subseteq N\colon a_T > 0\}$, $T^- = \{T \subseteq N\colon a_T < 0\}$, $T = T^+ \cup T^-$. The *incidence graph* of f is a directed bipartite graph $G = (T, N; E)$ in which an edge $e \in E$ is directed from $T \in T^+$ to $j \in N$ if $j \in T$, and an edge $e \in E$ is directed from $j \in N$ to $T \in T^-$ if $j \in T$. For any node $T \in T$, $I(T)$ denotes the set of edges $e \in E$ incident with T. For any node $j \in N$, $I^+(j)$ denotes the set of edges $e \in E$ directed to j, $I^-(J)$ denotes the set of edges $e \in E$ directed away from j and $I(j) = I^+(j) \cup I^-(j)$. For each edge $e \in E$, $T(e)$ denotes its end in T and $j(e)$ its end in N. The edge $e \in E$ corresponds to the occurence of the variable $x_{j(e)}$ in the term $a_{T(e)} \cdot \Pi_{j \in T(e)} x_j$ of f. Figure 1 illustrates the incidence graph of our pseudo-Boolean function (14) with the values a_T displayed next to the nodes T.

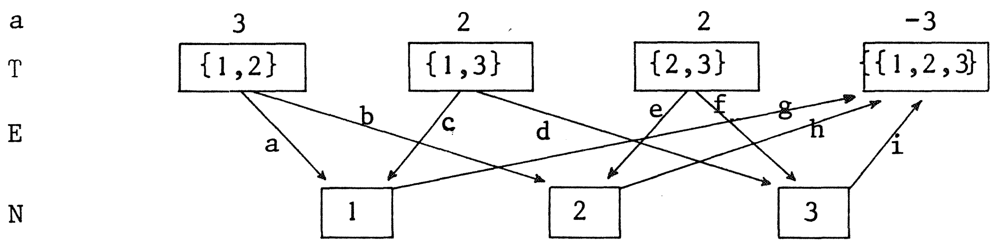

Figure 1

A *selector* of f is a vector $s = (e_T, T \in T)$, such that $e_T \in I(T)$, $\forall\, T \in T$. The corresponding *selection* of f is the vector $h(s) = (h_j, j \in N)$, where $h_j = \sum_{e_T \in I(j)} a_T$. S will denote the set of all selectors of f. In our example there are $\Pi_{T \in T} |T| = 2 \cdot 2 \cdot 2 \cdot 3 = 24$ selectors, which are listed below along with the corresponding selections (of which only 20 are distinct).

	Selectors				Selections		
	$e_{\{1,2\}}$	$e_{\{1,3\}}$	$e_{\{2,3\}}$	$e_{\{1,2,3\}}$	h_1	h_2	h_3
1	a	c	e	g	2	2	0
2	a	c	e	h	5	-1	0
3	a	c	e	i	5	2	-3
4	a	c	f	g	2	0	2
5	a	c	f	h	5	-3	2
6	a	c	f	i	5	0	-1
7	a	d	e	g	0	2	2
8	a	d	e	h	3	-1	2
9	a	d	e	i	3	2	-1
10	a	d	f	g	0	0	4
11	a	d	f	h	3	-3	4
12	a	d	f	i	3	0	1
13	b	c	e	g	-1	5	0
14	b	c	e	h	2	2	0
15	b	c	e	i	2	5	-3
16	b	c	f	g	-1	3	2
17	b	c	f	h	2	0	2
18	b	c	f	i	2	3	-1
19	b	d	e	g	-3	5	2
20	b	d	e	h	0	2	2
21	b	d	e	i	0	5	-1
22	b	d	f	g	-3	3	4
23	b	d	f	h	0	0	4
24	b	d	f	i	0	3	1

Selectors have been introduced in [27], where selections are called "linear factors", and where it is shown that if T^- is empty then $f(X)$ is the minimum of all the linear pseudo-Boolean functions $\sum_{j\in N} h_j x_j$, where h is a selection of f. This concept has been generalized by I. ROSENBERG in [43].

The *selectope* of f is the convex hull of all the selections of f. We give below a characterization of the selectope of f. Let $h = (h_j, j \in N)$ be any n-vector. A *flow* for h in G is a non-negative vector $z = (z_e, e \in E)$ satisfying the node equations

$$\sum_{e \in I(T)} z_e |a_T|, \qquad T \in T$$

$$\sum_{e \in I^+(j)} z_e - \sum_{e \in I^-(j)} z_e = H_j, \qquad j \in N.$$

Then:

(1) The selectope of f is the set of those n-vectors h for which there exists a flow in G.

(2) The selectope of f contains the core of f, equality holding if and only if all the nonlinear terms in (1) have nonnegative coefficients.

We remark that from here we obtain an efficient partial test for a non-negative vector h satisfying $h(1) = f(1)$ to be a core element of an un-linear pseudo-Boolean function f. Apply the maximal flow algorithm to G. If the value of this flow is less than $\sum_{T \in T^+} a_T$, h cannot be a core element of f. However if the value is $\sum_{T \in T^+} a_T m$ we do not have any conclusion. (For example, of all the 20 selections of the pseudo-Boolean function f in (14), only (2,2,0) is a core element of f.) It would be of interest to refine the test for that case.

A vector $Y \in B^n$ is said to be a *carrier* of a pseudo-Boolean function f on B^n if $f(X) = f(XY)$ for all $X \in B^n$. The product of carriers of f is a carrier of f, hence the product Y^* of all the carriers of f is the unique minimal carrier of f, and f effectively depends on x_j if and only if $y_j^* = 1$.

A mapping $\pi: B^n \to B^n$ is an *automorphism* if it is one-one and onto, and also conserves the operations $\vee$, $\cdot$ and $\bar{}$, i.e. $\pi(X \vee Y) = \pi(X) \vee \pi(Y)$, $\pi(XY) = \pi(X)\pi(Y)$, $\pi(\bar{X}) = \overline{\pi(X)}$. For convenience we shall write πX for $\pi(X)$. For any automorphism π on B^n and for any function f on B^n we define the function πf by $\pi f(X) = f(\pi^{-1}X)$ or equivalently by $\pi f(\pi X) = f(X)$. It can be seen that if π is an automorphism of B^n and X is a unit vector of B^n, then

πX is a unit vector of B^n. Hence π permutes the unit vectors of B^n and permits us to view π as a permutation of the variables $j \in N$ themselves. For $j \in N$, $k = \pi(j)$ is defined so that if X is the unit vector with $x_j = 1$, then πX is the unit vector with $(\pi X)_k = 1$. Thus
$(\pi X)_k = x_{\pi^{-1}k}$ for all $X \in B^n$.

We can now state the axiomatic definition of the Shapley value ([45]). Let F be the set of all pseudo-Boolean functions f on B^n such that $f(0) = 0$. A *Shapley value* is a mapping η: $F \to R^n$ satisfying the following axioms:

Axiom 1. For each automorphism π of B^n and for each $f \in F$,

$$\eta_{\pi(k)}[\pi f] = \eta_k[f], \qquad k=1,\dots,n.$$

Axiom 2. For each $f \in F$ and for each carrier Y of f,

$$\sum_{k=1}^{n} \eta_k[f] y_k = f(1).$$

(in particular then $\sum_{k=1}^{n} \eta_k[f] = f(1)$).

Axiom 3. For each $f, g \in F$

$$\eta[f + g] = \eta[f] + \eta[g].$$

The following theorem is due to SHAPLEY ([45]):
There exists a unique Shapley value, and it is given by the formula

$$\eta_k[f] = \sum_{T \subseteq N} \frac{a_T}{|T|}.$$

As an illustration, the Shapley value of (14) is:

$$\eta[f] = (\frac{3}{2} + \frac{2}{2} - \frac{3}{3},\ \frac{3}{2} + \frac{2}{2} - \frac{3}{3},\ \frac{2}{2} + \frac{2}{2} - \frac{3}{3}) = (\frac{3}{2}, \frac{3}{2}, 1).$$

Let f be a pseudo-Boolean function with $f(0) = 0$. Then the Shapley value of f is the arithmetic mean, over all the selectors of f, of the corresponding selections, i.e.

$$\frac{1}{|S|} \sum_{s \in S} h(s).$$

Also the following result holds: Let $f(0) = 0$. If $a_T \geq 0$ for all $T \subseteq N$ then f is superadditive and every selection of f as well as the Shapley value of f are core elements of f.

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