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K. HARDY AND I. JUHÁSZ

NORMALITY AND THE WEAK cb PROPERTY

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Normality and the weak cb property

by

K. Hardy and I. Juhász

ABSTRACT

It is demonstrated that the Alexandroff duplicate of a Dowker space is again a Dowker space which is not weak cb, while the existence of weak cb Dowker spaces is made manifest.

KEY WORDS AND PHRASES: *Dowker space, cb-space, weak cb-space, Alexandroff duplicate.*

A non-metrizable, first countable compact space was created by ALEXANDROFF in [1] and the construction has been subsequently generalized and employed (ENGELKING [2,3], JUHÁSZ [7,8]). The present note concentrates on some properties of the Alexandroff duplicate A(X) which, in particular, show that a normal space need not have the weak cb property, thus resolving the open question in MACK [11, p.240].

1. PRELIMINARIES

No separation axioms are implicitly assumed for the topological space X. The Hewitt-Nachbin realcompactification of a Tychonoff space X is denoted by υX. We will write $A_n \searrow \emptyset$ to indicate that (A_n) is a decreasing sequence of subsets of X such that $\bigcap_n A_n = \emptyset$. $\mathbb{N}$ denotes the natural numbers. A set A is regular closed if $A = cl_X int_X A$, and ∂A denotes the boundary of A.

<u>PROPOSITION 1.1</u>. (MACK [10,11]) *A space* X *is* cb (*weak* cb) *if and only if for each sequence* $A_n \searrow \emptyset$ *of closed (regular closed) subsets of* X, *there exists a sequence of zero sets* (Z_n) *with* $A_n \subseteq Z_n$ *and* $\bigcap_n Z_n = \emptyset$.

cb-spaces originated in HORNE [5] and were studied by MACK in [10]. Every normal, countably paracompact space is cb and every cb-space is countably paracompact. Weak cb-spaces were defined in [11]. They form a natural generalization of cb-spaces and include the Tychonoff pseudocompact spaces and all extremally disconnected spaces. Interest in weak cb-spaces is centered in the theorem ([11]) that for a Tychonoff space X, the Dedekind completion of C(X) is isomorphic to C(Y), for some space Y, if and only if υX is weak cb. It should be noted that if X is Tychonoff and weak cb, then any space T with $X \subseteq T \subseteq \upsilon X$ is weak cb. The converse fails in general (see HARDY & WOODS [4]) but the following result is evident and will be needed below.

<u>PROPOSITION 1.2</u>. *Let* X *be Tychonoff and consider the statements*

(a) *For any sequence* $A_n \searrow \emptyset$ *of regular closed sets in* X *we have* $\bigcap_n cl_{\upsilon X} A_n = \emptyset$

(b) *For any decreasing sequence* (A_n) *of regular closed sets in* X *we have* $\bigcap_n \mathrm{cl}_{\upsilon X} A_n = \mathrm{cl}_{\upsilon X} \bigcap_n A_n$

(c) *If* υX *is weak* cb, *then any space* T, *with* $X \subseteq T \subseteq \upsilon X$ *is weak* cb.

Then (a) *if and only if* (b); *and* (a) *or* (b) *implies* (c).

PROOF. We merely recall that if X is dense in T and A is a regular closed subset of X then $\mathrm{cl}_T A = B$ is the unique regular closed subset of T with $A = B \cap X$. □

According to a result of ISHIKAWA [6], a space X is countably paracompact if and only if for each sequence $A_n \searrow \emptyset$ of closed subsets of X, there exists a sequence (G_n) of open sets such that $A_n \subseteq G_n$ and $\bigcap_n \mathrm{cl}_X G_n = \emptyset$. The following observation will be useful below and may have independent interest.

PROPOSITION 1.3. *The following statements are equivalent*

(a) X *is countable paracompact.*

(b) *For each sequence* $F_n \searrow \emptyset$ *of closed nowhere dense subsets of* X, *there exists a sequence* (G_n) *of open sets such that* $F_n \subseteq G_n$ *and* $\bigcap_n \mathrm{cl}_X G_n = \emptyset$.

(c) *Each countable increasing cover* ([10]) *by dense open sets has a countable closed refinement whose interiors cover* X.

PROOF. It is enough to show (b) implies (a). Let $A_n \searrow \emptyset$ be an arbitrary sequence of closed sets and define a sequence of open sets (G_n) with $A_n \subseteq G_n$ and $\bigcap_n \mathrm{cl}_X G_n = \emptyset$ in the following manner:

(i) If $\mathrm{int}_X A_m = \emptyset$ for some $m \geq 1$, there exist open sets G_k with $A_k \subseteq G_k$, $k \geq m$ and $\bigcap_k \mathrm{cl}_X G_k = \emptyset$; put $G_n = X$ for $1 \leq n < m$. Now assume that $\mathrm{int}_X A_n \neq \emptyset$ for all n.

(ii) If a subsequence (A_{n_k}) exists with $A_{n_{k+1}} \subseteq \mathrm{int}_X A_{n_k}$, let $G_{n_{k+1}} = \mathrm{int}_X A_{n_k}$ and $G_n = X$ otherwise.

(iii) If there exists $m \geq 1$ such that $F_k = \partial A_k \cap \partial A_{k+1} \neq \emptyset$ for $k \geq m$ then $F_k \searrow \emptyset$ is a sequence of closed nowhere dense sets and there exists a sequence of open sets (U_k) with $F_k \subseteq U_k$ and $\bigcap_k \mathrm{cl}_X U_k = \emptyset$. Define $G_{k+1} = \mathrm{int}_X A_k \cup U_k$ for $k \geq m$ and $G_n = X$ for $1 \leq n \leq m$. □

In order to exploit the use of nowhere dense closed subsets, we venture to make the following

DEFINITION 1.4. X *is an* nd-*space if for each sequence* $F_n \searrow \emptyset$ *of closed nowhere dense sets, there exists a sequence of zero sets* (Z_n) *with* $F_n \subseteq Z_n$ *and* $\bigcap_n Z_n = \emptyset$.

Every cb-space is an nd-space. Since every zero set Z is a regular G_δ-set (a countable intersection of closed sets whose interiors contain Z), we may adapt the proof of Proposition 1.3 to conclude that every nd-space is countable paracompact. A space is cb if and only if it is both a weak cb and an nd-space. The example on p.240 of [11] is countably paracompact but not an nd-space. It is conjectured that an nd-space need not be cb, although an example at the present time is not forthcoming.

2. PROPERTIES OF A(X)

Recall the construction in [7]. Given an arbitrary topological space X, consider the set $A(X) = X \cup X'$, where X' is a disjoint copy of X. For any $x \in X$, let x' denote the corresponding point of X' and if $S \subseteq X$ define $S' = \{x' \mid x \in S\}$. A topology is introduced to A(X) by defining a base $\{B(z) \mid z \in A(X)\}$ as follows:

$$B(x') = \{\{x'\}\} \text{ and } B(x) = \{V \cup (V'\backslash\{x'\}) \mid V \in \mathcal{V}(x)\},$$

where $\mathcal{V}(x)$ is a neighbourhood base of x in X. The resulting space, also denoted by A(X), generalizes the original construction in ALEXANDROFF & URYSOHN [1] and is called the Alexandroff duplicate of X. It is clear that X is a closed, C-embedded subspace of A(X).

Many properties of X are shared with A(X). It has been noticed that A(X) is compact ([2]), α-compact (for any infinite cardinal α), realcompact and Tychonoff ([7]), if X has the corresponding property. We will now expand this list of properties.

Observe that a space is normal if and only if each pair of disjoint closed nowhere dense sets can be separated by disjoint open neighbourhoods.

PROPOSITION 2.1. X *is normal if and only if* $A(X)$ *is normal.*

PROOF. Let A and B be disjoint closed nowhere dense subsets of $A(X)$. Then A and B are closed and disjoint in X and can be separated by disjoint open sets U and V in X. The sets $U \cup U'$ and $V \cup V'$ are open disjoint neighbourhoods of A and B in $A(X)$. $\square$

PROPOSITION 2.2. X *is countably paracompact if and only if* $A(X)$ *is countably paracompact.*

PROOF. For the necessity, let $F_n \searrow \emptyset$ be a sequence of closed nowhere dense subsets of $A(X)$. Then $F_n \subseteq X$ and there exists a sequence (V_n) of open subsets of X with $F_n \subseteq U_n$ and $\bigcap_n cl_X U_n = \emptyset$. Define $G_n = U_n \cup U_n'$ and note that $cl_{A(X)} G_n = cl_X U_n \cup U_n'$, so that $F_n \subseteq G_n$ and $\bigcap_n cl_{A(X)} G_n = \emptyset$. $\square$

PROPOSITION 2.3. *If* $A(X)$ *is weak* cb *then both* X *and* $A(X)$ *are* cb.

PROOF. To show that X is cb, take a sequence $A_n \searrow \emptyset$ of closed sets in X. Then $B_n = A_n \cup A_n'$ is regular closed in $A(X)$ and $B_n \searrow \emptyset$. There exist zero sets W_n in $A(X)$ with $B_n \subseteq W_n$ amd $\bigcap_n W_n = \emptyset$. Then $Z_n = W_n \cap X$ is a zero set in X and $A_n \subseteq Z_n$ with $\bigcap_n Z_n = \emptyset$. If X is cb then both X and $A(X)$ are countably paracompact, hence $A(X)$ is cb. $\square$

One may show that $A(X)$ is countably compact if and only if X is. Furthermore, if X contains a C-embedded copy of $\mathbb{N}$, so does $A(X)$ so that $A(X)$ is pseudocompact implies that X is also. However, if X is pseudocompact (Tychonoff) but not countable compact then $A(X)$ is not weak cb, in particular, not pseudocompact.

3. DOWKER SPACES

A Dowker space is a normal Hausdorff space which is not countably paracompact. Such spaces exist within Zermelo-Fraenkel set theory; the axiom of choice implies the existence of a zero-dimensional P-space which is Dowker (RUDIN [12]) and more recently a certain combinatorial principle called $\diamond$ implies existence of a locally compact, first countable, hereditarily separable Dowker space (JUHÁSZ et al. [9]).

The open question in [11, p.240] may be phrased as follows: *Must every Dowker space have the weak cb property?* It follows from Propositions 2.1 and 2.2 that A(X) is a Dowker space if and only if X is such. Since no Dowker space can be even an nd-space, 2.3 implies that for any Dowker space X, the space A(X) answers the above question negatively. It may be of interest however that the Dowker space of M.E. RUDIN [12] is weak cb, as is now shown.

The reader is referred to [12] for details. With the same notation as in [12], define

$$F = \{f: \mathbb{N} \to \omega_\omega \mid f(n) \le \omega_n \text{ for all } n \in \mathbb{N}\}\,.$$

$$X = \{f \in F \mid \omega_1 \le \mathrm{cf}(f(n)) \le \omega_k \text{ for all } n \in \mathbb{N} \text{ and some } k \in \mathbb{N}\}\,,$$

$$X' = \{f \in F \mid \omega_1 \le \mathrm{cf}(f(n)) \text{ for all } n \in \mathbb{N}\}\,.$$

F carries a topology generated by the basic open-and-closed sets

$$(f,g] = \{h \in F \mid f(n) < h(n) \le g(n) \text{ for all } n \in \mathbb{N}\}\,.$$

Then $X \subseteq X' \subseteq F$ are subspaces and $\upsilon X = X'$ is paracompact, and hence a weak cb-space.

To show that X is weak cb, let $A_n \searrow \emptyset$ be a sequence of regular closed subsets of X and suppose $g \in \bigcap_n \mathrm{cl}_{\upsilon X} A_n$. We will define an increasing sequence $\{f_\alpha \in X \mid \alpha < \omega_1\}$ as follows:

1) Choose any $f_0 \in \mathrm{int}_X A_1$ with $f_0 \le g$.

2) Assume $f_\beta \in X$ is defined for all $\beta < \alpha$, and

   (a) if $\alpha = \beta + 1$, let $i \in \mathbb{N}$ be the smallest integer with $f_\beta \notin \mathrm{int}_X A_i$ and choose $f_\alpha \in (\mathrm{int}_X A_i) \cap (f_\beta, g]$.

   (b) if α is a limit ordinal, let $h_\alpha(n) = \sup\{f_\beta(n) \mid \beta < \alpha\}$ and choose $f_\alpha \in (\mathrm{int}_X A_1) \cap (h_\alpha, g]$.

Now define $f(n) = \sup\{f_\alpha(n) \mid \alpha < \omega_1\}$. Then $f \le g$ and $\mathrm{cf}(f(n)) = \omega_1$ for all $n \in \mathbb{N}$ implies that $f \in X$. However, $f \in A_k$ for all $k \in \mathbb{N}$: let $h < f$ and for each $n \in \mathbb{N}$ there is $f_{\alpha_n} \in \{f_\alpha \mid \alpha < \omega_1\}$ with $h(n) < f_{\alpha_n}(n)$. Let $\beta = \sup\{\alpha_n \mid n \in \mathbb{N}\}$ and then $f_{\beta+k} \in (\mathrm{int}_X A_k) \cap (h,f]$, that is

$f \in cl_X int_X A_k = A_k$.
We have a contradiction and so $A_n \searrow \emptyset$ implies $\bigcap_n cl_{\upsilon X} A_n = \emptyset$. Finally, apply Proposition 1.2 to infer that X is weak cb.

4. REMARKS

Since the Dowker space X in [12] is weak cb, it follows from [4] that $E(\upsilon X) = \upsilon E(X)$, where E(X) denotes the absolute of X (see for example [4, p.652]). Thus, $\upsilon E(X)$ is paracompact. However, it has been shown by E.K. VAN DOUWEN that E(X) is not normal. It would seem natural therefore to pose the following questions. 1) Is there a normal (non-paracompact) space X with normal absolute E(X); 2) Is there an extremally disconnected Dowker space; and ultimately 3) Is there a Dowker space X with Dowker absolute E(X).

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