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Department of Applied Mathematics Report AM-R8501 January

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Construction and Analysis of a Low Order Spectral Model of the Barotropic Potential Vorticity Equation in a Beta Channel

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A low order spectral model is derived from the barotropic potential vorticity equation in a beta channel. Distinction is made between the conservative and dissipative case. Both systems are analyzed by mathematical methods. The dissipative model is similar to that of Egger (1981), however, a different value for the β parameter and a constant width of the channel are taken, giving rise to new quantitative aspects.

Key words & phrases: barotropic potential vorticity equation, spectral model, flows, bifurcations.
Note: these investigations were supported by the Netherlands Foundation for the Technical Sciences (STW), future Technical Science Branch of the Netherlands Organization for the Advancement of Pure Research (ZWO).

1980 Mathematics subject classification: 35A35, 58F25.

Report AM-R8501
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

1. Introduction

In a previous report (de Swart 1984) a quasi-geostrophic potential vorticity equation has been derived for a barotropic flow over a large scale topography; it reads

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi + \frac{f_0 h}{H} + f) + \frac{f_0\delta_E}{2H}\nabla^2(\psi - \psi^*) = 0. \qquad (1\text{-}1)$$

Here ψ is the streamfunction, t is time, ∇^2 the two-dimensional Laplacian, J the Jacobian, f the Coriolis parameter, $f_0 = f(\phi_0)$ where ϕ_0 is a central latitude, H a scale height, h denotes the topography, δ_E the depth of the Ekman layer and ψ^* a forcing streamfunction.

A spectral model is constructed by development of ψ, ψ^* and h in eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ of the Laplace operator, with corresponding eigenvalues λ_i^2, on a two-dimensional domain with prescribed boundary conditions, which are orthonormalized by

$$\overline{(\phi_i, \phi_j)} = \delta_{ij}, \qquad (1\text{-}2)$$

where the bar denotes an average over the particular domain considered. Projection of equation (1-1) on these eigenfunctions results in an infinite number of nonlinear ordinary differential equations.

For practical applications the projection is restricted to a finite number (N) of eigenfunctions. Then it becomes convenient to write

$$\psi = \psi_r + \psi_u; \quad \psi^* = \psi_r^* + \psi_u^*; \quad h = h_r + h_u, \qquad (1\text{-}3)$$

where r and u denote the resolved and unresolved modes respectively, so e.g.

$$\psi_r = \sum_{i=1}^{N}\psi_i\phi_i; \quad \psi_u = \sum_{i=N+1}^{\infty}\psi_i\phi_i, \qquad (1\text{-}4)$$

Projection of equation (1-1) on the resolved modes gives the spectral model equations

$$\left.\begin{aligned}\frac{d\psi_i}{dt} = \frac{1}{\lambda_i^2}\{\sum_{j=1}^{N}\sum_{p=j+1}^{N} c_{ijp}[(\lambda_j^2-\lambda_p^2)\psi_j\psi_p + \frac{f_0}{H}(h_p\psi_j - h_j\psi_p)] + \sum_{j=1}^{N} b_{ij}\psi_j + \\ -\frac{f_0\delta_E}{2H}\lambda_i^2(\psi_i - \psi_i^*)\} + F_i^*; \quad i = 1,2, \ldots\ldots, N,\end{aligned}\right\} \qquad (1\text{-}5)$$

where

$$\left.\begin{aligned} c_{ijp} &= \overline{(\phi_i, J(\phi_j, \phi_p))}; \quad b_{ij} = \overline{(\phi_i, J(\phi_j, f))}, \\ F_i^* &= -\frac{1}{\lambda_i^2}\left\{\sum_{j=1}^{N}\sum_{p=N+1}^{\infty} + \sum_{j=N+1}^{\infty}\sum_{p=1}^{N} + \sum_{j=N+1}^{\infty}\sum_{p=N+1}^{\infty}\right\}\left\{c_{ijp}(\lambda_p^2\psi_p - \frac{f_0}{H}h_p)\psi_j\right\}.\end{aligned}\right\} \qquad (1\text{-}6)$$

The synoptic forcing terms $\{F_i^*\}_{i=1}^{N}$ are due to the projection of the Jacobians $J(\psi_r, \nabla^2\psi_u + \frac{f_0 h_u}{H})$, $J(\psi_u, \nabla^2\psi_r + \frac{f_0 h_r}{H})$ and $J(\psi_u, \nabla^2\psi_u + \frac{f_0 h_u}{H})$ on the resolved modes; they are unknowns by definition. Our primary aim is to analyse the nonlinear dynamics of the equations (1-5), which are assumed to model the large scale atmospheric flow. It is expected that this is a complicated problem, since the atmosphere is a chaotic system in the sense that small errors in the initial conditions rapidly grow during the time evolution, and lead to unpredictable behaviour, as pointed out by Lorenz (1984). Therefore a careful and systematic analysis is required. Our study consists of two parts: first the dynamics of a small number of modes will be analyzed, next the number of modes is increased and its consequences for the dynamics of the system will be investigated.

Concerning the effect of the unresolved- on the resolved modes, we expect that this will be negligibly small if N is large, an assumption made in almost all spectral models. Examples of such studies, in which the dynamical aspects of the large scale atmospheric flow are investigated, are the papers of Reinhold & Pierrehumbert (1982) for a two-layer model in a beta-channel, and Legras & Ghil (1984) for an equivalent barotropic model in spherical geometry.

However, as discussed by Foias et. al. (1983), theoretical research on the number of modes governing solutions of partial differential equations of fluid mechanics, show that N must be at least of the order 10^5 for (quasi) two-dimensional flows in order to neglect synoptic forcing terms. Hence they will have a considerable effect on the dynamics of low order spectral models.

One possible parametrization of the synoptic forcing terms follows from the ideas of Leith (1973), Hasselman (1976) and Madden (1976), who suggest that in time-averaged atmospheric models the effect of the transients on the low frequency variability can be modelled by stochastic terms. Adaption of this idea to synoptic forcing terms has been proposed by Egger (1981), and is confirmed by the data analysis of Egger and Schilling (1983). Nevertheless, there is evidence that the effect is not entirely random since there is a dynamical feedback between different parts of the spectrum (Wallace & Blackman, 1983). The parametrization of the noise in terms of physical processes will be a subject of future research.

2. Application to the β-channel

Consider the barotropic potential vorticity equation in a rectangular channel with length L and width B, as shown in Figure 1. Here $f = f_0 + \beta_0 y$.

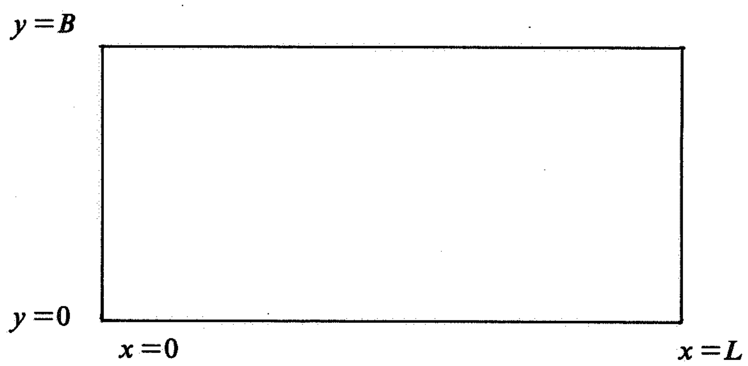


Figure 1. Geometry of the domain.

We investigate the existence of travelling wave solutions in the x-direction. At the boundaries $y=0$ and $y=B$ the meridional velocity is zero. Furthermore no circulation may develop around the boundaries. The resulting boundary conditions for the streamfunction $\psi = \psi(x,y,t)$ are derived in detail by Philips (1954), and are summarized by Vickroy & Dutton (1979). They read

$$\int_0^L \psi \, dx = 0, \tag{2-1}$$

$$\frac{\partial \psi}{\partial x} = 0 \qquad \text{at} \quad y=0 \quad \text{and} \quad y=B, \tag{2-2}$$

$$\int_0^L \frac{\partial^2 \psi}{\partial t \, \partial y} \, dx = 0 \qquad \text{at} \quad y=0 \quad \text{and} \quad y=B. \tag{2-3}$$

We now have constructed a well-defined mathematical problem. However, it is not a realistic model of the atmosphere because of the artificial boundary conditions at the side-walls, which suppress all coupling with outer regions. A different model is obtained by rewriting equation (1-1) in spherical coordinates and then constructing a spectral model. This is done by Wiin-Nielsen (1979), Källén (1981, 1982) and Legras & Ghil (1984). However, these models carry conflicts in their physical formalism with them as well. For example, quasi-geostrophy is assumed to be valid over the whole sphere, but it is well-known that this is a poor representation in the tropics. As we wish to study a simple model of the atmospheric flow which roughly describes the physical problem, we do not wonder about this aspect.

The eigenfunctions $\phi_i(x,y)$ for this problem can be calculated from substitution of

$$\phi(x,y) = F(x)G(y) \tag{2-4}$$

in the Helmholtz equation, resulting in the general solutions

$$F(x) = \alpha_1 e^{i\mu x} + \alpha_2 e^{-i\mu x}, \tag{2-5}$$

$$G(y) = \gamma_1 e^{i\sqrt{\lambda^2-\mu^2}y} + \gamma_2 e^{-i\sqrt{\lambda^2-\mu^2}y}, \tag{2-6}$$

where μ is a separation constant and $\alpha_1,\alpha_2,\gamma_1$ and γ_2 are integration constants. Application of the periodicity condition (2-1) gives

$$\mu = \frac{2\pi n}{L} \equiv k_n; \quad n=0,1,2, \ldots . \tag{2-7}$$

At first we put $n=0$. Then the remaining conditions result in the set of eigenfunctions

$$\phi_q^{(1)}(y) = \sqrt{2}\cos(l_q y); \quad l_q = \frac{q\pi}{B}; \quad q=1,2, \ldots , \tag{2-8}$$

with eigenvalues

$$\lambda^2 = l_q^2, \tag{2-9}$$

where l_q is a meridional wave number. The corresponding streamfunction modes describe a purely zonal flow.

In the case $\mu \neq 0$ solutions have wave number k_n. The boundary condition (2-2) gives eigenvalues

$$\lambda^2 = k_n^2 + l_m^2; \quad m=1,2, \ldots; \quad n=1,2, \ldots . \tag{2-10}$$

From the orthonormality condition it finally follows that there are two other set of eigenfunctions, viz.

$$\phi_{mn}^{(2)}(x,y) = 2\cos(k_n x)\sin(l_m y); \quad m=1,2, \ldots; \quad n=1,2, \ldots , \tag{2-11}$$

$$\phi_{rs}^{(3)}(x,y) = 2\sin(k_s x)\sin(l_r y); \quad r=1,2, \ldots; \quad s=1,2, \ldots . \tag{2-12}$$

The full set of orthonormalised eigenfunctions is given by (2-8), (2-11) and (2-12).

Next we investigate the interaction coeffieients c_{ijp} and the b_{ij}'s, both defined in (1-6). It can easily be shown, using partial integration plus boundary conditions and the antisymmetry property of the Jacobian, that

$$c_{ijp} = c_{jpi} = c_{pij} = -c_{ipj} \quad \text{and} \quad b_{ij} = -b_{ij}, \tag{2-13}$$

(Charney & DeVore, 1979). Furthermore the interation coefficients are only nonzero if the three eigenfunctions in its definition (1-6) are taken from three different sets. It is sufficient to calculate c_{ijp} for

$$\phi_i = \phi_q^{(1)}; \quad \phi_j = \phi_{mn}^{(2)}; \quad \phi_p = \phi_{rs}^{(3)},$$

resulting in

$$c_{ijp} = \begin{cases} \dfrac{-2\sqrt{2}k_n}{B}\,\delta_{ns}\left\{\dfrac{(r+m)^2}{(r+m)^2-q^2} - \dfrac{(r-m)^2}{(r-m)^2-q^2}\right\} & ; \quad q+m+r \ \text{odd}, \\ 0 & ; \quad q+m+r \ \text{even}. \end{cases} \tag{2-14}$$

The b_{ij}'s are nonzero if the two eigenfunctions in its definition are taken from the sets (2-11) and (2-12).

For $\phi_i = \phi_{mn}^{(2)}$ and $\phi_j = \phi_{rs}^{(3)}$ we find

$$b_{ij} = \beta_0 k_n \delta_{mr} \delta_{ns}. \tag{2-15}$$

The spectral model equations (1-5) can now be developed for any number of modes. Here we take only three eigenfunctions into account, viz. $(k \equiv k_1,\ l \equiv l_1)$

$$\left.\begin{aligned} \phi_1 &= \sqrt{2}\cos(ly)\,; & \lambda_1^2 &= l^2, \\ \phi_2 &= 2\cos(kx)\sin(ly)\,; & \lambda_2^2 &= k^2+l^2, \\ \phi_3 &= 2\sin(kx)\sin(ly)\,; & \lambda_3^2 &= k^2+l^2, \end{aligned}\right\} \tag{2-16}$$

which represent a basic zonal flow and two planetary wave modes respectively. The model is further simplified by setting

$$\psi^* = \psi_1^*\phi_1; \quad h = h_0\cos(kx)\sin(ly) = \tfrac{1}{2}h_0\phi_2. \tag{2-17}$$

Thus the forcing only acts upon the zonal flow, and the orography is fully described by one eigenfunction. The model equations become

$$\frac{d\psi_1}{dt} = \frac{1}{l^2}\left\{\frac{8\sqrt{2}k}{3B}\,\frac{f_0 h_0}{2H}\,\psi_3 - \frac{f_0\delta_E}{2H}\,l^2(\psi_1-\psi_1^*)\right\} + F_1^*, \tag{2-18}$$

$$\frac{d\psi_2}{dt} = \frac{1}{k^2+l^2}\left\{\frac{-8\sqrt{2}k^3}{3B}\,\psi_1\psi_3 + \beta_0 k\psi_3 - \frac{f_0\delta_E}{2H}\,(k^2+l^2)\psi_2\right\} + F_2^*, \tag{2-19}$$

$$\frac{d\psi_3}{dt} = \frac{1}{k^2+l^2}\left\{\frac{8\sqrt{2}k^3}{3B}\,\psi_1\psi_2 - \beta_0 k\psi_2 - \frac{8\sqrt{2}k}{3B}\,\frac{f_0 h_0}{2H}\,\psi_1 - \frac{f_0\delta_E}{2H}\,(k^2+l^2)\psi_3\right\} + F_3^*. \tag{2-20}$$

These equations can be simplified by introduction of a new time

$$\tau = \frac{4\sqrt{2}}{3\pi}\,t. \tag{2-21}$$

Furthermore, new variables

$$\left.\begin{aligned} v_1 &= \frac{2lk^2}{k^2+l^2}\,\psi_1; & v_1^* &= \frac{2lk^2}{k^2+l^2}\,\psi_1^*, \\ v_2 &= k\psi_2; & v_3 &= -k\psi_3 \\ F_1 &= \frac{3\pi}{4\sqrt{2}}\,\frac{2lk^2}{k^2+l^2}\,F_1^*; \quad F_2 = \frac{3\pi}{4\sqrt{2}}\,kF_2^*; & F_3 &= -\frac{3\pi}{4\sqrt{2}}\,kF_3^* \end{aligned}\right\}. \tag{2-22}$$

are defined. Here v_1 is a zonal velocity, v_1^* a forcing velocity and v_2 and v_3 are wave velocities. Application of all these transformations to (2-18) - (2-20) results in

$$\dot{v}_1 = -\alpha\frac{f_0 h_0}{H}\,v_3 - C(v_1 - v_1^*) + F_1, \tag{2-23}$$

$$\dot{v}_2 = k(v_1 - c_R)v_3 - Cv_2 + F_2, \tag{2-24}$$

$$\dot{v}_3 = -k(v_1 - c_R)v_2 + \frac{f_0 h_0}{2H}\,v_1 - Cv_3 + F_3, \tag{2-25}$$

where

$$\alpha = \frac{2k^2}{k^2+l^2}\,; \quad C = \frac{3\pi}{4\sqrt{2}}\,\frac{f_0\delta_E}{2H}\,; \quad c_R = \frac{\beta_*}{k^2+l^2}; \quad \beta_* = \frac{3\pi}{4\sqrt{2}}\,\beta_0, \tag{2-26}$$

and a dot denotes differentiation with respect to τ. Note that c_R is the phase speed of a free Rossby wave mode.

In subsequent sections we will analyse the equations, neglecting the effect of the synoptic forcing terms. Distinction is made between the cases $C=0$ (Section 3) and $C>0$ (Section 4).

3. The unforced, nondissipative system

In this section we consider the equations (2-23) - (2-25) without forcing ($v_1^*=0$) and dissipation ($C=0$). They read

$$\dot{v}_1 = -\alpha\frac{f_0 h_0}{H}\,v_3, \tag{3-1}$$

$$\dot{v}_2 = k(v_1 - c_R)v_3, \tag{3-2}$$

$$\dot{v}_3 = -k(v_1 - c_R)v_2 + \frac{f_0 h_0}{2H}\,v_1. \tag{3-3}$$

These equations can be completely analyzed in terms of standard integrals, which gives some insight in the particular effects of the nonlinear terms.

It is well-known that spectral models of the barotropic potential vorticity equation, without forcing and dissipation, possesses two constants of motion, viz. the kinetic energy and the potential enstrophy, the latter being the squared relative potential vorticity. For the equations (3-1) - (3-3) it easily yields two constants of motion, viz.

$$E = \tfrac{1}{2}v_1^2 + \alpha(v_2^2 + v_3^2) \tag{3-4}$$

and

$$G = (\tfrac{1}{2}v_1 - c_R)v_1 + \frac{\alpha f_0 h_0}{kH}\,v_2. \tag{3-5}$$

The former is clearly the kinetic energy, the latter is a combination of the energy and the potential enstrophy in spectral terms; they are specified by the initial conditions.

From the spectral model equations we can now deduce one equation for v_1. By differentiating (3-1) and substituting (3-3) and (3-5) we obtain

$$\ddot{v}_1 = -\tfrac{1}{2}k^2 v_1^3 + \tfrac{3}{2}k^2 c_R v_1^2 + \left[k^2(G - c_R^2) - \frac{\alpha f_0^2 h_0^2}{2H^2}\right]v_1 - k^2 c_R G \equiv P_3(v_1). \tag{3-6}$$

Multiplying with $\dot{v}_1$ and next integrating, we obtain

$$\frac{dv_1}{\sqrt{P_4(v_1)}} = \pm d\tau, \quad \text{or} \quad \tau = \pm\int_{v_1(0)}^{v_1}\frac{dx}{\sqrt{P_4(x)}}, \tag{3-7}$$

where

$$P_4(x) = -\tfrac{1}{4}k^2x^4 + k^2 c_R x^3 + \left[k^2(G - c_R^2) - \tfrac{1}{2}\alpha\left(\frac{f_0 h_0}{H}\right)^2\right]x^2 - 2k^2 c_R G x + A \tag{3-8}$$

is a quartic in x, with A an integration constant being a function of E and G. The sign on the right hand side depends on whether $\dot{v}_1$ is positive or negative so that τ will increase monotonically.

The global analysis of equation (3-7) is quite difficult, because of the possible complicated structure of the quartic. Here we will present qualitative results, which will give insight in the behaviour of the solutions.

We search for real solutions of (3-7), hence positive values of $P_4(x)$ are required. Since the quartic in (3-8) tends to $-\infty$ for large values of $|x|$ it must have real zeros. From algebraic considerations we then know that there will be two or four real roots.

First we assume that all roots are distinct. Then the integral in (3-7) is of the elliptic type and can be analyzed in terms of standard integrals. It appears that, by means of suitable transformations and reduction methods (see Bowman, 1953), the solution $v_1(\tau)$ can be expressed in terms of Jacobian elliptic functions. As a consequence $v_1(\tau)$ is bounded and periodic in τ, and from the constants of motions it follows the same results for $v_2(\tau)$ and $v_3(\tau)$. Obviously the nonlinear terms in the spectral equations (3-1) - (3-3) bring about a continuous energy transfer between the different wave modes and the zonal flow.

Next we consider the case that the quartic in (3-7) has multiple zeros. As can be seen from geometrical considerations, the only case resulting in different nontrival behaviour occurs if $P_4(x)$ has four real roots $a_1<a_2=a_3<a_4$, thus $\gamma\equiv a_2=a_3$ is a double zero. For this value of x both the quartic and its derivative with respect to x become zero, i.e.

$$P_4(x)\Big|_{x=\gamma} = 0; \quad \frac{dP_4(x)}{dx}\Big|_{x=\gamma} = 2P_3(x)\Big|_{x=\gamma} = 0. \tag{3-9}$$

From the equations (3-1) - (3-3), (3-6) and (3-7) it then follows $\dot v_1=\dot v_2=\dot v_3=0$, thus if $v_1=\gamma$, with γ a double zero of the quartic $P_4(x)$, the corresponding flow is steady. By similar arguments it also can be shown that the opposite assertion holds.

We will now determine the characteristics of the solution of (3-7) for this case. Let v_1 be in the interval $[a_1,\gamma]$; the case that v_1 is in $[\gamma,a_4]$ is not fundamentally different. Although $v_1(\tau)$ may approach and leave a_1 once, it must proceed towards γ eventually, as $\dot v_1$ nowhere changes sign on the interval. The quartic may be written

$$P_4(x) = \tfrac{1}{4}k^2(a_4-x)(\gamma-x)^2(x-a_1), \tag{3-10}$$

and from (3-7) it follows

$$\tau(\gamma-\epsilon) = \frac{2}{k}\int_{v_1(0)}^{\gamma-\epsilon} \frac{dx}{(\gamma-x)\sqrt{(a_4-x)(x-a_1)}}. \tag{3-11}$$

The transformation $\xi=\gamma-x$ gives

$$\tau(\gamma-\epsilon) = \frac{2}{k}\int_{\epsilon}^{\gamma-v_1(0)} \frac{d\xi}{\xi\sqrt{(a_4-\gamma+\xi)(\gamma-a_1-\xi)}} \geqslant D\int_{\epsilon}^{\gamma-v_1(0)} \frac{d\xi}{\xi\sqrt{a+\xi}}, \tag{3-12}$$

where

$$D = \frac{2}{k\sqrt{\gamma-a_1}}; \quad a = a_4-\gamma. \tag{3-13}$$

The integral on the right hand side of (3-12) can be performed by application of the transformation $y=\sqrt{a+\xi}$, and next splitting fractions. It then appears that in the limit $\epsilon\to 0$, $\tau(\gamma-\epsilon)$ becomes infinite, hence $v_1(\tau)$ asymptotically tends to γ, which corresponds to a steady state.

With this we have obtained a class of aperiodic solutions. However, as they require special choices of the coefficients of the quartic in (3-8), the solution of (3-7) will in general be periodic, as was already mentioned by Charney & DeVore (1979).

For three-component conservative spectral models in spherical geometry the results differ in the sense that the solutions will in general be aperiodic, although their associated spectra are periodic. Details are discussed by Dutton (1976) and Lupini & Pellacani (1984).

4. The dissipative system

In this section we study the full nonlinear spectral model equations (2-23) - (2-25). They form a dynamical system of the type

$$\dot x = f_\mu(x), \tag{4-1}$$

where $x=(v_1,v_2,\ldots,v_N)$, and $f_\mu(x)$ is a vectorfield which depends on x and parameters $\mu=(\mu_1,\mu_2,\ldots,\mu_m)$. Note that $f_\mu(x)$ does not depend explicitly on time. Such systems are called autonomous, with the property that if $x=(\tau,x_0)$ is a solution such that $x(0,x_0)=x_0$ then $x(\tau-\tau_0,x_0)$ also satisfies the equations (4-1). The set of solution curves in the phase space is called the flow generated by $f_\mu(x)$.

To make a local analysis of the system we first determine the possible steady states $\hat{x}$, governed by

$$f_\mu(\hat{x}) = 0. \tag{4-2}$$

As μ varies the implicit function theorem implies that these steady states are described by smooth functions of μ away from points at which the Jacobian derivative of $f_\mu(\hat{x})$ with respect to $\hat{x}$, denoted $Df_\mu(\hat{x})$, has a zero eigenvalue. The graph of each of these functions is a branch of equilibria of (4-1). At an equilibrium where $Df_\mu(\hat{x})$ has a zero eigenvalue several branches of equilibria come together. The set of parameter values for which $Df_\mu(\hat{x})$ has a zero eigenvalue forms a hypersurface in the parameter space, which is called the bifurcation set.

The stability of the steady states can be analyzed from the linearized equations for small perturbations about these states. Substitution of

$$x = \hat{x}+x' \tag{4-3}$$

in (4-1), and expansion of the vectorfield in Taylor series about $\hat{x}$, results in

$$\dot{x}' = Df_\mu(\hat{x})x'+O(|x'|^2). \tag{4-4}$$

If all eigenvalues of matrix $Df_\mu(\hat{x})$ have negative real parts the steady state is stable. If at least one eigenvalue has a positive real part the steady state is called unstable. The points $(\hat{x},\mu)$ for which the real part of some eigenvalue vanishes are the bifurcation points.

The case of a zero eigenvalue has already been discussed. If some eigenvalues are purely imaginary a periodic solution branches off from a steady state curve in that point. This is called a Hopf bifurcation.

Finally we state that the divergence of the vectorfield

$$\nabla\cdot f_\mu(x) \tag{4-5}$$

gives some qualitative insight in the behaviour of the flow. In the first place the time evolution of an infinitesimal volume element δV in the phase space is governed by the equation

$$\frac{d}{dt}\delta V = (\nabla\cdot f_\mu)\delta V, \tag{4-6}$$

and furthermore the sum of the eigenvalues near a steady state is equal to $\nabla\cdot f_\mu(x)$ in that point. More details about dynamical systems can be found in e.g. Guckenheimer & Holmes (1983).

We now apply this theory to the spectral model equations (2-23) - (2-25). It appears that the divergence of the vectorfield is here

$$\frac{\partial\dot{v}_1}{\partial v_1}+\frac{\partial\dot{v}_2}{\partial v_2}+\frac{\partial\dot{v}_3}{\partial v_3} = -3C, \tag{4-7}$$

which is always negative. According to (4-6) it means that a volume element in the phase space is contracted, and thus for $\tau\to\infty$ the flow will tend to set of points in phase space with lower dimension. As a further consequence there are no steady states with the real part of all eigenvalues positive.

The steady states $(\hat{v}_1,\hat{v}_2,\hat{v}_3)$ of this model follow from (4-2). Elimination of $\hat{v}_2$ and $\hat{v}_3$ results in

$$\hat{v}_1^3+a_2\hat{v}_1^2+a_1\hat{v}_1+a_0 = 0, \tag{4-8}$$

where

$$\left.\begin{aligned} a_2 &= -(2c_R+v_1^*),\\ a_1 &= c_R^2+2c_Rv_1^*+\tfrac{1}{2}\alpha\left(\frac{f_0h_0}{H}\right)^2+\frac{C^2}{k^2};\\ a_0 &= -\left(c_R^2+\frac{C^2}{k^2}\right)v_1^* . \end{aligned}\right\} \tag{4-9}$$

The eigenvalue equation for the linearized equation (4-4) reads

$$(\lambda+c)^3+p_1(\lambda+c)+p_0 = 0, \tag{4-10}$$

where

$$p_1 = k^2(\hat{v}_1-c_R)^2-\alpha\frac{f_0h_0}{H}\,(k\hat{v}_2-\frac{f_0h_0}{2H});$$
$$p_0 = -\alpha\frac{f_0h_0}{H}\,k^2\hat{v}_3(\hat{v}_1-c_R). \tag{4-11}$$

From (4-8) it follows that the model has either one- or three real steady states. From the preceding theory we conclude that the transition from one- to three steady states occurs if both the cubic in (4-8) and its derivative with respect to $\hat{v}_1$ vanish. Elimination of $\hat{v}_1$ finally results in the bifurcation set

$$q^3+r^2 = 0, \tag{4-12}$$

where

$$q = \frac{1}{3}a_1-\frac{1}{9}a_2^2;\quad r = \frac{1}{6}(a_1a_2-3a_0)-\frac{1}{27}\,a_2^3. \tag{4-13}$$

In the region of the parameter space where q^3+r^2 has negative values, called the catastrophe set, three real steady states occur.

The qualitative behaviour of $\hat{v}_1$ as a function of the parameters in (4-8) is fully determined by two of them, since e.g. the quadratic term can be suppressed by a suitable transformation. For that reason we choose two free parameters, viz. the zonal wave number k and the forcing velocity v_1^*.

To calculate the bifurcation- and catastrophe set in the L,v_1^*-parameter space ($L=2\pi k^{-1}$) we have to specify the numerical constants in the equations (2-23) - (2-25). We first investigate the choices of Egger (1981). He takes $f_0=1.10^{-4}s^{-1}$, $h_0=500m$, $H=1.10^4m$, $C=1.10^{-6}s^{-1}$; $\beta_*=1{,}6.10^{-11}m^{-1}s^{-1}$ and $B=\frac{1}{2}L$. In figure 2 the bifurcation set of this model in the L,v_1^*-parameter space, enclosing the catastrophe set, is shown by the solid lines.

We have developed a different model by taking $\beta_*=2{,}67.10^{-11}m^{-1}s^{-1}$, resulting from the choice $\beta_0=1{,}6.10^{-11}m^{-1}s^{-1}$, which is a representable value for midlatitude atmospheric flow. Furthermore a constant width of the channel is assumed, in our model $B=1{,}5.10^6m$. The bifurcation set of our model in the L,v_1^* parameter space, enclosing the catastrophe set, is shown in Figure 2 by the dashed curves.

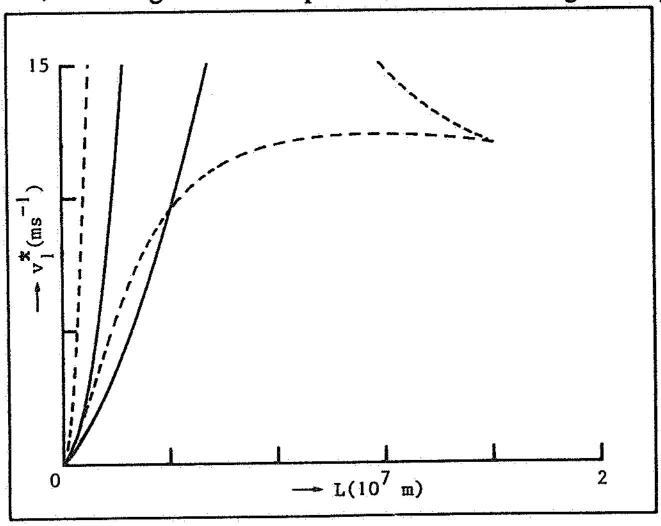


Figure 2. Bifurcation set of the Egger model (full line) and of our model (dotted line).

From this we conclude that the results of the two models are fundamentally different. We first treat the

Egger model. There it appears that the region of possible zonal wave-lengths, for which three real steady states occur, increases with increasing values of the forcing velocity. This can also be seen from figure 3, which shows the real equilibrium solutions $\hat{v}_1$ as functions of the free parameters.

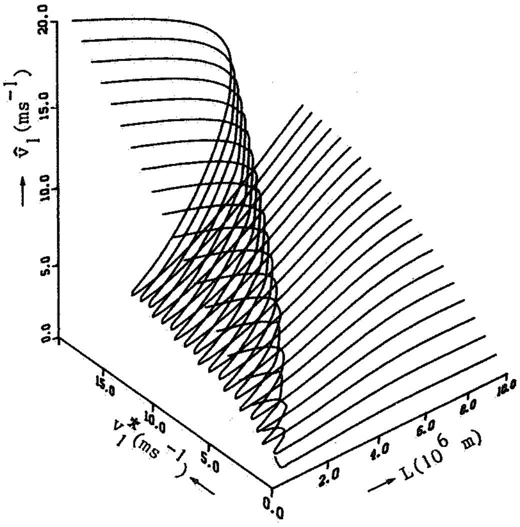
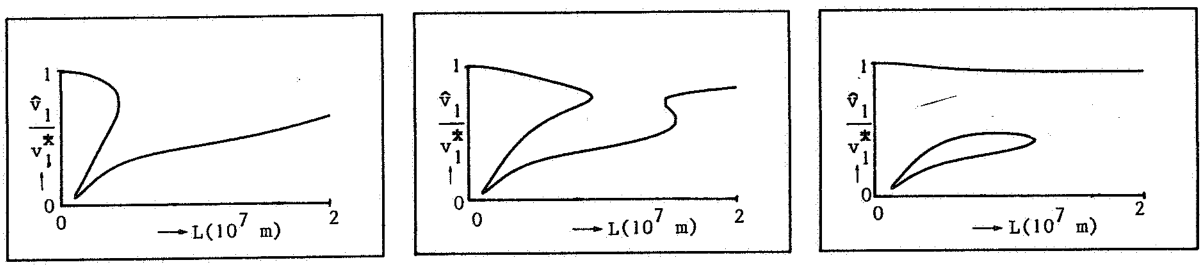


Figure 3. Bifurcation diagram of the Egger model.

The bifurcation set of our model (see Figure 2) is more complicated. Up to $v_1^* = (12{,}003 \pm 0{,}001) ms^{-1}$ there is one region of possible zonal wave-lengths where three real steady states occur, and it increases with increasing v_1^*. Due to numerical inaccuracy the transition values could not be calculated exactly. An example of the behaviour of the equilibrium solution $\hat{v}_1/v_1^*$ as a function of the zonal wave-length is shown in figure 4^a for $v_1^* = 10\ ms^{-1}$. At $v_1^* = (12{,}003 \pm 0.001) ms^{-1}$ and $L = (1{,}60 \pm 0{,}01) \cdot 10^7 m$ a new catastrophe set arises. Next for $(12{,}003 \pm 0{,}001) ms^{-1} \leqslant v_1^* \leqslant (12{,}342 \pm 0{,}001) ms^{-1}$ two separated regions exist, where three real steady states occur. In figure 4^b $\hat{v}_1/v_1^*$ is shown as a function of L for $v_1^* = 12{,}2\ ms^{-1}$. At $v_1^* = (12{,}342 \pm 0{,}001) ms^{-1}$ and $L = (1{,}18 \pm 0{,}01) \cdot 10^7 m$ the two regions join each other. For larger values of the forcing velocity one region is left over, which becomes smaller if v_1^* is increased, and eventuality disappears at $v_1^* = (41{,}420 \pm 0{,}001) ms^{-1}$, $L = (3{,}12 \pm 0{,}01) \cdot 10^6 m$. Figure 4^c shows $\hat{v}_1/v_1^*$ as a function of L for $v_1^* = 15 ms^{-1}$.



a) $v_1^* = 10\,ms^{-1}$

b) $v_1^* = 12{,}2\,ms^{-1}$

c) $v_1^* = 15\,ms^{-1}$

Figure 4. The equilibrium solution $\hat{v}_1/v_1^*$ as a function of L for different values of v_1^*.

The difference between the results of the Egger model and our model becomes even more clear if the stability of the steady states is investigated. From the preceding results it appears that for realistic values of the forcing velocity there is one equilibrium E1 for small zonal wave-lengths. It is characterized by $\hat{v}_1 \sim v_1^*$, positive $\hat{v}_2$ and $\hat{v}_3 \sim 0\,ms^{-1}$, and a linear stability analysis shows that it is always stable. With increasing L two further equilibria appear. The intermediate one (E2) appears to be unstable and is of little dynamical significance in the deterministic model. The low one (E3) is stable; it is characterized by small values of $\hat{v}_1$, negative $\hat{v}_2$ and positive $\hat{v}_3$. Mathematically spoken E1 and E3 are stable spiral points, and E2 is a saddle point.

Now consider again the model equations (2-23) - (2-25), where the free parameters are assumed to be slowly varying functions of time, i.e. they vary on a time scale which is much larger than the adjustment time scale of the model. For arbitrary initial conditions the system will always reach an equilibrium state after a sufficiently long time. For the Egger model it then appears that each time the bifurcation set is crossed the equilibrium solution suddenly transits from the one- to the other stable steady state, a phenomenon which is called a castatrophe. But for our model this is not always the case: as long as v_1^* is larger than $(12{,}342 \pm 0.001)ms^{-1}$ the equilibrium solution will always be represented by E1 after the bifurcation set has been crossed one time. Catastrophes can only occur for smaller values of v_1^*.

Finally our model results show that, apart from the bifurcation set, there are no points in the parameter space at which the stability of some steady state changes. This e.g. means that there are no Hopf bifurcations.

As can be seen from the equations (2-23) - (2-25) the existence of multiple equilibria is solely due to the presence of orography. Furthermore the instability of the steady state E2 is due to the interaction of the orography and the zonal flow. Generally there are more instability mechanisms for barotropic flows, viz. barotropic instability of the zonal flow and wave instabilities. However, a necessary condition for barotropic instability is that $\beta_0 - d^2\bar{u}/dy^2$ vanishes somewhere in the beta channel, where $\bar{u}$ is a steady zonal flow (Holton, 1979). In the three-component spectral model the equilibrium zonal flow profiles do not satisfy this condition, and hence the mechanism is absent. Similarly, wave instabilities cannot occur, since they require at least three different wave modes (Lorenz (1972), Gill (1974), Coaker (1977)). The present mechanism, causing the unstable steady state E2, is called orographic instability.

5. Construction of a stochastic model

In order to let the dynamics of the large scale atmospheric flow be modelled by the equations (2-23) - (2-25), we have to choose realistic values for the free parameters. We take $k = (2\pi/3.10^6)m^{-1}$ and

$v_1^* = 10\,ms^{-1}$. As can be seen from figure 2 the model then has three real steady states, from which two are stable. The numerical values are shown in table 1, together with the values of the Egger model.

Table 1

	our model			Egger model		
	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_3$
E1	9.33	1.76	0.13	9.55	1.47	0.09
E2	4.49	3.34	1.10	2.75	2.80	1.45
E3	2.26	-2.52	1.55	1.36	-1.69	1.73

The differences are entirely due to the different values of β_* in both models.

In figure 5 the streamfunction patterns of the stable equilibria E1 and E3 of our model are shown.

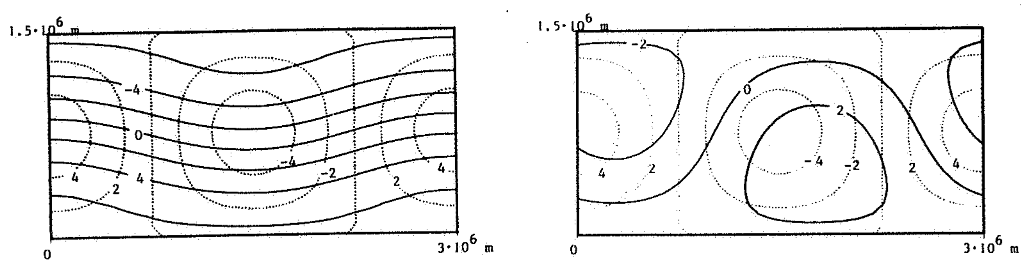


Figure 5. Streamfunction patterns ($10^6 m^2 s^{-1}$, solid lines) for E1 and E3. The dotted lines represents the orography ($10^2 m$).

The equilibrium E1 corresponds to a strong zonal flow, while the flow pattern of E3 has a large meridional component. Both equilibria resemble well-known large scale preference states in the atmosphere (Charney & DeVore, 1979).

The two attraction domains of the stable steady states in the three-dimensional phase space are separated by a two-dimensional hypersurface, called the separatrix. Its position can only be found by numerical analysis. In figure 6 sections of the separatrix with various planes v_3=constant are presented.

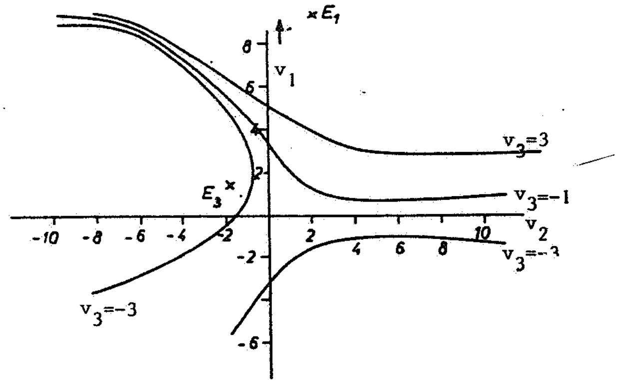


Figure 6. Intersection curves of the separatrix with various planes v_3=constant.

From our model results it appears that, for large values of τ, nearly all trajectories approach one of the two stable steady states, depending on the initial conditions. Exceptions are trajectories starting at points which lie on the separatrix; they spiral to the unstable steady state. However in the phase space these points form a set of measure zero, which is not of dynamical significance.

So far we have not considered the effect of the synoptic forcing terms $F_i (i=1,2,3)$ in the equations (2-23) - (2-25). As argued in the introduction a crude parametrization of this effect can be attained by letting the F_i stochastic forcing terms. In a symbolic form the new model reads

$$\dot{x} = f(x) + \eta(t), \tag{5-1}$$

where $f(x)$ is the deterministic vectorfield and the components of η are stochastic terms.

The stochastic forcing will drive the system away from the steady states, but these states will continue to attract the trajectories. Therefore the latter will become erratic paths in the phase space. The system will be kicked around and thus be capable of passing from the one- to the other attraction domain.

We ask for the typical residence time of the system in a certain attraction domain, and for the probability that at a certain time the system will be in a certain part of the phase space. Since we argue that the equations (5-1) resemble in some way the dynamical climatology of the atmosphere, the first quantity is a measure of the typical life of a large scale preference state. The probability distribution over the phase space is an analogy with the relative frequency distribution of the possible flow states.

Since the model is no longer deterministic, statistical methods have to be involved in order to describe the behaviour of the solutions. This will be done in a next report.

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