Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

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BS-R9603 1996

Report BS-R9603
ISSN 0924-0659

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Abstract

This paper develops an abstract theory for mathematical morphology on complete lattices. The approach is based upon the idea that objects are only known through information provided by a given collection of measurements (called evaluations in this paper). This abstract approach leads in a natural way to the concept of convolution lattice (where 'convolution' has to be understood in the sense of an abstract Minkowski addition), the morphological slope transform, and the notion of 'random lattice element'.

AMS Subject Classification (1991): 06A23, 06F99, 60D05, 68U10

Keywords & Phrases: image processing, mathematical morphology, complete lattice, measurements, adjunction, dilation, erosion, opening, closing, Minkowski addition and subtraction, supremal convolution, evaluation, convolution lattice, support function, slope transform, random set.

Note: The second author was supported in part by the Netherlands Organization for Scientific Research (NWO).

1. Introduction

In many practical situations, physical objects are accessible only through a finite number of measurements. For example, one can get information about an unknown set X by testing, for any test set B in a given collection $\mathcal{T}$, whether or not $X \cap B = \varnothing$. This is Kendall's approach to random set theory [8]. This simple observation, i.e., that objects are only known through measurements, is used in this paper to build an abstract theory of mathematical morphology on complete lattices.

The last decade, complete lattices have manifested themselves as a convenient mathematical framework for morphological image processing. Within this framework, classical approaches towards binary and grey-scale morphology happen to be only particular cases of a much more general theory [4, 6, 14, 15, 18].

The approach adopted in this paper is based essentially upon two assumptions: (i) the objects under study are elements of a given complete lattice; (ii) such objects are not known explicitly but only through a given collection of measurements. Such measurements, called *evaluations* in this paper, are represented by mappings from a subset of the complete lattice (a so-called *sup-generating family*) into another complete lattice modelling the space of values. In practice, this value lattice will be 'smaller' than the object lattice, and often, it will have some additional structure, e.g. that of a vector space. In many practical cases, the value lattice will be the set of extended reals.

As the only available information about the objects comes from the evaluations, we can use only this information to build morphological operators. If the value lattice is endowed with a group or semigroup operation (e.g. Minkowski addition), we can define a 'similar' operation on the object lattice by exploiting the evaluation family. It turns out that in this way we are able to build a general theory for mathematical morphology which includes most of the known special cases. The prototype example is the case where the object lattice comprises all subsets of $\mathbb{R}^d$, the value lattice is $\overline{\mathbb{R}}$, the extended reals, and the measurements are given by the *support function*. Recently, it has been observed that the support function can be regarded as the binary slope transform [2, 5, 9, 10]. The approach to morphology advocated in this paper leads us immediately towards a definition of the morphological slope transform in an abstract context.

We conclude with a brief description of the further contents of this paper. The next section recalls some elementary concepts in mathematical morphology. Section 3 discusses convolution lattices, which were first introduced by Heijmans and Ronse [6]. However, our nomenclature is new. In Section 4 it is explained how families of evaluations can be used to define adjunctions (dilations and erosions) on the object lattice. Some examples are given in Section 5. If the value lattices possesses additional structure, in particular, if it is a convolution lattice, then it is possible to define Minkowski-type operations on the object lattice; this is the subject of Section 6. In Section 7 we investigate under which conditions the object lattice has the structure of a convolution lattice. Then, in Section 8, we discuss the abstract slope transform. Finally, in Section 9 we explain how our approach based on evaluations can also be used to define random lattice elements; such a definition includes the concept of a random closed set as a special case.

2. Morphology on complete lattices: a reminder

In this section, we recall some basic concepts from the theory of morphology on complete lattices; refer to [6, 15, 18] for details. A comprehensive account can be found in [4].

A set $\mathcal{L}$ with a partial ordering $\leq$ is called a *complete lattice* if every subset $\mathcal{H}$ of $\mathcal{L}$ has an infimum (greatest lower bound) $\bigwedge \mathcal{H}$ and supremum (least upper bound) $\bigvee \mathcal{H}$. The least and greatest element of $\mathcal{L}$ are respectively denoted by O and I. Some examples of complete lattices encountered in this paper are:

- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with the usual ordering of reals.
- $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$.
- $\mathcal{P}(E)$, the power set of a set E ordered by set inclusion.
- $\mathcal{F}(\mathbb{R}^d)$, the closed subsets of $\mathbb{R}^d$, ordered by set inclusion.
- $\mathcal{C}(\mathbb{R}^d)$, the convex subsets of $\mathbb{R}^d$, ordered by set inclusion.
- $\mathrm{Fun}(E, \mathcal{T})$, with $\mathcal{T}$ being a complete lattice, denotes the set of all functions mapping E into $\mathcal{T}$. It is a complete lattice under the partial ordering '$f \leq g$ if $f(x) \leq g(x)$ (in $\mathcal{T}$) for every $x \in E$'.

Let $\mathcal{L}, \mathcal{M}$ be complete lattices. An operator $\psi : \mathcal{L} \to \mathcal{M}$ is said to be *increasing* when $X \leq X'$ in $\mathcal{L}$ implies that $\psi(X) \leq \psi(X')$ in $\mathcal{M}$. An operator $\varepsilon : \mathcal{L} \to \mathcal{M}$ is called *erosion* if

$\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i)$, for every collection $\{X_i \mid i \in I\} \subseteq \mathcal{L}$. An operator $\delta : \mathcal{M} \to \mathcal{L}$ is called *dilation* if $\delta(\bigvee_{i\in I} Y_i) = \bigvee_{i\in I} \delta(Y_i)$, for every collection $\{Y_i \mid i \in I\} \subseteq \mathcal{M}$. Erosions and dilations are increasing operators. The pair (ε, δ), where $\varepsilon : \mathcal{L} \to \mathcal{M}$ and $\delta : \mathcal{M} \to \mathcal{L}$, is said to be an *adjunction* if

$$\delta(Y) \leq X \iff Y \leq \varepsilon(X), \qquad X \in \mathcal{L},\ Y \in \mathcal{M}. \tag{2.1}$$

Denote by $\mathrm{id}_{\mathcal{L}}$ the identity operator on $\mathcal{L}$, that is, $\mathrm{id}_{\mathcal{L}}(X) = X$ for $X \in \mathcal{L}$. The following results can be found, e.g., in [4].

2.1. Proposition. *Let $\mathcal{L}, \mathcal{M}$ be complete lattices. If (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$, then ε is an erosion and δ is a dilation. The following identities hold:*

$$\varepsilon\delta\varepsilon = \varepsilon \quad \text{and} \quad \delta\varepsilon\delta = \delta. \tag{2.2}$$

Furthermore,

$$\delta\varepsilon \leq \mathrm{id}_{\mathcal{L}} \quad \text{and} \quad \varepsilon\delta \leq \mathrm{id}_{\mathcal{M}}. \tag{2.3}$$

2.2. Proposition. *If $\varepsilon : \mathcal{L} \to \mathcal{M}$ is an erosion, then there exists a unique dilation $\delta : \mathcal{M} \to \mathcal{L}$ such that (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$. Dually, if $\delta : \mathcal{M} \to \mathcal{L}$ is a dilation, then there exists a unique erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ such that (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$.*

2.3. Proposition. *If $(\varepsilon_1, \delta_1)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}$ and $(\varepsilon_2, \delta_2)$ is an adjunction between $\mathcal{M}$ and $\mathcal{N}$, then $(\varepsilon_2\varepsilon_1, \delta_1\delta_2)$ is an adjunction between $\mathcal{L}$ and $\mathcal{N}$.*

An operator α on $\mathcal{L}$ is called *opening* if it is increasing, anti-extensive ($\alpha \leq \mathrm{id}$) and idempotent ($\alpha^2 = \alpha$). An operator β is called *closing* if it is increasing, extensive ($\beta \geq \mathrm{id}$) and idempotent.

2.4. Proposition. *If α_i, $i \in I$, are openings, then $\bigvee_{i\in I} \alpha_i$ is an opening as well. Dually, if β_i, $i \in I$, are closings, then $\bigwedge_{i\in I} \beta_i$ is also a closing.*

2.5. Proposition. *If (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$, then $\delta\varepsilon$ is an opening on $\mathcal{L}$ and $\varepsilon\delta$ is a closing on $\mathcal{M}$.*

As an example we discuss the support function as this will play an important role in the sequel.

2.6. Example: Support function.
In the literature [7, 13, 16] the support function for convex sets has been thoroughly investigated. Here we will extend its definition to arbitrary subsets of $\mathbb{R}^d$.

The *support function* $h(X, \cdot)$ of $X \in \mathcal{P}(\mathbb{R}^d)$ is defined as

$$h(X, v) = \bigvee_{x\in X} \langle x, v\rangle, \quad v \in \mathbb{R}^d.$$

Here $\langle x, v\rangle$ is the inner product of x and v. The operator $\sigma : \mathcal{P}(\mathbb{R}^d) \to \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$ given by $\sigma(X) = h(X, \cdot)$ is called the *slope transform for sets* [5]. In [5] it is shown that $\sigma(X)$ is lower semi-continuous (l.s.c.) and sublinear. Define, for $a \in \mathbb{R}^d$ and $r \in \overline{\mathbb{R}}$, the closed halfspace

$$\mathbb{H}^-(a, r) = \{x \in \mathbb{R}^d \mid \langle a, x\rangle \leq r\}.$$

Let the operator $\sigma^{\leftarrow} : \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}}) \to \mathcal{P}(\mathbb{R}^d)$ be given by

$$\sigma^{\leftarrow}(f) = \bigcap_{v\in\mathbb{R}^d} \mathbb{H}^-(v, f(v)).$$

In [5] it is shown that $(\sigma^{\leftarrow}, \sigma)$ constitutes an adjunction between $\mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$ and $\mathcal{P}(\mathbb{R}^d)$, and that the closing $\sigma^{\leftarrow}\sigma$ on $\mathcal{P}(\mathbb{R}^d)$ is given by

$$\sigma^{\leftarrow}\sigma(X) = \overline{\mathrm{co}}(X),$$

the closed convex hull of X.

3. Convolution lattices

The theory developed in this section is largely based upon the work of Heijmans and Ronse [6]; see also [4]. The concept of a convolution lattice, however, is new.

Let $\mathcal{L}$ be a complete lattice and let ℓ be a sup-generating family in $\mathcal{L}$. The latter means that every element in $\mathcal{L}$ can be written as a supremum of elements of ℓ. For an element $X \in \mathcal{L}$ we define

$$\ell(X) = \{x \in \ell \mid x \leq X\}.$$

Then $X = \bigvee \ell(X)$, for every $X \in \mathcal{L}$. Assume that $\oplus$ is a commutative group operation on ℓ. We say that $\oplus$ is *order-preserving* if $x \leq y$ implies that $x \oplus h \leq y \oplus h$, for $x, y, h \in \ell$. We define

$$X \oplus h = \bigvee \{x \oplus h \mid x \in \ell(X)\}, \quad X \in \mathcal{L},\ h \in \ell.$$

This notation is justified by the observation that $x \oplus h = \bigvee\{x' \oplus h \mid x' \in \ell(x)\}$, for $x, h \in \ell$: see [6]. It is evident that, for a given $h \in \ell$, the operator $X \mapsto X \oplus h$ on $\mathcal{L}$ is increasing iff the group operation $\oplus$ is order-preserving. Let o denote the unit element of ℓ with respect to $\oplus$, i.e., $x \oplus o = o \oplus x = x$ for $x \in \ell$. For $x \in \ell$ we define $-x$ as the inverse element of x, i.e., $x \oplus -x = -x \oplus x = o$.

In [6] it was observed that further assumptions have to be made if one wants to get useful results. Towards that goal the so-called "Basic Assumption" was formulated. Here we will present a slightly different, but equivalent formulation of this assumption.

3.1. Basic Assumption. ℓ is a sup-generating family in $\mathcal{L}$ with an order-preserving group operation $\oplus$ such that $(X \oplus h) \oplus -h = X$, for every $X \in \mathcal{L}$ and $h \in \ell$.

In [6] the starting point is a family of automorphisms on $\mathcal{L}$ which leave ℓ invariant and which is (simply) transitive on ℓ. In the present context, these automorphisms are the mappings $X \mapsto X \oplus h$, where h ranges over ℓ. We define

$$X_h = X \oplus h.$$

We also define the (generalized) *Minkowski addition* and *subtraction*

$$X \oplus Y = \bigvee_{y \in \ell(Y)} X_y, \tag{3.1}$$

$$X \ominus Y = \bigwedge_{y \in \ell(Y)} X_{-y}, \tag{3.2}$$

for $X, Y \in \mathcal{L}$. Alternatively, we might call $X \oplus Y$ the *convolution* of X and Y. The following result holds [4, 6].

3.2. Proposition. *Let the Basic Assumption be satisfied. For $x, y \in \ell$ and $X, Y, Z \in \mathcal{L}$:*

$$X \oplus Y = Y \oplus X = \bigvee\{x \oplus y \mid x \in \ell(X), y \in \ell(Y)\};$$
$$X \ominus Y = \bigvee\{h \in \ell \mid Y_h \leq X\};$$
$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z);$$
$$(X \ominus Y) \ominus Z = X \ominus (Y \oplus Z).$$

The choice of ℓ is, to a certain extent, arbitrary. We illustrate this point by means of the following example. Let $\mathcal{L} = \mathcal{F}(\mathbb{R}^d)$, the closed subsets of $\mathbb{R}^d$, and let ℓ consist of all singletons with rational coordinates. If we consider the usual vector addition on ℓ, then the Basic Assumption holds. Alternatively, we can choose ℓ to be the collection of all singletons. It is easy to see that both choices lead to the same Minkowski addition $\overline{\oplus}$ on $\mathcal{F}(\mathbb{R}^d)$, namely $X \overline{\oplus} Y = \overline{X \oplus Y}$, where $\oplus$ is the 'classical' Minkowski addition and $\overline{Z}$ denotes the closure of Z.

To obtain a sup-generating family in $\mathcal{L}$ which is less arbitrary, we define the *completion* of ℓ, denoted by $\overline{\ell}$, as the set of elements of $\mathcal{L}$ which are invertible with respect to $\oplus$. It is evident that $\ell \subseteq \overline{\ell}$. If $\overline{\ell}(X)$ denotes the set $\{x \in \overline{\ell} \mid x \leq X\}$, then $X \oplus h = \bigvee \{x \oplus h \mid x \in \overline{\ell}(X)\}$, as one can easily show. Replacing ℓ by $\overline{\ell}$, the Basic Assumption still holds, and the resulting Minkowski addition and subtraction coincide with the original one. Also Proposition 3.2 remains valid in this case.

3.3. Definition. If $\mathcal{L}$ is a complete lattice with a sup-generating family ℓ, and if $\oplus$ is a commutative group operation on ℓ such that the Basic Assumption holds, and such that every element of $\mathcal{L}$ which is invertible with respect to $\oplus$ lies in ℓ, then we shall call the triple $(\mathcal{L}, \ell, \oplus)$ a *convolution lattice.*

Let $(E, +)$ be a commutative group, let $\mathcal{P}(E)$ be the power set of E, and denote by $\{E\}$ the collection of singletons $\{x\}$, $x \in E$. It is evident that $\{E\}$ is a sup-generating family in $\mathcal{P}(E)$. Defining $\{x\} \oplus \{y\} = \{x + y\}$, we get that $(\mathcal{P}(E), \{E\}, \oplus)$ is a convolution lattice.

3.4. Examples.

(a) $(\{0,1\}, \{1\}, \vee)$, where '$\vee$' is the logical OR, is a convolution lattice.

(b) $(\overline{\mathbb{R}}, \mathbb{R}, +)$, where '+' is the usual addition of real numbers, is a convolution lattice. In this case we write '+' rather than '$\oplus$'. A straightforward computation shows that $-\infty + t = -\infty$ for $t \in \overline{\mathbb{R}}$, and that $+\infty + t = +\infty$ for $t \in \overline{\mathbb{R}}$, $t \neq -\infty$.

(c) $(\mathcal{P}(\mathbb{R}^d), \{\mathbb{R}^d\}, \oplus)$, where '$\oplus$' is the vector addition on $\mathbb{R}^d$, is a convolution lattice.

Also $(\mathcal{C}(\mathbb{R}^d), \{\mathbb{R}^d\}, \oplus)$, where $\mathcal{C}(\mathbb{R}^d)$ is the family of convex subsets of $\mathbb{R}^d$, is a convolution lattice.

Let $\mathcal{F}(\mathbb{R}^d)$ be the closed subsets of $\mathbb{R}^d$, and let $X \overline{\oplus} Y = \overline{X \oplus Y}$ for two elements $X, Y \in \mathcal{F}(\mathbb{R}^d)$, then $(\mathcal{F}(\mathbb{R}^d), \{\mathbb{R}^d\}, \overline{\oplus})$ is a convolution lattice.

(d) Let $w : \mathbb{R} \to \mathbb{R}$ be a bijective mapping, and define $x \dot{+} y = w^{-1}(w(x) + w(y))$, $x, y \in \mathbb{R}$. Here '+' is the usual addition, and w^{-1} is the inverse of w. It is evident that $(\mathbb{R}, \dot{+})$ is a commutative group. Define, for $X, Y \subseteq \mathbb{R}$,

$$X \dot{\oplus} Y = \{x \dot{+} y \mid x \in X,\ y \in Y\},$$

then $(\mathcal{P}(\mathbb{R}), \{\mathbb{R}\}, \dot{\oplus})$ is a convolution lattice. We can take, for example,

$$w(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

We return to this situation in Example 6.7.

(e) Consider the complete lattice of functions $\mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$. For $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$ the pulse function $f_{x,t}$ is defined as the function which equals t at x and $-\infty$ elsewhere. We denote the family of all pulse functions, which is a sup-generating family in $\mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$, by $\mathrm{PF}(\mathbb{R}^d, \overline{\mathbb{R}})$. Defining

$$f_{x,s} \oplus f_{y,t} = f_{x+y,s+t},$$

we arrive at the Minkowski addition given by

$$(F \oplus G)(x) = \bigvee_{y \in \mathbb{R}^d} F(x - y) + G(y)$$

for $F, G \in \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$; see [6]. This operation is sometimes called *supremal convolution* [5]. It follows easily that $(\mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}}), \mathrm{PF}(\mathbb{R}^d, \overline{\mathbb{R}}), \oplus)$ is a convolution lattice.

(f) The pulse functions $\mathrm{PF}(\mathbb{R}^d, \overline{\mathbb{R}})$ also constitute a sup-generating family in the complete lattice of concave u.s.c. (upper semi-continuous) functions, denoted by $\mathrm{Fun}_u(\mathbb{R}^d, \overline{\mathbb{R}})$. In this case, the Minkowski addition is given by

$$(F \oplus G)(x) = \overline{\bigvee_{y \in \mathbb{R}^d} F(x-y) + G(y)};$$

here $\overline{F}$ is the u.s.c. hull of F, that is, the infimum of all u.s.c. functions above F. The triple $(\mathrm{Fun}_u(\mathbb{R}^d, \overline{\mathbb{R}}), \mathrm{PF}(\mathbb{R}^d, \overline{\mathbb{R}}), \oplus)$ is a convolution lattice.

4. From evaluations to adjunctions

Throughout this section we assume that $\mathcal{L}$ and $\mathcal{M}$ are complete lattices.

4.1. Definition. A mapping $u : \ell \to \mathcal{M}$ is called an *evaluation* if it satisfies the condition

$$x \le \bigvee_{i \in I} x_i \;\Rightarrow\; u(x) \le \bigvee_{i \in I} u(x_i), \tag{4.1}$$

for all $x, x_i \in \ell$, $i \in I$.

If $\mathcal{L}$ is an atomic lattice and ℓ is the set of atoms [1, 4], then every mapping $u : \ell \to \mathcal{M}$ is an evaluation. For non-atomic lattices this is not true in general. For example, let $\mathcal{L} = \mathcal{M} = \overline{\mathbb{R}}$ and $\ell = \mathbb{R}$, and define $u(x) = 1$ if x is rational and 0 otherwise. It is easy to see that u is not an evaluation.

Assume that $u : \ell \mapsto \mathcal{M}$ is an evaluation; define the mappings $\delta_u : \mathcal{L} \mapsto \mathcal{M}$ and $\varepsilon_u : \mathcal{M} \mapsto \mathcal{L}$ as follows:

$$\delta_u(X) = \bigvee \{u(x) \mid x \in \ell(X)\}, \tag{4.2}$$

$$\varepsilon_u(Y) = \bigvee \{x \in \ell \mid u(x) \le Y\}. \tag{4.3}$$

Note that $\delta_u(x) = u(x)$ for all $x \in \ell$.

4.2. Proposition. *If $u : \ell \to \mathcal{M}$ is an evaluation, then $(\varepsilon_u, \delta_u)$ is an adjunction between $\mathcal{M}$ and $\mathcal{L}$. Conversely, if (ε, δ) is an adjunction between $\mathcal{M}$ and $\mathcal{L}$, then $u : \ell \mapsto \mathcal{M}$ defined by $u(x) = \delta(x)$ is an evaluation, and $\delta = \delta_u$.*

PROOF. First, we show that $(\varepsilon_u, \delta_u)$ is an adjunction, given that u is an evaluation. We must show that $\delta_u(X) \le Y \iff X \le \varepsilon_u(Y)$ for $X \in \mathcal{L}$ and $Y \in \mathcal{M}$. To prove '$\Rightarrow$', assume that $\bigvee \{u(x) \mid x \in \ell(X)\} \le Y$. Thus $u(x) \le Y$ for $x \in \ell(X)$, which implies that $x \le \varepsilon_u(Y)$. We conclude that $X \le \varepsilon_u(Y)$. The proof of '$\Leftarrow$' is similar.

To finish the proof note that if $x \le \bigvee_{i \in I} x_i$, then

$$\delta(x) \le \delta(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} \delta(x_i),$$

whence it follows that $u(x) = \delta(x)$ is an evaluation. It is obvious that $\delta = \delta_u$. ∎

Let $\mathbb{U}$ be a collection of evaluations. From Proposition 2.5 we know that $\varepsilon_u\delta_u$ is a closing on $\mathcal{L}$ for every $u \in \mathbb{U}$. Now Proposition 2.4 yields that the operator $X \mapsto \langle X \rangle_{\mathbb{U}}$ given by

$$\langle X \rangle_{\mathbb{U}} = \bigwedge_{u \in \mathbb{U}} \varepsilon_u\delta_u(X) \tag{4.4}$$

is a closing on $\mathcal{L}$, too. We refer to it as $\mathbb{U}$-*closing*. We show that

$$\langle X \rangle_{\mathbb{U}} = \bigvee \{x \in \ell \mid u(x) \leq \delta_u(X) \text{ for every } u \in \mathbb{U}\}. \tag{4.5}$$

Denote the right-hand expression by Y. For a fixed $u \in \mathbb{U}$:

$$Y \leq \bigvee \{x \in \ell \mid \ u(x) \leq \delta_u(X)\} = \varepsilon_u\delta_u(X).$$

Thus, $Y \leq \bigwedge_{u \in \mathbb{U}} \varepsilon_u\delta_u(X) = \langle X \rangle_{\mathbb{U}}$. For the converse, let $h \in \ell$ satisfy $h \leq \bigwedge_{u \in \mathbb{U}} \varepsilon_u\delta_u(X)$. Then $h \leq \varepsilon_u\delta_u(X)$ for all $u \in \mathbb{U}$, that is, $\delta_u(h) = u(h) \leq \delta_u(X)$. This yields $h \leq Y$, and (4.5) has been established.

Next, we show that

$$\delta_u(\langle X \rangle_{\mathbb{U}}) = \delta_u(X), \quad X \in \mathcal{L},\ u \in \mathbb{U}. \tag{4.6}$$

As $X \leq \langle X \rangle_{\mathbb{U}}$, it follows that $\delta_u(X) \leq \delta_u(\langle X \rangle_{\mathbb{U}})$. On the other hand,

$$\delta_u(\langle X \rangle_{\mathbb{U}}) = \delta_u(\bigwedge_{v \in \mathbb{U}} \varepsilon_v\delta_v(X)) \leq \delta_u\varepsilon_u\delta_u(X) = \delta_u(X),$$

where we used (2.2).

If the lattice $\mathcal{L}$ contains the objects under study, then the evaluations represent the available information and may be regarded as measurements. Therefore, $\langle X \rangle_{\mathbb{U}}$ represents an element of $\mathcal{L}$ which is retrievable from measurements of X. In the context of random sets, the closing $\langle X \rangle_{\mathbb{U}}$ has been used in [12].

The family of $\mathbb{U}$-*closed elements* in $\mathcal{L}$ is denoted by $\mathcal{L}_{\mathbb{U}}$, i.e.,

$$\mathcal{L}_{\mathbb{U}} = \left\{X \in \mathcal{L} \mid \ \langle X \rangle_{\mathbb{U}} = X\right\}.$$

Being the invariance domain of a closing, the family $\mathcal{L}_{\mathbb{U}}$ is closed under infima [4].

4.3. Proposition. *For every $u \in \mathbb{U}$ and $Y \in \mathcal{M}$ the element $\varepsilon_u(Y)$ is $\mathbb{U}$-closed. Furthermore,*

$$\langle X \rangle_{\mathbb{U}} = \bigwedge \{\varepsilon_u(Y) \mid Y \in \mathcal{M},\ u \in \mathbb{U} \text{ and } X \leq \varepsilon_u(Y)\}. \tag{4.7}$$

PROOF. To prove the first statement observe that $\langle \varepsilon_u(Y) \rangle_{\mathbb{U}} \geq \varepsilon_u(Y)$. On the other hand,

$$\langle \varepsilon_u(Y) \rangle_{\mathbb{U}} = \bigwedge_{v \in \mathbb{U}} \varepsilon_v\delta_v\varepsilon_u(Y) \leq \varepsilon_u\delta_u\varepsilon_u(Y) = \varepsilon_u(Y),$$

by (2.2).

To prove (4.7), put $X' = \bigwedge \{\varepsilon_u(Y) \mid Y \in \mathcal{M},\ u \in \mathbb{U} \text{ and } X \leq \varepsilon_u(Y)\}$. From the previous statement we derive that $X \leq \varepsilon_u(Y)$ implies that $\langle X \rangle_{\mathbb{U}} \leq \varepsilon_u(Y)$. Therefore, it is evident that $\langle X \rangle_{\mathbb{U}} \leq X'$. To prove '$\geq$', observe that $X' \leq \bigwedge \{\varepsilon_u\delta_u(X) \mid \ u \in \mathbb{U}\} = \langle X \rangle_{\mathbb{U}}$. ∎

4.4. Proposition. *Let $\bar{\mathbb{U}}$ consist of all evaluations which are suprema of elements of $\mathbb{U}$. For every $X \in \mathcal{L}$,*

$$\langle X \rangle_{\mathbb{U}} = \langle X \rangle_{\bar{\mathbb{U}}}\,.$$

PROOF. It is easy to see that $\langle X \rangle_{\bar{\mathbb{U}}} \subseteq \langle X \rangle_{\mathbb{U}}$. To prove the converse, define $Y = \langle X \rangle_{\mathbb{U}}$. Let $u \in \bar{\mathbb{U}}$ be given by $u = \bigvee_{i \in I} u_i$, where $u_i \in \mathbb{U}$. Then $Y \le \varepsilon_{u_i}\delta_{u_i}(X)$ for $i \in I$, that is, $\delta_{u_i}(Y) \le \delta_{u_i}(X)$. Now taking the supremum over $i \in I$ at both sides and using that $\bigvee_{i \in I} \delta_{u_i} = \delta_u$, we get $\delta_u(Y) \le \delta_u(X)$. Therefore, $Y \le \varepsilon_u \delta_u(X)$. As this holds for every $u \in \bar{\mathbb{U}}$ we conclude that $Y \le \langle X \rangle_{\bar{\mathbb{U}}}$. ∎

4.5. Definition. The family $\mathbb{U}$ and the corresponding closing are called *unbiased* if $\langle x \rangle_{\mathbb{U}} = x$ for all $x \in \ell$.

4.6. Example. Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$, $\mathcal{M} = [0, \infty]$, and suppose that $\mathbb{U}$ contains only one element, namely $u(x) = \|x\|$. Then $\langle x \rangle_{\mathbb{U}}$ is the smallest closed disk centered at the origin which contains x. This closing is biased.

4.7. Proposition. *$\mathbb{U}$ is unbiased if and only if for any two elements $x, y \in \ell$ with $x \not\le y$ there exists an evaluation $u \in \mathbb{U}$ such that $u(x) \not\le u(y)$.*

PROOF. 'if': suppose that the latter condition holds and that $\langle y \rangle_{\mathbb{U}} \ne y$ for some $y \in \ell$. Then there exists an element $x \in \ell$ with $x \not\le y$ such that $x \le \langle y \rangle_{\mathbb{U}} = \bigwedge_{u \in \mathbb{U}} \varepsilon_u \delta_u(y)$. Thus $u(x) \le u(y)$ for all $u \in \mathbb{U}$, a contradiction.

'only if': suppose that $\mathbb{U}$ is unbiased and let $x, y \in \ell$ with $x \not\le y$. Suppose that $u(x) \le u(y)$ for all $u \in \mathbb{U}$. Then $x \le \langle y \rangle_{\mathbb{U}} = y$, a contradiction. ∎

5. Evaluations and $\mathbb{U}$-closings: some examples

In this section we present a number of examples of evaluation families and the corresponding $\mathbb{U}$-closings.

5.1. Example: Support function. (Continuation of Example 2.6)
Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^d)$ and $\ell = \{\mathbb{R}^d\}$. Often, we shall denote a singleton by x rather than by $\{x\}$, where $x \in \mathbb{R}^d$. Let $\mathcal{M} = \overline{\mathbb{R}}$ and consider for $v \in \mathbb{R}^d$ the evaluation $u_v(x) = \langle x, v \rangle$, where $\langle x, v \rangle$ is the inner product of x and v. The corresponding dilation $\delta_{u_v} : \mathcal{L} \mapsto \mathcal{M}$ is given by

$$\delta_{u_v}(X) = h(X, v) = \bigvee_{x \in X} \langle x, v \rangle.$$

The mapping $v \mapsto \delta_{u_v}(X)$ is known as the *support function* of X; see Example 2.6. The adjoint erosion is given by

$$\varepsilon_{u_v}(c) = \{x \in \mathbb{R}^d \mid \langle x, v \rangle \le c\} = \mathbb{H}^-(v, c), \quad c \in \overline{\mathbb{R}}.$$

Thus, for finite c, the set $\varepsilon_{u_v}(c)$ is a halfspace with outer normal vector v. We denote the class of all linear functions by $\mathbb{U}$, i.e., $\mathbb{U} = \{u_v \mid v \in \mathbb{R}^d\}$. It is easy to show that

$$\langle X \rangle_{\mathbb{U}} = \overline{\text{co}}(X),$$

the closed convex hull of X.

If $\mathbb{U} = \{u_v \mid v = v_1, \ldots, v_n\}$, then $\langle X \rangle_{\mathbb{U}}$ is the smallest convex polyhedron containing X and having faces orthogonal to $v_1, \ldots, v_n$. This closing is unbiased if and only if the only solution of "$\langle x, v_i \rangle \le 0$ for all $i = 1, \ldots, n$" is $x = 0$.

5.2. Example: Kendall's trapping system.
Let E be a topological space; define $\mathcal{L} = \mathcal{P}(E)$, $\ell = \{E\}$, and $\mathcal{M} = \{0,1\}$. For a set $B \subseteq E$ we define the evaluation $u_B : \ell \to \mathcal{M}$ by

$$u_B(x) = \begin{cases} 1, & x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

The dilation $\delta_{u_B} : \mathcal{P}(E) \to \{0,1\}$ and erosion $\varepsilon_{u_B} : \{0,1\} \to \mathcal{P}(E)$ are, respectively, given by

$$\delta_{u_B}(X) = \begin{cases} 1, & X \cap B \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

$$\varepsilon_{u_B}(0) = B^c\ , \quad \varepsilon_{u_B}(1) = E.$$

Here B^c is the complement of B.

Now consider the family of evaluations u_B, where B lies in a given family $\mathcal{T} \subseteq \mathcal{P}(E)$. In other words:

$$\mathbb{U}_{\mathcal{T}} = \{u_B \mid\ B \in \mathcal{T}\}\ .$$

Since

$$\varepsilon_{u_B}\delta_{u_B}(X) = \begin{cases} E, & X \cap B \neq \varnothing, \\ B^c, & \text{otherwise,} \end{cases}$$

the closing $\langle X \rangle_{\mathbb{U}_{\mathcal{T}}}$ is given by

$$\langle X \rangle_{\mathbb{U}_{\mathcal{T}}} = \bigcap_{X \cap B = \varnothing} B^c\ .$$

This concept goes back to Kendall [8], who used the term "trapping system" for $\mathcal{T}$ and called a set X $\mathcal{T}$-closed if $\langle X \rangle_{\mathbb{U}_{\mathcal{T}}} = X$.

Clearly, $\langle X \rangle_{\mathbb{U}_{\mathcal{T}}} = X$ for all X if $\mathcal{T} = \mathcal{P}(E)$. If $\mathcal{T}$ comprises the open subsets of E, then $\langle X \rangle_{\mathbb{U}_{\mathcal{T}}} = \overline{X}$ is the topological closure of X. If $E = \mathbb{R}^d$ and $\mathcal{T}$ is the family of all open half-spaces, then $\langle X \rangle_{\mathbb{U}_{\mathcal{T}}}$ is the closed convex hull of X; see Example 5.1.

Note that Proposition 4.4 implies that the $\mathcal{T}$-closure remains the same if the family $\mathcal{T}$ is enlarged by all possible unions of its elements. For example, we can choose for $\mathcal{T}$ the base of the topology instead of the family of all open sets.

5.3. Example: Translation invariant morphology.
Let $(E, +)$ be a commutative group. Define $\mathcal{L} = \mathcal{M} = \mathcal{P}(E)$, and let ℓ comprise the singletons of E. Given $X \subseteq E,\ h \in E$, we define the translate X_h by

$$X_h = \{x + h \mid x \in X\}.$$

For $B \subseteq E$ we define the evaluation $u_B : \ell \to \mathcal{M}$ by $u_B(x) = B_x$. Then

$$\delta_{u_B}(X) = X \oplus B = \bigcup_{x \in X} B_x = \bigcup_{b \in B} X_b.$$

The adjoint erosion is given by

$$\varepsilon_{u_B}(X) = X \ominus B = \bigcap_{b \in B} X_{-b}.$$

The pair $(\varepsilon_{u_B}, \delta_{u_B})$ is a translation invariant adjunction, well-known from mathematical morphology: see [4, Ch.4] and [11, 17]. If $\mathbb{U} = \{u_B\}$, then $\langle X \rangle_{\mathbb{U}} = X \bullet B$ is the classical morphological closing of X by B.

If $E = \mathbb{R}^d$ and B is bounded, then $\mathbb{U} = \{u_B\}$ is unbiased. However, if B is a closed halfspace, then $\langle X \rangle_{\mathbb{U}} = X \oplus B$. For singletons this yields $\langle \{x\} \rangle_{\mathbb{U}} = B_x$, hence the family $\mathbb{U} = \{u_B\}$ is biased in this case.

5.4. Example: Slope transform for functions.
Consider the complete lattice $\mathcal{L} = \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$ with sup-generating family $\ell = \mathrm{PF}(\mathbb{R}^d, \overline{\mathbb{R}})$; see also Example 3.4(e). Furthermore, let $\mathcal{M} = \overline{\mathbb{R}}$. For $v \in \mathbb{R}^d$ we define the mapping $u_v : \ell \to \mathcal{M}$ by

$$u_v(f_{x,t}) = t - \langle x, v \rangle.$$

It is obvious that u_v defines an evaluation. The corresponding adjunction is given by

$$\delta_{u_v}(F) = \bigvee_{x \in \mathbb{R}^d} F(x) - \langle x, v \rangle, \quad F \in \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}}),$$
$$\big(\varepsilon_{u_v}(c)\big)(x) = \langle x, v \rangle + c, \quad c \in \overline{\mathbb{R}}.$$

Observe that the graph of the function $\varepsilon_{u_v}(c)$ is a hyperplane. It follows that $\varepsilon_{u_v}\delta_{u_v}(F)$ is the smallest affine function $x \mapsto \langle x, v \rangle + c$ which lies above F. This yields that

$$\langle F \rangle_{\mathbb{U}} = \overline{\mathrm{cc}}(F),$$

the u.s.c. concave upper envelope of F; see [5]. We will return to this case in Example 8.4

6. From evaluations to convolutions

Assume that $\mathcal{L}$ is a complete lattice with a sup-generating family ℓ, that $(\mathcal{M}, m, \oplus)$ is a convolution lattice, and that $\mathbb{U}$ is a collection of evaluations mapping ℓ into $\mathcal{M}$. We can define a binary operation on $\mathcal{L}$ in the following way:

$$X_1 \dot{\boxplus} X_2 = \bigwedge_{u \in \mathbb{U}} \varepsilon_u\big(\delta_u(X_1) \oplus \delta_u(X_2)\big), \quad X_1, X_2 \in \mathcal{L}. \tag{6.1}$$

A number of properties of '$\oplus$' carry over to '$\dot{\boxplus}$', e.g., the operation $(X_1, X_2) \mapsto X_1 \dot{\boxplus} X_2$ is

- increasing with respect to both arguments;
- commutative.

But for other properties, such as associativity, this is not true in general, as shown by the following example.

6.1. Example. Let $\mathcal{L} = \mathcal{M} = \overline{\mathbb{R}}$ and $\ell = m = \mathbb{R}$. Define $u : \mathbb{R} \to \mathbb{R}$ by

$$u(x) = \begin{cases} x - 1, & x \le 0, \\ x + 1, & x > 0; \end{cases}$$

see Figure 1.

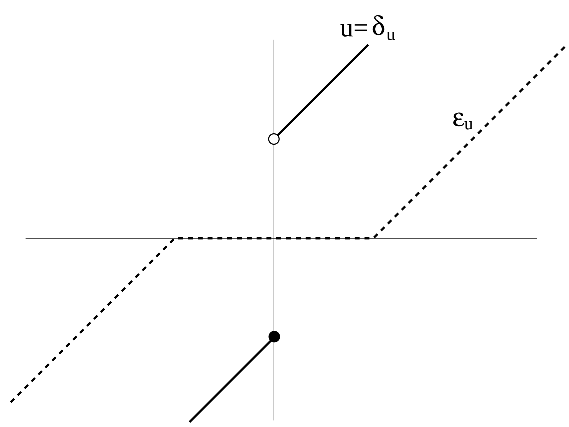


Fig. 1. An evaluation for which the resulting operation $\dot{\oplus}$ is not associative.

It is easy to see that u is an evaluation. The corresponding dilation δ_u and erosion ε_u are given by

$$\delta_u(x)=\begin{cases}-\infty, & x=-\infty,\\ x-1, & x\le 0,\\ x+1, & x>0,\\ +\infty, & x=+\infty,\end{cases}\qquad \varepsilon_u(x)=\begin{cases}-\infty, & x=-\infty,\\ x+1, & x\le -1,\\ 0, & -1\le x\le 1,\\ x-1, & x\ge 1,\\ +\infty, & x=+\infty.\end{cases}$$

A straightforward computation shows that

$$\begin{aligned}&-1\,\dot{\boxplus}\,(1\,\dot{\boxplus}\,1)=-1\,\dot{\boxplus}\,3=1,\\ &(-1\,\dot{\boxplus}\,1)\,\dot{\boxplus}\,1=0\,\dot{\boxplus}\,1=0.\end{aligned}$$

Therefore, $\dot{\boxplus}$ is not associative.

There is an alternative way to define a binary operation on $\mathcal{L}$:

$$X_1\,\dot{\oplus}\,X_2=\bigvee\{x_1\,\dot{\boxplus}\,x_2\mid x_1\in\ell(X_1),\ x_2\in\ell(X_2)\},\quad X_1,X_2\in\mathcal{L}. \tag{6.2}$$

Then

$$x_1\,\dot{\oplus}\,x_2=x_1\,\dot{\boxplus}\,x_2,\quad x_1,x_2\in\ell.$$

Since $x_1\,\dot{\oplus}\,x_2\le X_1\,\dot{\boxplus}\,X_2$ for $x_1\in\ell(X_1),x_2\in\ell(X_2)$, we find that

$$X_1\,\dot{\oplus}\,X_2\le X_1\,\dot{\boxplus}\,X_2,\quad X_1,X_2\in\mathcal{L}. \tag{6.3}$$

6.2. Definition. The family $\mathbb{U}$ is said to be *linear* if

$$\delta_u(x_1\,\dot{\boxplus}\,x_2)=u(x_1)\oplus u(x_2),\qquad x_1,x_2\in\ell.$$

6.3. Proposition. *If $\mathbb{U}$ is linear, then*

$$\delta_u(X_1\,\dot{\boxplus}\,X_2)=\delta_u(X_1\,\dot{\oplus}\,X_2)=\delta_u(X_1)\oplus\delta_u(X_2), \tag{6.4}$$

$$\langle X_1\,\dot{\oplus}\,X_2\rangle_{\mathbb{U}}=X_1\,\dot{\boxplus}\,X_2, \tag{6.5}$$

for $X_1,X_2\in\mathcal{L}$.

PROOF. By the fact that $\mathbb{U}$ is linear, we get that

$$\begin{aligned}\delta_u(X_1\,\dot{\oplus}\,X_2)&=\bigvee\{\delta_u(x_1\,\dot{\boxplus}\,x_2)\mid x_1\in\ell(X_1),x_2\in\ell(X_2)\}\\&=\bigvee\{\delta_u(x_1)\oplus\delta_u(x_2)\mid x_1\in\ell(X_1),x_2\in\ell(X_2)\}\\&=\delta_u(X_1)\oplus\delta_u(X_2).\end{aligned}$$

Therefore,

$$\langle X_1\,\dot{\oplus}\,X_2\rangle_{\mathbb{U}}=\bigwedge_{u\in\mathbb{U}}\varepsilon_u(\delta_u(X_1)\oplus\delta_u(X_2))=X_1\,\dot{\boxplus}\,X_2.$$

It remains to prove that $\delta_u(X_1\,\dot{\boxplus}\,X_2)=\delta_u(X_1)\oplus\delta_u(X_2)$. Firstly,

$$\begin{aligned}\delta_u(X_1\,\dot{\boxplus}\,X_2)&=\delta_u\Big(\bigwedge_{v\in\mathbb{U}}\varepsilon_v(\delta_v(X_1)\oplus\delta_v(X_2))\Big)\\&\le\delta_u\varepsilon_u(\delta_u(X_1)\oplus\delta_u(X_2))\\&\le\delta_u(X_1)\oplus\delta_u(X_2).\end{aligned}$$

On the other hand,

$$\begin{aligned}\delta_u(X_1\,\dot{\boxplus}\,X_2)&=\delta_u(\langle X_1\,\dot{\oplus}\,X_2\rangle_{\mathbb{U}})\ge\delta_u(X_1\,\dot{\oplus}\,X_2)\\&=\delta_u(X_1)\oplus\delta_u(X_2).\end{aligned}$$

This concludes the proof. ∎

6.4. Proposition. *If $\mathbb{U}$ is linear, then the operation $\dot{\boxplus}$ is associative.*

PROOF. Suppose that $\mathbb{U}$ is linear. Then

$$\begin{aligned}(X_1 \dot{\boxplus} X_2) \dot{\boxplus} X_3 &= \bigwedge_{u \in \mathbb{U}} \varepsilon_u(\delta_u(X_1 \dot{\boxplus} X_2) \oplus \delta_u(X_3)) \\ &= \bigwedge_{u \in \mathbb{U}} \varepsilon_u(\delta_u(X_1) \oplus \delta_u(X_2) \oplus \delta_u(X_3)) \\ &= X_1 \dot{\boxplus} (X_2 \dot{\boxplus} X_3).\end{aligned}$$

This proves the result. ∎

Observe that the family $\{u\}$ in Example 6.1 is not linear since $\delta_u(-1 \dot{\boxplus} 1) = \delta_u(0) = -1$ and $u(-1) + u(1) = 0$.

6.5. Question. What can be said about associativity of $\dot{\oplus}$ under the assumption that $\mathbb{U}$ is linear? In all examples that we have considered and for which $\mathbb{U}$ is linear, the operation $\dot{\oplus}$ is associative. As a matter of fact, the question appears nontrivial even in the case where $\mathbb{U}$ contains only one evaluation.

We discuss a number of examples. All of them deal with linear evaluation families.

6.6. Example. (Continuation of Example 5.1)
Let $\mathcal{M} = \overline{\mathbb{R}}$ be endowed with the usual addition operation. Then $\{x_1\} \dot{\boxplus} \{x_2\} = \{x_1 + x_2\}$ for two singletons in $\mathcal{L} = \mathcal{P}(\mathbb{R}^d)$. It follows that $X_1 \dot{\oplus} X_2$ is the usual Minkowski sum of two sets $X_1, X_2 \subseteq \mathbb{R}^d$, and that $X_1 \dot{\boxplus} X_2$ is the closed convex hull of the Minkowski sum of X_1 and X_2, that is

$$X_1 \dot{\boxplus} X_2 = \overline{\text{co}}(X_1 \oplus X_2).$$

It is evident that $\mathbb{U}$ is linear.

6.7. Example. (Continuation of Example 3.4(d))
Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$ and $\ell = \{\mathbb{R}^2\}$, and let $\mathcal{M}$ be the convolution lattice $(\overline{\mathbb{R}}, \mathbb{R}, +)$. Assume that $w : \mathbb{R} \to \mathbb{R}$ is a bijective mapping. Consider the evaluations $\mathbb{U} = \{u_{a,b} \mid a, b \in \mathbb{R}\}$ from $\{\mathbb{R}^2\}$ to $\mathbb{R}$ given by

$$u_{a,b}(x, y) = aw(x) + bw(y).$$

Writing $\varepsilon_{a,b}$ for $\varepsilon_{u_{a,b}}$, we have

$$\varepsilon_{a,b}(c) = \{(x, y) \in \mathbb{R}^2 \mid aw(x) + bw(y) \le c\},$$

for $c \in \overline{\mathbb{R}}$. An easy computation shows that

$$\begin{aligned}\{(x_1, y_1)\} \dot{\boxplus} \{(x_2, y_2)\} &= \bigcap_{a,b \in \mathbb{R}} \{(x, y) \mid aw(x) + bw(y) \le aw(x_1) + aw(x_2) + bw(y_1) + bw(y_2)\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid w(x) = w(x_1) + w(x_2) \text{ and } w(y) = w(y_1) + w(y_2)\} \\ &= \{(x_1 \dot{+} x_2, y_1 \dot{+} y_2)\},\end{aligned}$$

where $x_1 \dot{+} x_2 = w^{-1}(w(x_1) + w(x_2))$; cf. Example 3.4(d). It follows immediately that the family $\mathbb{U}$ is linear (without any further restrictions on w).

Using Proposition 4.3 we derive the following expression for the $\mathbb{U}$-closing:

$$\langle X \rangle_{\mathbb{U}} = \bigwedge \{\varepsilon_{a,b}(c) \mid X \subseteq \varepsilon_{a,b}(c),\ a, b \in \mathbb{R},\ c \in \overline{\mathbb{R}}\}.$$

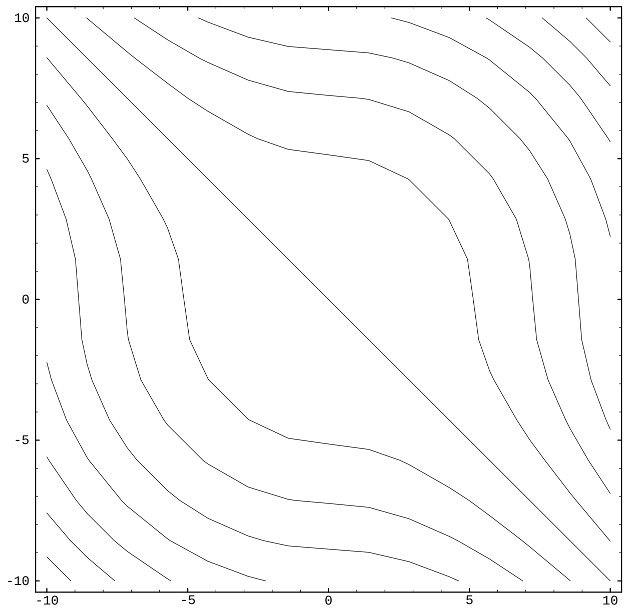


Fig. 2. Generalized halfspaces are determined by an inequality of the form $\text{sign}(x)\cdot x^2+\text{sign}(y)\cdot y^2 \le c$.

We can interpret $\varepsilon_{a,b}(c)$ as a 'generalized halfspace' with 'normal vector' $(a,b)^T$. If $w(x)=x$, then $\varepsilon_{a,b}(c)$ is the affine halfspace bounded by the straight line $ax+by=c$. The halfspaces $\varepsilon_{a,b}(c)$ for $a=b=1$ and $w(x)=\text{sign}(x)\cdot x^2$ are depicted in Figure 2.

6.8. Example. (Continuation of Example 5.2)
Consider Example 5.2 with $\mathcal{T}$ equal to the family of open subsets of E. In Example 3.4(a) we have seen that $(\{0,1\},\{1\},\oplus)$, where '$\oplus$' represents '$\vee$', the logical OR, is a convolution lattice. Let, for $B\in\mathcal{T}$, $u_B,\delta_{u_B},\varepsilon_{u_B}$ be as in Example 5.2.

Let $X_1,X_2\subseteq E$ and $B\in\mathcal{T}$; then $\delta_{u_B}(X_1)\oplus\delta_{u_B}(X_2)=0$ iff $X_1\cap B=X_2\cap B=\varnothing$, which holds iff $X_1\cup X_2\subseteq B^c$. Thus we get that

$$\begin{aligned} X_1 \dot{\boxplus} X_2 &= \bigcap\{B^c \mid B \text{ open and } X_1\cup X_2\subseteq B^c\} \\ &= \bigcap\{C \mid C \text{ closed and } X_1\cup X_2\subseteq C\} \\ &= \overline{X_1\cup X_2} = \overline{X_1}\cup\overline{X_2}. \end{aligned}$$

Furthermore, $X_1 \dot{\oplus} X_2 = X_1\cup X_2$, i.e., inclusion (6.3) is strict.

It is evident that the evaluation family $\mathbb{U}$ is linear and that both operations $\dot{\oplus}$ and $\dot{\boxplus}$ are associative.

6.9. Example. (Continuation of Example 5.3)
Let $\mathcal{L}=\mathcal{P}(E)$ and $\ell=\{E\}$, and let $\mathcal{M}$ be the convolution lattice $(\mathcal{P}(E),\{E\},\oplus)$. Let $\mathbb{U}=\{u_B\}$,

where $u_B(x) = B_x$, and $B \subseteq E$ is given. Now

$$\begin{aligned} X_1 \dot{\boxplus} X_2 &= \varepsilon_{u_B}(\delta_{u_B}(X_1) \oplus \delta_{u_B}(X_2)) \\ &= (X_1 \oplus B \oplus X_2 \oplus B) \ominus B = (X_1 \oplus X_2 \oplus B) \bullet B. \end{aligned}$$

Thus $\{x_1\} \dot{\boxplus} \{x_2\} = B'_{x_1+x_2}$, where $B' = B \bullet B$. This implies that $X_1 \dot{\oplus} X_2 = X_1 \oplus X_2 \oplus B'$. It is easy to verify that $\mathbb{U}$ is linear. Observe also that the operation $\dot{\oplus}$ is associative.

6.10. Example. (Continuation of Example 5.4)
The triple $(\overline{\mathbb{R}}, \mathbb{R}, +)$, where $+$ is the extended addition, is a convolution lattice; see Example 3.4(b). Let f_{x_1,t_1}, f_{x_2,t_2} be two pulse functions, then

$$\begin{aligned} f_{x_1,t_1} \dot{\boxplus} f_{x_2,t_2} &= \bigwedge_{v \in \mathbb{R}^d} \varepsilon_{u_v}\big(\delta_{u_v}(f_{x_1,t_1}) + \delta_{u_v}(f_{x_2,t_2})\big) \\ &= \bigwedge_{v \in \mathbb{R}^d} \varepsilon_{u_v}(t_1 + t_2 - \langle x_1 + x_2, v\rangle) \\ &= \bigwedge_{v \in \mathbb{R}^d} (x \mapsto \langle x - x_1 - x_2, v\rangle + t_1 + t_2). \end{aligned}$$

Here $x \mapsto f(x)$ denotes the function which maps x onto $f(x)$. In other words, $f_{x_1,t_1} \dot{\boxplus} f_{x_2,t_2}$ is the infimum of all hyperplanes through $(x_1 + x_2, t_1 + t_2)$ that is

$$f_{x_1,t_1} \dot{\boxplus} f_{x_2,t_2} = f_{x_1+x_2,t_1+t_2}.$$

In Example 3.4(e) we have seen that the resulting operation $\dot{\oplus}$ on $\mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$ is the supremal convolution

$$(F \dot{\oplus} G)(x) = \bigvee_{y \in \mathbb{R}^d} F(x - y) + G(y).$$

Using (6.5) we get that

$$F \dot{\boxplus} G = \langle F \dot{\oplus} G \rangle_{\mathbb{U}} = \overline{\mathrm{cc}}(F \dot{\oplus} G).$$

$\overline{\mathrm{cc}}(F)$ is the u.s.c. concave upper envelope of F; cf. Example 5.4.

6.11. Example. Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$, $\ell = \{\mathbb{R}^2\}$, and take for $\mathcal{M}$ the convolution lattice $(\overline{\mathbb{R}}, \mathbb{R}, +)$. Let $\mathbb{U} = \{u\}$ with $u(x) = \|x\|$, the norm of x. Then

$$\begin{aligned} \delta_u(X) &= r(X) = \sup_{x \in X} \|x\| \\ \varepsilon_u(r) &= \begin{cases} rB, & r \geq 0, \\ \varnothing, & r < 0, \end{cases} \end{aligned}$$

where B is the closed unit ball in $\mathbb{R}^2$. Thus $\langle X \rangle_{\mathbb{U}} = r(X)B$, which means in particular that $\mathbb{U}$ is biased. (See also Example 4.6.) It is easy to show that

$$X_1 \dot{\boxplus} X_2 = X_1 \dot{\oplus} X_2 = (r(X_1) + r(X_2))B.$$

The family $\mathbb{U}$ is linear, and the operation $\dot{\oplus}$ is associative.

7. When is $\mathcal{L}$ a convolution lattice?

An interesting question is the following: when is $(\mathcal{L}, \ell, \dot{\oplus})$ a convolution lattice, given that $(\mathcal{M}, m, \oplus)$ is one? Note that in general ℓ may be too big, e.g., $\ell = \mathcal{L}$. Consider the following conditions:

(I) There exists an element $\dot{o} \in \ell$ such that $u(\dot{o}) = o$, for all $u \in \mathbb{U}$; here o is the zero group element of m.

(II) For every $x \in \ell$ there exists an element $\dot{-}x \in \ell$ such that $u(x) \oplus u(\dot{-}x) = o$, for $u \in \mathbb{U}$.

Condition (I) does not hold in Example 6.9 and condition (II) does not hold in Example 6.11.

7.1. Lemma. *Assume that $(\mathcal{M}, m, \oplus)$ is a convolution lattice, that $\mathbb{U}$ is linear and unbiased, and that conditions* (I)–(II) *hold. Then*

(a) *The element $\dot{o}$ is unique and equals $\bigwedge_{u \in \mathbb{U}} \varepsilon_u(o)$.*

(b) *$u(x) \in m$ if $x \in \ell$ and $u \in \mathbb{U}$.*

(c) *For every $x \in \ell$, the element $\dot{-}x$ given in* (II) *is unique, namely $\dot{-}x = \bigwedge_{u \in \mathbb{U}} \varepsilon_u(-u(x))$, and it is the inverse of x in $\mathcal{L}$ in the sense that $x \dot{\oplus} \dot{-}x = \dot{o}$.*

Proof. (a): If $u(\dot{o}) = o$, then $\varepsilon_u \delta_u(\dot{o}) = \varepsilon_u(o)$. Therefore, $\dot{o} = \langle \dot{o} \rangle_{\mathbb{U}} = \bigwedge_{u \in \mathbb{U}} \varepsilon_u \delta_u(\dot{o}) = \bigwedge_{u \in \mathbb{U}} \varepsilon_u(o)$.

(b): Condition (II) implies that $u(x)$ is invertible in $\mathcal{M}$, for $x \in \ell$, $u \in \mathbb{U}$. As m is complete, this means in particular that $u(x) \in m$.

(c): Assume that $u(x) \oplus u(\dot{-}x) = o$ for $u \in \mathbb{U}$. Then $u(\dot{-}x) = -u(x)$, hence

$$\dot{-}x = \langle \dot{-}x \rangle_{\mathbb{U}} = \bigwedge_{u \in \mathbb{U}} \varepsilon_u \delta_u(\dot{-}x) = \bigwedge_{u \in \mathbb{U}} \varepsilon_u(-u(x)).$$

Since $u(x) \oplus u(\dot{-}x) = u(x \dot{\oplus} \dot{-}x) = o$, we get that $x \dot{\oplus} \dot{-}x = \dot{o}$. ∎

7.2. Proposition. *Assume that the family $\mathbb{U}$ is linear and unbiased and that conditions* (I)–(II) *hold, then the triple $(\mathcal{L}, \ell, \dot{\oplus})$ satisfies the Basic Assumption.*

Proof. We must show that $(X \dot{\oplus} h) \dot{\oplus} \dot{-}h = X$ for $X \in \mathcal{L}$ and $h \in \ell$; here $X \dot{\oplus} h = \bigvee_{x \in \ell(X)} x \dot{\oplus} h$. We show that the pair $X \mapsto X \dot{\oplus} h$, $X \mapsto X \dot{\oplus} \dot{-}h$ forms an adjunction on $\mathcal{L}$, i.e.,

$$X \dot{\oplus} h \le Y \iff X \le Y \dot{\oplus} \dot{-}h.$$

Note that $X \dot{\oplus} h \le Y$ means that $x \dot{\oplus} h \le Y$ for $x \in \ell(X)$, hence $(x \dot{\oplus} h) \dot{\oplus} \dot{-}h \le Y \dot{\oplus} \dot{-}h$. Using that $\dot{\oplus}$ is associative on ℓ (see Proposition 6.4), this yields $x \le Y \dot{\oplus} \dot{-}h$ for $x \in \ell(X)$. Therefore, $X \le Y \dot{\oplus} \dot{-}h$. The reverse implication is proved similarly. Now, $X \mapsto (X \dot{\oplus} h) \dot{\oplus} \dot{-}h$ is a closing, whence it follows that $X \le (X \dot{\oplus} h) \dot{\oplus} \dot{-}h$. On the other hand, $X \mapsto (X \dot{\oplus} \dot{-}h) \dot{\oplus} h$ is an opening, hence $(X \dot{\oplus} \dot{-}h) \dot{\oplus} h \le X$. Substituting $\dot{-}h$ for h, we get $(X \dot{\oplus} h) \dot{\oplus} \dot{-}h \le X$, whence equality follows. ∎

Now we follow the procedure described in Section 3. Let $\bar{\ell}$ be the invertible elements of $\mathcal{L}$ with respect to $\dot{\oplus}$, then $(\mathcal{L}, \bar{\ell}, \dot{\oplus})$ is a convolution lattice. An evaluation $u : \ell \to \mathcal{M}$ can be 'completed' by putting $\overline{u}(\overline{x}) = \delta_u(\overline{x})$, for $\overline{x} \in \bar{\ell}$. Then $\overline{u}$ is an evaluation, too. With this modification, the family $\mathbb{U}$ is a convolution system in the sense of the following definition.

7.3. Definition. Let $(\mathcal{L}, \ell, \dot{\oplus})$ and $(\mathcal{M}, m, \oplus)$ be two convolution lattices. If $\mathbb{U}$ is an unbiased family of evaluations from ℓ into $\mathcal{M}$ which satisfies

$$u(\dot{o}) = o$$
$$u(x \dot{\oplus} y) = u(x) \oplus u(y), \quad x, y \in \ell,$$

for all $u \in \mathbb{U}$, then we say that $\mathbb{U}$ is a *convolution system* between $(\mathcal{L}, \ell, \dot{\oplus})$ and $(\mathcal{M}, m, \oplus)$.

We point out that the condition $u(\dot{o}) = o$ in this definition cannot be omitted because of Example 6.9.

If $\mathbb{U}$ is a convolution system, then

$$\langle x \dot\oplus y\rangle_{\mathbb{U}} = \bigwedge_{u\in\mathbb{U}} \varepsilon_u(u(x \dot\oplus y)) = \bigwedge_{u\in\mathbb{U}} \varepsilon_u(u(x) \oplus u(y)),$$

and since $\mathbb{U}$ is unbiased, this gives

$$x \dot\oplus y = \bigwedge_{u\in\mathbb{U}} \varepsilon_u(u(x) \oplus u(y)).$$

7.4. Examples.

(a) $\mathbb{U} = \{u_v \mid v \in I\!R^d\}$ with $u_v(x) = \langle x, v\rangle$ is a convolution system between $(\mathcal{P}(I\!R^d), \{I\!R^d\}, \oplus)$ and $(\overline{I\!R}, I\!R, +)$; cf. Examples 2.6, 5.1, 6.6.

(b) $\mathbb{U} = \{u_v \mid v \in I\!R^d\}$ with $u_v(f_{x,t}) = t - \langle x, v\rangle$ is a convolution system between $(\mathrm{Fun}(I\!R^d, \overline{I\!R}), \mathrm{PF}(I\!R^d, \overline{I\!R}), \dot\oplus)$ and $(\overline{I\!R}, I\!R, +)$, with '$\dot\oplus$' being supremal convolution; cf. Examples 3.4(e), 5.4 and 6.10.

7.5. Proposition. *Let $\mathbb{U}$ be a convolution system between $(\mathcal{L}, \ell, \dot\oplus)$ and $(\mathcal{M}, m, \oplus)$. The following identities hold:*

$$\delta_u(X \dot\oplus h) = \delta_u(X) \oplus u(h), \tag{7.1}$$

$$\varepsilon_u(Y \oplus u(h)) = \varepsilon_u(Y) \dot\oplus h, \tag{7.2}$$

$$\langle X \dot\oplus h\rangle_{\mathbb{U}} = \langle X\rangle_{\mathbb{U}} \dot\oplus h, \tag{7.3}$$

for $X \in \mathcal{L}$, $Y \in \mathcal{M}$, $h \in \ell$, $u \in \mathbb{U}$.

From Proposition 6.3 we know that $\delta_u(X \dot\oplus A) = \delta_u(X) \oplus \delta_u(A)$, for $X, A \in \mathcal{L}$. Taking adjoints on both sides (see Proposition 2.3), we find that

$$\varepsilon_u(Y \ominus \delta_u(A)) = \varepsilon_u(Y) \dot\ominus A. \tag{7.4}$$

Here $\dot\ominus$ is given by (3.2), i.e.

$$X \dot\ominus A = \bigwedge_{a\in\ell(A)} X \dot\oplus \dot{-}a,$$

where $\dot{-}a$ is the inverse of a in ℓ with respect to $\dot\oplus$, that is, $a \dot\oplus \dot{-}a = \dot{o}$.

In Example 5.1, the $\mathbb{U}$-closing is the closed convex hull operation. It is well-known [13, 16] that the Minkowski sum of two compact convex sets is compact and convex. The following example shows that this is not true for general $\mathbb{U}$-closed sets.

7.6. Example. This example shows that X_1, X_2 $\mathbb{U}$-closed does <u>not</u> imply that $X_1 \dot\oplus X_2$ is $\mathbb{U}$-closed.

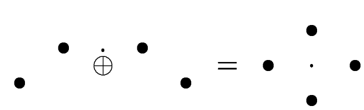
Let $\mathcal{L} = \mathcal{P}(\mathbb{Z}^2)$, $\ell = \{\mathbb{Z}^2\}$, consider also the convolution lattice $(\overline{\mathbb{Z}}, \mathbb{Z}, +)$, and let $\mathbb{U} = \{u_1, u_2, u_3, u_4\}$, where

$$u_1(x,y) = x + y,\ u_2(x,y) = -x + y,\ u_3(x,y) = -x - y,\ u_4(x,y) = x - y.$$

Note that singletons in $\{\mathbb{Z}^2\}$ are denoted by (x, y) rather than by $\{(x, y)\}$. It is not difficult to prove that $\mathbb{U}$ is linear, unbiased, and satisfies (I)-(II). However

$$\{(-1,-1),(0,0)\} \dot\oplus \{(1,0),(0,1)\} = \{(-1,0),(1,0),(0,-1),(0,1)\},$$

or graphically:



The two sets at the left hand-side are $\mathbb{U}$-closed, but the one at the right hand-side is not.

8. Slope transform

The *slope transform* is defined as the operator $\Sigma : \mathcal{L} \mapsto \mathrm{Fun}(\mathbb{U}, \mathcal{M})$ given by

$$\Sigma(X)(u) = \delta_u(X), \quad X \in \mathcal{L},\ u \in \mathbb{U}. \tag{8.1}$$

For two evaluations $u, v \in \mathbb{U}$ we define $(u \oplus v)(x) = u(x) \oplus v(x)$, for $x \in \ell$. It is easy to show that

$$\Sigma(X)(u \oplus v) \leq \Sigma(X)(u) \oplus \Sigma(X)(v).$$

For $X_1, X_2 \in \mathcal{L}$ we define $\Big(\Sigma(X_1) \oplus \Sigma(X_2)\Big)(u) = \Sigma(X_1)(u) \oplus \Sigma(X_2)(u)$, for $u \in \mathbb{U}$.

It is easy to see that

$$\Sigma(X_1 \dot{\boxplus} X_2) \leq \Sigma(X_1) \oplus \Sigma(X_2), \qquad X_1, X_2 \in \mathcal{L}. \tag{8.2}$$

Namely,

$$\begin{aligned} \Sigma(X_1 \dot{\boxplus} X_2)(u) &= \delta_u(X_1 \dot{\boxplus} X_2) \\ &= \delta_u\Big(\bigwedge_{v \in \mathbb{U}} \varepsilon_v(\delta_v(X_1) \oplus \delta_v(X_2))\Big) \\ &\leq \delta_u\left(\varepsilon_u(\delta_u(X_1) \oplus \delta_u(X_2))\right) \\ &\leq \delta_u(X_1) \oplus \delta_u(X_2). \end{aligned}$$

Here we have used that $\delta_u \varepsilon_u \leq \mathsf{id}_{\mathcal{L}}$.

For a function $S : \mathbb{U} \mapsto \mathcal{M}$ we define

$$\Sigma^{\leftarrow}(S) = \bigwedge_{u \in \mathbb{U}} \varepsilon_u(S(u)), \tag{8.3}$$

which is an element of $\mathcal{L}$. Therefore $\Sigma^{\leftarrow}$ is an operator which maps $\mathrm{Fun}(\mathbb{U}, \mathcal{M})$ into $\mathcal{L}$.

8.1. Proposition. *The pair* $(\Sigma^{\leftarrow}, \Sigma)$ *defines an adjunction between* $\mathrm{Fun}(\mathbb{U}, \mathcal{M})$ *and* $\mathcal{L}$.

Proof. We show that $\Sigma(X) \leq S \iff X \leq \Sigma^{\leftarrow}(S)$, for $X \in \mathcal{L}$ and $S \in \mathrm{Fun}(\mathbb{U}, \mathcal{M})$.

'$\Rightarrow$': $\Sigma(X) \leq S$ means that $\delta_u(X) \leq S(u)$, hence $X \leq \varepsilon_u(S(u))$, for every $u \in \mathbb{U}$. This implies that $X \leq \bigwedge_{u \in \mathbb{U}} \varepsilon_u(S(u)) = \Sigma^{\leftarrow}(S)$.

'$\Leftarrow$': if $X \leq \Sigma^{\leftarrow}(S) = \bigwedge_{u \in \mathbb{U}} \varepsilon_u(S(u))$, then $X \leq \varepsilon_u(S(u))$, hence $\delta_u(X) \leq S(u)$ for $u \in \mathbb{U}$. This means that $\Sigma(X) \leq S$. ∎

We call $\Sigma^{\leftarrow}$ the *inverse slope transform*. Note however that $\Sigma^{\leftarrow}$ is not an inverse in the usual sense of the word. The composition $\Sigma^{\leftarrow}\Sigma$ is a closing on $\mathcal{L}$. It follows from (4.4) that

$$\Sigma^{\leftarrow}\Sigma(X) = \langle X \rangle_{\mathbb{U}}, \quad X \in \mathcal{L}. \tag{8.4}$$

8.2. Theorem. *If* $\mathbb{U}$ *is linear then*

$$\Sigma(X_1 \dot{\boxplus} X_2) = \Sigma(X_1 \dot{\oplus} X_2) = \Sigma(X_1) \oplus \Sigma(X_2), \qquad X_1, X_2 \in \mathcal{L}. \tag{8.5}$$

Proof. Note that the second equality in (8.5) is a reformulation of the first statement in Proposition 6.3. By using (6.5) and (8.4) we find

$$\begin{aligned} \Sigma(X_1 \dot{\boxplus} X_2) &= \Sigma(\langle X_1 \dot{\oplus} X_2 \rangle_{\mathbb{U}}) = \Sigma\Sigma^{\leftarrow}\Sigma(X_1 \dot{\oplus} X_2) \\ &= \Sigma(X_1 \dot{\oplus} X_2) = \Sigma(X_1) \oplus \Sigma(X_2), \end{aligned}$$

where we have also used that $(\Sigma^{\leftarrow}, \Sigma)$ is an adjunction. ∎

8.3. Example: Support function. (Continuation of Examples 5.1 and 6.6).
The operator $\Sigma : \mathcal{P}(\mathbb{R}^d) \to \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$ given by

$$\Sigma(X)(v) = \bigvee_{x \in X} \langle x, v \rangle$$

is the *slope transform for sets*; see [5]. It coincides with the support function if one restricts to convex sets [7, 13, 16]. For some theoretical results on this transform the reader may refer to [5].

8.4. Example: Slope transform. (Continuation of Examples 5.4 and 6.10).
The operator $\Sigma : \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}}) \to \mathrm{Fun}(\mathbb{R}^d, \overline{\mathbb{R}})$ given by

$$\Sigma(F)(v) = \bigvee_{x \in \mathbb{R}^d} F(x) - \langle x, v \rangle$$

is the (*upper*) *slope transform* for functions: see [9, 10] and [5]. In the latter reference it is explained in considerable detail that this transform is closely related to the (Young-Fenchel) conjugate [7, 13, 16]. The adjoint $\Sigma^{\leftarrow}$ can be computed explicitly:

$$\Sigma^{\leftarrow}(S)(x) = \bigwedge_{v \in \mathbb{R}^d} \varepsilon_{u_v}(S(v)) = \bigwedge_{v \in \mathbb{R}^d} S(v) + \langle x, v \rangle .$$

As observed above, $\Sigma^{\leftarrow}$ is not an inverse of Σ in the usual sense of the word, but $\Sigma^{\leftarrow}\Sigma(F) = \langle F \rangle_{\mathbb{U}} = \overline{\mathrm{cc}}(F)$, the u.s.c. concave hull of F. In particular, if the original function F is u.s.c. and concave, then $\Sigma^{\leftarrow}\Sigma(F) = F$. Further theoretical results about this slope transform can be found in [5].

9. Applications to random elements

The introduced concept allows to define random elements with values in $\mathcal{L}$. The construction resembles Kendall's [8] approach to define a random set. For this, assume that $\mathcal{M} = \overline{\mathbb{R}}$ is endowed with the usual addition and a Borel σ-algebra $\mathcal{B}$, and $m = \mathbb{R}$. We also assume that the family of evaluations $\mathbb{U}$ is separable, i.e. there exists a countable subfamily $\mathbb{U}_0 \subseteq \mathbb{U}$ such that $\mathbb{U} = \bar{\mathbb{U}}_0$, see Proposition 4.4.

9.1. Definition. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be an abstract probability space. Then $\Xi : \Omega \mapsto \mathcal{L}$ is said to be a *random element in* $\mathcal{L}$ *with respect to* $\mathbb{U}$ if $\delta_u(\Xi) : \Omega \mapsto \mathcal{M}$ is $(\mathcal{A}, \mathcal{B})$-measurable for all $u \in \mathbb{U}$.

Thus, Ξ is a random element in $\mathcal{L}$ with respect to $\mathbb{U}$ if $\delta_u(\Xi)$ is a random variable for all $u \in \mathbb{U}$. In the framework of Example 5.2 we get Kendall's definition of a random closed set [8]. Furthermore, if $\mathcal{T}$ in Example 5.2 comprises the open subsets of E, then our definition coincides with Matheron's definition of a random closed set [11].

The following result follows from (4.6).

9.2. Proposition. *If* Ξ *is a random element in* $\mathcal{L}$ *with respect to* $\mathbb{U}$*, then its* $\mathbb{U}$*-closing* $\langle \Xi \rangle_{\mathbb{U}}$ *is a random element as well.*

By separability, the distribution of Ξ is determined by joint distributions of $(\delta_{u_1}(\Xi), \dots, \delta_{u_n}(\Xi))$, which play the same role as finite-dimensional distributions in the theory of stochastic processes. Note that from the distributional point of view, Ξ is indistinguishable from $\langle \Xi \rangle_{\mathbb{U}}$ if only information about $\delta_u(\Xi)$, $u \in \mathbb{U}$, is available.

9.3. Definition. A random element Ξ in $\mathcal{L}$ is said to be *integrable* if $\delta_u(\Xi)$ is integrable (has a finite expectation $\mathbf{E}[\delta_u(\Xi)]$) for all $u \in \mathbb{U}$. If Ξ is integrable, we define the $\mathbb{U}$-*expectation* of Ξ as follows:

$$\mathbf{E}_{\mathbb{U}}\Xi = \bigwedge_{u\in\mathbb{U}} \varepsilon_u(\mathbf{E}[\delta_u(\Xi)])\,. \tag{9.1}$$

Note that when Ξ is deterministic, i.e., $\Xi = X$ with probability one, then

$$\mathbf{E}_{\mathbb{U}}\Xi = \langle X\rangle_{\mathbb{U}}.$$

It is obvious that $\mathbf{E}_{\mathbb{U}}\Xi \le \varepsilon_u\big(\mathbf{E}[\delta_u(\Xi)]\big)$, for $u \in \mathbb{U}$. This yields that

$$\delta_u(\mathbf{E}_{\mathbb{U}}\Xi) \le \mathbf{E}[\delta_u(\Xi)], \quad u \in \mathbb{U}. \tag{9.2}$$

9.4. Example. Let $\mathcal{L} = \mathcal{F}(\mathbb{R}^d)$, and let $\delta_u(\Xi) = h(\Xi, u)$ be the support function of Ξ; see Example 2.6. If

$$\mathbf{E}[\sup\{\|x\| \mid x \in \Xi\}] < \infty,$$

then Ξ is integrable. Furthermore, $\mathbf{E}[h(\Xi,u)] = h(\mathbf{E}_{\mathbb{U}}\Xi, u)$, and the set $\mathbf{E}_{\mathbb{U}}\Xi$ is called the *Aumann expectation* of Ξ [19, 20]. Then

$$\mathbf{E}_{\mathbb{U}}\Xi = \bigcap_{u\in\mathbb{U}} \big\{x \in \mathbb{R}^d \mid \langle x,u\rangle \le \mathbf{E}h(\Xi,u)\big\}$$

i.e., (9.1) holds for for $\mathbb{U} = \mathbb{S}^{d-1}$, and the Aumann expectation appears to be a particular case of (9.1).

9.5. Example. Let $\mathcal{L} = \mathcal{F}(E)$ be the space of all closed subsets of a metric space (E, d), let $\ell = \{E\}$, and let $\mathcal{M} = \overline{\mathbb{R}}$. Put $\mathbb{U} = \{u_p \mid p \in E\}$ where $u_p(x) = d(p,x)$, for $x \in E$. Then

$$\delta_{u_p}(\Xi) = \sup\{d(p,x) \mid x \in \Xi\} = d_H(p,\Xi),$$

where $d_H(p,\Xi)$ is the Hausdorff distance between p and Ξ [11, 13, 16]. Note that for $X \in \mathcal{L}$, its $\mathbb{U}$-closing $\langle X\rangle_{\mathbb{U}}$ is the intersection of all closed balls which contains X. If $\mathbf{E}[d_H(p_0,\Xi)] < \infty$ for some $p_0 \in E$, then Ξ is integrable. In this case,

$$\mathbf{E}_{\mathbb{U}}\Xi = \bigcap_{p\in E} \{x \in E \mid d(p,x) \le \mathbf{E}[d_H(p,\Xi)]\}\,,$$

which is exactly the so-called *Doss expectation* of Ξ; see [3].

The following result follows from Proposition 4.3 and the fact that infima of $\mathbb{U}$-closed sets are $\mathbb{U}$-closed.

9.6. Proposition. *If Ξ is integrable, then $\mathbf{E}_{\mathbb{U}}\Xi \in \mathcal{L}$ is $\mathbb{U}$-closed, and $\mathbf{E}_{\mathbb{U}}\Xi = \mathbf{E}_{\mathbb{U}}\langle\Xi\rangle_{\mathbb{U}}$.*

The binary operation introduced in Section 6 can be applied to random elements in $\mathcal{L}$.

9.7. Theorem. *If $\mathbb{U}$ is linear and Ξ_1, Ξ_2 are random elements in $\mathcal{L}$ with respect to $\mathbb{U}$, then $\Xi_1 \dot{\oplus} \Xi_2$ is a random element, and*

$$\mathbf{E}_{\mathbb{U}}(\Xi_1 \dot{\oplus} \Xi_2) \ge \mathbf{E}_{\mathbb{U}}\Xi_1 \dot{\oplus} \mathbf{E}_{\mathbb{U}}\Xi_2.$$

PROOF. By Proposition 6.3, $\delta_u(\Xi_1 \dot\oplus \Xi_2) = \delta_u(\Xi_1) \oplus \delta_u(\Xi_2)$. Now $\Xi_1 \dot\oplus \Xi_2$ is a random element because a sum of two random variables is again a random variable (note that "$\oplus$" is the usual addition in $\mathcal{M} = \overline{\mathbb{R}}$).

To establish the inequality, we observe that for $Y_1, Y_2 \in \overline{\mathbb{R}}$ and $u \in \mathbb{U}$

$$Y_1 \oplus Y_2 \geq \delta_u \varepsilon_u(Y_1) \oplus \delta_u \varepsilon_u(Y_2) = \delta_u(\varepsilon_u(Y_1) \dot\oplus \varepsilon_u(Y_2)).$$

This yields that $\varepsilon_u(Y_1 \oplus Y_2) \geq \varepsilon_u(Y_1) \dot\oplus \varepsilon_u(Y_2)$. Thus we derive that

$$\begin{aligned}
\mathbf{E}_{\mathbb{U}}(\Xi_1 \dot\oplus \Xi_2) &= \bigwedge_{u \in \mathbb{U}} \varepsilon_u(\mathbf{E}\delta_u(\Xi_1 \dot\oplus \Xi_2)) \\
&= \bigwedge_{u \in \mathbb{U}} \varepsilon_u(\mathbf{E}\delta_u(\Xi_1) \oplus \mathbf{E}\delta_u(\Xi_2)) \\
&\geq \bigwedge_{u \in \mathbb{U}} \Big[\varepsilon_u(\mathbf{E}\delta_u(\Xi_1)) \dot\oplus \varepsilon_u(\mathbf{E}\delta_u(\Xi_2))\Big] \\
&\geq \bigwedge_{u \in \mathbb{U}} \varepsilon_u(\mathbf{E}\delta_u(\Xi_1)) \dot\oplus \bigwedge_{u \in \mathbb{U}} \varepsilon_u(\mathbf{E}\delta_u(\Xi_2)) \\
&= \mathbf{E}_{\mathbb{U}}(\Xi_1) \dot\oplus \mathbf{E}_{\mathbb{U}}(\Xi_2).
\end{aligned}$$

This concludes the proof. ∎

In general, we do not have equality in Theorem 9.7. For, let $\Xi_i = X_i$ with probability one. Then $\mathbf{E}_{\mathbb{U}}\Xi_i = \langle X_i \rangle_{\mathbb{U}}$, and equality in Theorem 9.7 would imply that

$$\langle X_1 \dot\oplus X_2 \rangle_{\mathbb{U}} = \langle X_1 \rangle_{\mathbb{U}} \dot\oplus \langle X_2 \rangle_{\mathbb{U}}.$$

We have seen in Example 7.6 that this equality does not hold in general.

9.8. Proposition. *Assume that $\mathbb{U}$ is linear, that Ξ_1, Ξ_2 are random elements in $\mathcal{L}$ with respect to $\mathbb{U}$, and that the following conditions hold:*
(i) $\delta_u(\mathbf{E}_{\mathbb{U}}\Xi) = \mathbf{E}[\delta_u(\Xi)], \quad u \in \mathbb{U}$,
(ii) *$X_1 \dot\oplus X_2$ is $\mathbb{U}$-closed if X_1, X_2 are $\mathbb{U}$-closed.*
Then we have the following identity:

$$\mathbf{E}_{\mathbb{U}}(\Xi_1 \dot\oplus \Xi_2) = \mathbf{E}_{\mathbb{U}}\Xi_1 \dot\oplus \mathbf{E}_{\mathbb{U}}\Xi_2.$$

PROOF. We derive

$$\begin{aligned}
\mathbf{E}_{\mathbb{U}}(\Xi_1 \dot\oplus \Xi_2) &= \bigwedge_{u \in \mathbb{U}} \varepsilon_u(\mathbf{E}\delta_u(\Xi_1) \oplus \mathbf{E}\delta_u(\Xi_2)) \\
&= \bigwedge_{u \in \mathbb{U}} \varepsilon_u\Big(\delta_u(\mathbf{E}_{\mathbb{U}}\Xi_1) \oplus \delta_u(\mathbf{E}_{\mathbb{U}}\Xi_2)\Big) \\
&= \bigwedge_{u \in \mathbb{U}} \varepsilon_u\Big(\delta_u(\mathbf{E}_{\mathbb{U}}\Xi_1 \dot\oplus \mathbf{E}_{\mathbb{U}}\Xi_2)\Big) \\
&= \langle \mathbf{E}_{\mathbb{U}}\Xi_1 \dot\oplus \mathbf{E}_{\mathbb{U}}\Xi_2 \rangle_{\mathbb{U}} \\
&= \mathbf{E}_{\mathbb{U}}\Xi_1 \dot\oplus \mathbf{E}_{\mathbb{U}}\Xi_2,
\end{aligned}$$

where we have used condition (ii) and (i), respectively, as well as the fact that $\mathbf{E}_{\mathbb{U}}\Xi_1$ and $\mathbf{E}_{\mathbb{U}}\Xi_2$ are closed; see Proposition 9.6. The first identity was taken from the proof of Theorem 9.7. ∎

Note that (9.2) says that the inequality '$\leq$' in (i) is automatically satisfied.

Acknowledgements

I. Molchanov is grateful to the Centrum voor Wiskunde en Informatica (CWI, Amsterdam) for its hospitality.

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