stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 175/82 AUGUSTUS

H.A. WILBRINK & A.E. BROUWER

A CHARACTERIZATION OF TWO CLASSES OF SEMI PARTIAL GEOMETRIES BY THEIR PARAMETERS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 51E30

A characterization of two classes of semi partial geometries by their parameters *)

by

H.A. Wilbrink & A.E. Brouwer

ABSTRACT

We show that, under mild restrictions on the parameters, semi-partial geometries with $\mu = \alpha^2$ or $\mu = \alpha(\alpha+1)$ are determined by their parameters.

KEY WORDS & PHRASES: *Semi-partial geometry, partial geometry, strongly regular graph*

---

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let X be a (finite) nonempty set and L a set of subsets of X. Elements of X are called *points*, elements of L are called *lines*. The pair (X,L) is called a *partial linear space* if any two distinct points are on at most one line.

Two distinct points x and y are called *collinear* if there exists $L \in L$ such that $x,y \in L$, *noncollinear* otherwise. Two distinct lines L and M are called *concurrent* if $|L \cap M| = 1$.

We write $x \sim y$ ($x \not\sim y$) to denote that x and y are collinear (noncollinear). Similarly $L \sim M$ ($L \not\sim M$) means $|L \cap M| = 1$ ($|L \cap M| = 0$).

If $x \sim y$ ($L \sim M$) we denote by xy (LM) the line (point) incident with x and y (L and M).

For a nonincident point-line pair (x,L) we define:

$$[L,x] := \{y \in X | y \in L,\ y \sim x\},$$

$$[x,L] := \{M \in L | x \in M,\ L \sim M\}.$$

Given positive integers s,t,α,μ, the partial linear space (X,L) is called a *semi-partial geometry* (s.p.g) with parameters s,t,α,μ if:

(i) every line contains s+1 points,

(ii) every point is on t+1 lines,

(iii) for all $x \in X$, $L \in L$, $x \notin L$ we have $|[x,L]| \in \{0,\alpha\}$,

(iv) for all $x,y \in X$ with $x \not\sim y$ the number of points z such that $x \sim z \sim y$ equals μ.

A semi-partial geometry which satisfies $|[x,L]| = \alpha$ for all $x \in X$, $L \in L$ with $x \notin L$, or equivalently which satisfies $\mu = \alpha(t+1)$, is also called a *partial geometry* (p.g).

The *point-graph* of the partial linear space (X,L) is the graph with vertex set X, two distinct vertices x and y being adjacent iff $x \sim y$. The point-graph of a semi-partial geometry is easily seen to be strongly regular. Let (X,L) be a semi-partial geometry.

For $x,y \in X$, $x \not\sim y$ we define

$$[x,y] := \{L \in \mathcal{L} \mid x \in L,\ |[L,y]| = \alpha\}.$$

It is easy to see that $\alpha = s+1$ iff any two distinct points are collinear iff $(X,\mathcal{L})$ is a Steiner system $S(2,s+1,|X|)$. We shall always assume $s \geq \alpha$, hence noncollinear points exist.

Let $x,y \in X$, $x \not\sim y$. Then $\mu = |[x,y]|\alpha$ and $|[x,y]| \geq |[x,L]| = \alpha$ if $L \in [y,x]$. Hence, $\mu \geq \alpha^2$ and

$$(*) \qquad \mu = \alpha^2 \iff \forall K \in [x,y],\ L \in [y,x]:\ K \sim L,$$

$$(*\,*) \qquad \mu = \alpha(\alpha+1) \iff \text{every line } K \in [x,y] \text{ intersect every line } L \in [y,x] \text{ but one.}$$

This is the basic observation we use in showing that, under mild restrictions on the parameters, semi partial geometries with $\mu = \alpha^2$ or $\mu = \alpha(\alpha+1)$ satisfy the Diagonal Axiom (D).

(D) : Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$.
Then also $x_2 \sim x_4$.

From DEBROEY [1], it then follows that such a semi-partial geometry is known.

2. SEMI-PARTIAL GEOMETRIES WITH $\mu = \alpha^2$.

Our first theorem deals with the case $\alpha = 1$, $\mu = 1$.

THEOREM 1. *Every strongly regular graph with parameters* $(n,k,\lambda,\mu = 1)$ *is the point-graph of a s.p.g. with* $s = \lambda+1$, $t = \frac{k}{\lambda+1} - 1$, $\alpha=1$, $\mu=1$.

PROOF. Let (X,E) be a strongly regular graph with $\mu = 1$, and let $x \in X$. Since two nonadjacent points in $\Gamma(x)$ cannot have a common neighbour in $\Gamma(x)$, the induced subgraph on $\Gamma(x)$ in the union of cliques. This induced subgraph has valency λ, so it is the union of $\frac{k}{\lambda+1}$ cliques of size $\lambda+1$. □

Next we deal with the case $\alpha = 2$, $\mu = 4$.

THEOREM 2. *Let* (X,L) *be a s.p.g. with parameters* s,t, $\alpha = 2$, $\mu = 4$. *Then* (X,L) *satisfies* (D).

PROOF. Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$. If $x_2 \not\sim x_4$, then we can apply (*) to the points x_2 and x_4. Since $x_1x_4 \in [x_4,x_2]$ and $x_2x_3 \in [x_2,x_4]$, x_1x_4 and x_2x_3 intersect in a point $\neq x_2,x_3$. Now $3 \leq |[x_1,x_2x_3]| \leq \alpha=2$, a contradiction. □

Let U be a set containing t+3 elements. Then we denote by $U_{2,3}$ the s.p.g. which has as points the 2-subsets of U, as lines the 3-subsets of U together with the natural incidence.

The parameters are $s=2$, t, $\alpha=2$, $\mu=4$.

DEBROEY [1] showed that a s.p.g. with $t>1$, $\alpha=2$, $\mu=4$ satisfying (D) is isomorphic to a $U_{2,3}$. Hence we have the following theorem.

THEOREM 3. *A s.p.g. with* $t>1$, $\alpha=2$, $\mu=4$ *is isomorphic to a* $U_{2,3}$. □

REMARK. A s.p.g. with $t=1$, $\alpha=2$, $\mu=4$ is isomorphic to the geometry of edges and vertices of the complete graph K_{s+2}.

We now consider the case $\alpha>2$. For the remainder of this section let (X,L) be a s.p.g with $\alpha>2$ and $\mu = \alpha^2$.

LEMMA 1. *Let* $x \in X$, $L \in L$, $x \notin L$ *such that* $[L,x] = \{z_1,\ldots,z_\alpha\}$. *Let* M *be a line through* z_1 *intersecting* xz_2 *in a point* $u \neq x,z_2$. *Suppose there exists* $y \in L$, $y \neq z_1,\ldots,z_\alpha$ *with* $u \not\sim y$. *Then* M *intersects* xz_i *for all* $i = 1,\ldots,\alpha$ (*see figure* 1).

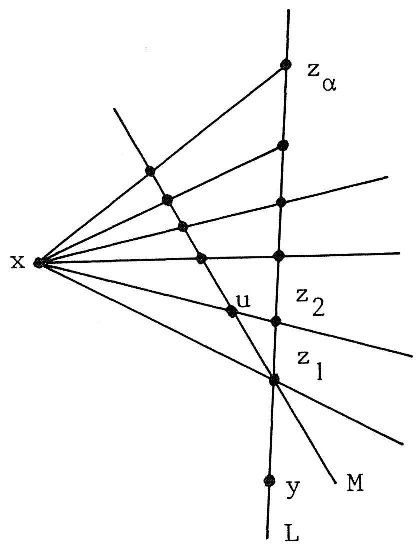

Figure 1.

PROOF. By (*) applied to x and y, the α lines $L = L_1, L_2, \ldots, L_\alpha$ of $[y,x]$ intersect the α lines $xz_1, \ldots, xz_\alpha$ of $[x,y]$. In particular $L_1, \ldots, L_\alpha$ intersect xz_2. Hence $[y,u] = [y,x] = \{L_1, \ldots, L_\alpha\}$.

Since $M \in [u,y]$, M intersects $L_1, \ldots, L_\alpha$ in points $v_1 = z_1, v_2, \ldots, v_\alpha$ respectively. If $x \sim v_i$ for all i, then the $\alpha+1$ points $u, v_1 v_2, \ldots, v_\alpha$ on M are all collinear with x, a contradiction. Hence $x \not\sim v_i$ for some i. Since L_i intersects $xz_1, \ldots, xz_\alpha$ it follows that $[x,v_i] = [x,y] = \{xz_1, \ldots, xz_\alpha\}$. Since $M \in [v_i, x]$, M intersects all lines in $[x,v_i]$. □

LEMMA 2. *Let* $x \in X$, $L \in L$, $x \notin L$ *such that* $[L,x] = \{z_1, \ldots, z_\alpha\}$. *Let* M *be a line through* z_1 *intersecting* xz_2 *in a point* $u \neq x, z_2$. *If* $s > \alpha$, *then* M *intersects* xz_i *for all* $i = 1, \ldots, \alpha$.

PROOF. Assume that M intersects xz_i, $i = 1, \ldots, \beta$ $(2 \le \beta < \alpha)$ in points $u_1 = z_1$, $u_2 = u, \ldots, u_\beta$ respectively and does not intersect $xz_{\beta+1}, \ldots, xz_\alpha$. Take $y \in L$, $y \neq z_1, \ldots, z_\alpha$. By lemma 1 $y \sim u_i$, $i = 1, \ldots, \beta$.

Since $|[M,x]| = \alpha$, there is a $v \in M$ such that $v \sim x$, $v \neq u_1, \ldots, u_\beta$. Also $v \sim z$ for all $z \in \bigcup_{i=1}^{\beta} [yu_i, x]$, for if $v \not\sim z$ for some $z \in [yu_i, x]$, then $vx \in [v,z]$ and $yu_i \in [z,v]$. Hence $vx \sim yu_i$ and so yu_i intersects the $\alpha+1$ lines $xv, xz_1, \ldots, xz_\alpha$ through x, a contradiction. The points of $\bigcup_{i=1}^{\beta} [yu_i, x]$ are therefore on the α lines $M = vz_1, vz_2, \ldots, vz_\alpha$ of $[v,y]$.

Since $s > \alpha$ we can take $y' \in L$ such that $y' \neq y, z_1, \ldots, z_\alpha$.

Now if $z \in [yu_2, x]$, then $z \sim y'$. Indeed, as shown z is on some vz_i and since vz_i intersects at most $\alpha-1$ of the lines $xz_1, \ldots, xz_\alpha$, it follows from Lemma 1 that every point of intersection of vz_i and a line xz_j, so in particular z, is collinear with y'.

But now we have $|[yu_2, y']| \ge |[yu_2, x] \cup \{y\}| = \alpha+1$, a contradiction. □

LEMMA 3. *Let* $x \in X$, $L \in L$, $x \notin L$ *such that* $[L,x] = \{z_1, \ldots, z_\alpha\}$. *If* $s > \alpha$, *then every line* M *not through* x *which intersects two lines of* $[x,L] = \{xz_1, \ldots, xz_\alpha\}$ *also intersects* L *and all lines of* $[x,L]$.

PROOF. The number of pairs $(u,v) \neq (z_1, z_2)$ such that $u \in xz_1$, $v \in xz_2$, $u, v \neq x$, $u \sim v$ equals $s(\alpha-1)-1$. Every line $M \neq xz_1, \ldots, xz_\alpha$ which intersects L and $xz_1, \ldots, xz_\alpha$ gives rise to such a pair (u,v). By (*) and lemma 2 the number of these lines equals $(s+1-\alpha)(\alpha-1) + \alpha(\alpha-2) = s(\alpha-1)-1$. □

Let $L_1, L_2 \in L$ intersect in a point x. If L is any line intersecting L_1 and L_2 not in x, we let $L_3, L_4, \ldots, L_\alpha$ be the other lines in [x,L]. By lemma 3, $L_3, L_4, \ldots, L_\alpha$ are independent of the choice of L. Put

$$L(L_1,L_2) := \{L_1,L_2,\ldots,L_\alpha\} \cup \{L \in L \mid L \sim L_1,L_2, LL_1 \neq x \neq LL_2\},$$

$$X(L_1,L_2) := \bigcup_{L \in L(L_1,L_2)} L$$

<u>LEMMA 4</u>. *Let* $L_1, L_2 \in L$, $L_1 \sim L_2$. *If* $s > \alpha$, *then* $\langle L_1,L_2\rangle := (X(L_1,L_2), L(L_1,L_2))$ *is a partial geometry (in fact a dual design) with parameters* $\tilde{s} = s$, $\tilde{t} = \alpha-1$, $\tilde{\alpha} = \alpha$.

<u>PROOF</u>. Clearly two points are on at most one line and each line contains s+1 points. Using (*) and Lemma 3 it follows immediately that every point $x \in X(L_1,L_2)$ is on α lines of $L(L_1,L_2)$ so $\tilde{t}+1 = \alpha$. It also follows immediately that any two lines of $L(L_1,L_2)$ intersect, hence $\tilde{\alpha} = \tilde{t}+1 = \alpha$. □

Notice that for $M_1, M_2 \in L(L_1,L_2)$, $M_1 \neq M_2$, $M_1 \sim M_2$ we have $\langle M_1,M_2\rangle = \langle L_1,L_2\rangle$. Notice also that for any two noncollinear points x and y of $\langle L_1,L_2\rangle$ there are $\tilde{\mu} = \tilde{\alpha}(\tilde{t}+1) = \alpha^2 = \mu$ points $z \in X(L_1,L_2)$ collinear with both x and y, i.e. the common neighbours of x and y in (X,L) are the common neighbours of x and y in $\langle L_1,L_2\rangle$.

<u>THEOREM 4</u>. *Let* (X,L) *be a s.p.g. with parameters* $s,t,\alpha(>2)$, $\mu = \alpha^2$. *If* $s > \alpha$ *and* $t \geq \alpha$, *then* (X,L) *satisfies* (D).

<u>PROOF</u>. Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$.

Suppose $x_2 \not\sim x_4$. Since $x_2 \sim x_1 \sim x_4$ it follows that

$$x_1 \in \langle x_4x_3, x_2x_3\rangle \qquad (\dagger)$$

In (X,L) there are $\lambda = s-1 + (\alpha-1)t$ points collinear with both x_1 and x_3. In $\langle x_4x_3,x_2x_3\rangle$ there are $\tilde{\lambda} = \tilde{s}-1 + (\tilde{\alpha}-1)\tilde{t} = (s-1) + (\alpha-1)^2$ points collinear with both x_1 and x_3. Since $t \geq \alpha = \tilde{t}+1$ it follows that $\tilde{\lambda} < \lambda$ and so there exists $x_5 \in X \backslash X(x_4x_3,x_2x_3)$ such that $x_1 \sim x_5 \sim x_3$. Now application of

(†)

to x_1,x_5,x_3,x_4 yields $x_5 \sim x_4$,
to x_1,x_2,x_3,x_5 yields $x_5 \sim x_2$,
to x_4,x_1,x_2,x_5 yields $x_2 \sim x_4$. □

DEBROEY [1] showed that a s.p.g. with parameters $s,t,\alpha(>2)$, $\mu = \alpha^2$ satisfying (D) is of the following type: the "points" are the lines of PG(d,q), the "lines" are the planes in PG(d,q) for some prime power q and $d \in \mathbb{N}$, $d \geq 4$. In this case $s = q(q+1)$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q+1$, $\mu = (q+1)^2$.

THEOREM 5. *Let* (X,L) *be a s.p.g. with parameters* $s,t,\alpha(>2)$, $\mu = \alpha^2$. *If* $s > \alpha$ *and* $t \geq \alpha$, *then* (X,L) *is isomorphic to the s.p.g. consisting of the lines and planes in* PG(d,q). *In particular* $s = q(q+1)$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q+1$, $\mu = (q+1)^2$.

The only interesting case remaining is $s = \alpha$. Now if (X,E) is a Moore graph of valency r, i.e. a strongly regular graph with $\lambda = 0$, $\mu = 1$, then $(X,\{\Gamma(x) \mid x \in X\})$ is easily seen to be a s.p.g. with parameters $s = t = \alpha = r-1$, $\mu = (r-1)^2$ (here $\Gamma(x) = \{y \in X \mid (x,y) \in E\}$). The point graph of this s.p.g. is the complement of (X,E). Such a s.p.g. does not satisfy (D) for $r > 2$. From the following theorem follows immediately that a s.p.g. with $\mu = \alpha^2$, $s = \alpha$ is necessarily of this type.

THEOREM 6. *Let* (X,L) *be a s.p.g. with* $t \geq \alpha$, $\mu = \alpha^2$ *and* $s = \alpha$. *Then* $t = \alpha$.

PROOF. Let $x,y \in X$, $x \not\sim y$. Let $[x,y] = \{L_1,\ldots,L_\alpha\}$, $[y,u] = \{M_1,\ldots,M_\alpha]$ and put $z_{ij} = L_iM_j$, $i,j = 1,\ldots,\alpha$ (see figure 2).

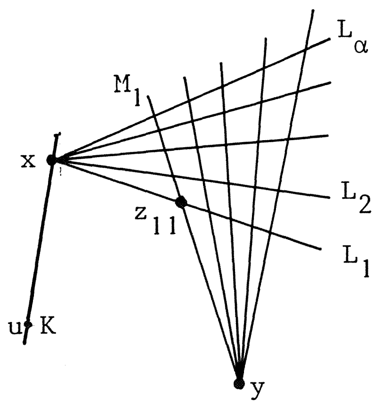

Figure 2.

The number of $(z_{ij},z_{k\ell})$ with $i \neq k$, $j \neq \ell$, $z_{ij} \sim z_{k\ell}$ equals $\alpha^2 \cdot (\alpha-1)(\alpha-2)$. Now let K be a line through x, $K \neq L_1,\ldots,L_\alpha$, and let u be a point on K, $u \neq x$.

Then u is collinear with $(\alpha-1)$ of the α points $z_{i,1},\ldots,z_{i,\alpha}$, for $i = 1,\ldots,\alpha$. Since $u \not\sim y$, u is collinear with all of $z_{1,j},\ldots,z_{\alpha,j}$ or with none, for $j = 1,\ldots,\alpha$.

It follows that there are α lines through u intersecting $(\alpha-1)$ of the α lines $M_1,\ldots,M_\alpha$. Hence each point $u \neq x$ on K gives rise to $\alpha(\alpha-1)(\alpha-2)$ pairs $(z_{ij},z_{k\ell})$ as described, so K gives rise to all $\alpha^2(\alpha-1)(\alpha-2)$ pairs $(z_{ij},z_{k\ell})$.

Suppose $t > \alpha$, then we can find two such lines K and K'. It follows that for $u \in K$, the α lines through u intersecting $(\alpha-1)$ of the α lines $M_1,\ldots,M_\alpha$ also intersect K'. But now $|[u,K']| = \alpha+1$, a contradiction. □

3. SEMI-PARTIAL GEOMETRIES WITH $\mu = \alpha(\alpha+1)$.

In this section (X,L) is a semi-partial geometry with parameters s,t,α and $\mu = \alpha(\alpha+1)$.

If $x,y \in X$, $x \not\sim y$ we shall always denote the $\alpha+1$ lines in $[x,y]$ by $K_1,\ldots,K_{\alpha+1}$, and the $(\alpha+1)$ lines in $[y,x]$ by $L_1,\ldots,L_{\alpha+1}$. By (**) we can number these lines in such a way that $K_i \cap L_i = \emptyset$, $i = 1,\ldots,\alpha+1$ and $K_i \cap L_j \neq \emptyset$, $i,j = 1,\ldots,\alpha+1$, $i \neq j$ (see figure 3).

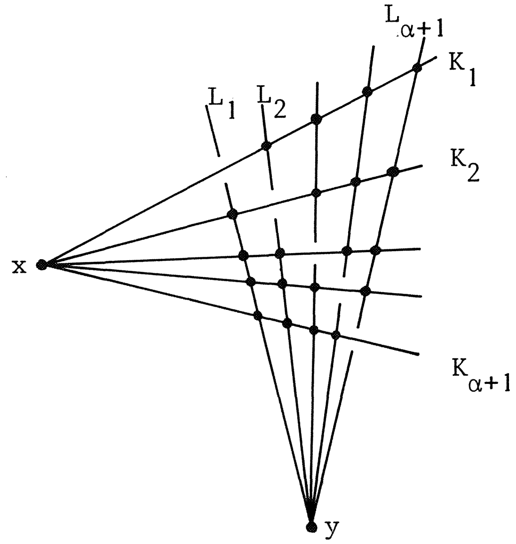

Figure 3.

Again our aim will be to show that the diagonal axiom (D) holds. We first deal with the case $\alpha = 2$.

LEMMA 5. *If* $\alpha = 2$ *and* $t > s$, *then a set of* 3 *collinear points not on one line can be extended to a set of* 4 *collinear points no 3 on a line.*

PROOF. Let x, a and b be three distinct collinear points not on one line. There are t-1 lines $\neq$ xa,ab through a and on each of those lines there is a point $y_i \sim b$, $y_i \neq a$, $i = 1,\dots,t-1$. Suppose $y_i \not\sim x$ for all $i = 1,\dots,t-1$. Now for each $i = 1,\dots,t-1$, $ay_i \not\sim xb$ (for otherwise $|[a,xb]| \geq 3$) and $by_i \not\sim xa$. Also $xa,xb \in [x,y_i]$ and $ay_i,by_i \in [y_i,x]$. Hence, by (**) there is a third line through y_i intersecting xa and xb in points u_i and v_i respectively. Clearly $u_i \neq u_j$ if $i \neq j$, for $u_i = u_j$ implies $x,v_i,v_j \in [u_i,xb]$. Thus xa contains $t+1 > s+1$ points (namely $x,a,u_1,\dots,u_{t-1}$), a contradiction. □

LEMMA 6. *Suppose* $\alpha = 2$. *If* x_1,x_2,x_3,x_4 *are four distinct collinear points, no three on a line, then no point can be collinear with exactly three of these four points.*

PROOF. Suppose x_5 is collinear with x_2,x_3,x_4 and $x_1 \not\sim x_5$. Clearly $x_5 \notin x_2x_3, x_2x_4,x_3x_4$. Hence $\{x_1x_2,x_1x_3,x_1x_4\} = [x_1,x_5]$ and $\{x_5x_2,x_5x_3,x_5x_4\} = [x_5,x_1]$ so x_5x_2 has to intersect x_1x_3 or x_1x_4 by (**). But then $|[x_2,x_1x_3]|$ or $|[x_2,x_1x_4]| > 2$, a contradiction. □

LEMMA 7. *Same hypothesis as in lemma 6. Then the only points collinear with exactly two points of* $\{x_1,x_2,x_3,x_4\}$ *are the points on the lines* x_1x_j, $i \neq j$.

PROOF. Suppose $x_5 \sim x_1,x_4$ and $x_5 \not\sim x_2,x_3$, $x_5 \notin x_1x_4$ (see figure 4).

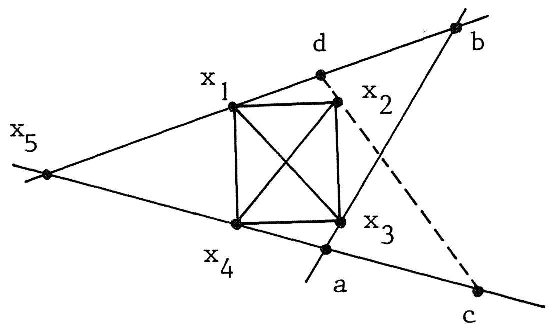

Figure 4.

Apply (**) to x_3 and x_5 to get a line ab through x_3 with $a \in x_5x_4$, $b \in x_5x_1$. Similarly (**) applied to x_5 and x_2 gives us a line cd through x_2 with $c \in x_5x_4$, $d \in x_5x_1$. Clearly $b \not\sim c$ so we can apply (**) to b and c. It follows that $ab \cap cd = \emptyset$. Also $x_2 \not\sim a$ and (**) applied to x_2 and a yields: $ab \cap cd \neq \emptyset$ or $ab \cap x_2x_4 \neq \emptyset$. Hence $ab \cap x_2x_4 \neq \emptyset$, a contradiction since $\{x_2,x_4\} = [x_2x_4,x_3]$. □

THEOREM 7. *If* (X,L) *is a s.p.g with parameters* $s,t,\alpha = 2$, $\mu = 6$ and $t>s$, *then* (X,L) *satisfies* (D).

PROOF. Let x_1,x_2,x_3 and x_4 be four distinct points no three on a line such that $x_4{\sim}x_1{\sim}x_2{\sim}x_3{\sim}x_4{\sim}x_2$. By Lemma 5 there exists $x_5{\sim}x_2,x_3,x_4$.

By Lemmas 6 and 7 $x_1{\sim}x_3,x_5$. □

REMARK. If (X,L) is a s.p.g but not a partial geometry, then $t \geq s$ (see DEBROEY & THAS [2]). Using the integrality conditions for the multiplicities of the eigenvalues of a strongly regular graph it follows that a s.p.g with s=t, α=2 and μ=6 satisfies $(8s^2-24s+25) \mid \{8(s+1)(2s^3-9s^2+19s-30)\}^2$. From this one easily deduces an upper bound for s. The remaining cases were checked by computer and only s=t=28 survived. Thus, every s.p.g which is not a partial geometry satisfies (D) or has s=t=28 (and 103125 points).

We now turn to the case $\alpha \geq 3$. We shall make two additional assumptions in this case. The first assumption is $\alpha \neq 3$, the second assumption is $s \geq f(\alpha)$ where f is defined in Lemma 9. Notice that this bound on s is used only in the proof of Lemma 9.

LEMMA 8. *Let* $x,y \in X$, $x \not\sim y$ *and suppose* $[x,y] = [K_1,\ldots,K_{\alpha+1}]$, $[y,x] = [L_1,\ldots,L_{\alpha+1}]$ *such that* $K_i \cap L_i = \emptyset$, $i = 1,\ldots,\alpha+1$. *If* M *is a line intersecting* $\sigma \geq 1$ *lines of* $[x,y]$, $\tau \geq 1$ *lines of* $[y,x]$ *and* $\sigma < \tau$, *then* $\sigma = \alpha-1$ *and* $\tau = \alpha$.

PROOF. Since $\sigma < \tau$, there exists a point of intersection u of M with a line $L_i \in [y,x]$ such that u is not on one of the lines of $[x,y]$. Then $u \not\sim x$ and so, applying (**) to u and x, it follows that $M \in [u,x]$ intersects $\alpha-1$ of the α lines $K_1,K_2,\ldots,K_{i-1},K_{i+1},\ldots,K_{\alpha+1} \in [x,u]$. Thus $\alpha-1 \leq \sigma < \tau \leq \alpha$, which proves our claim. □

LEMMA 9. *Let* $x \in X$ *and* $L \in \mathcal{L}$ *such that* $x \notin L$ *and* x *is collinear with* α *points* $z_2, z_3, \ldots, z_{\alpha+1}$ *on* L. *Let* M *be a line through* $z_{\alpha+1}$ *meeting* xz_α *in a point* $u \neq x, z_\alpha$. *Suppose* $s \geq f(\alpha)$ *where* $f(4) = 12$, $f(5) = 16$, $f(6) = f(7) = 17$, $f(8) = 18$, $f(9) = 19$, $f(10) = 21$, $f(11) = 23$, $f(\alpha) = 2\alpha$ $(\alpha \geq 12)$. *Then* M *intersects at least* $\alpha-1$ *lines of* $[x,L]$.

PROOF. Suppose M does not meet at least two lines of $[x,L]$, xz_2 and xz_3, say. Since $s \geq 2\alpha$ we can find $y \in L$ such that $x \not\sim y \not\sim u$. Let $[x,y] = \{K_1, K_2 = xz_2, \ldots, K_{\alpha+1} = xZ_{\alpha+1}\}$ and $[y,x] = \{L_1 = L, L_2, L_3, \ldots, L_{\alpha+1}\}$ with $K_i \cap L_i = \emptyset$.

Looking at u and y we find that M intersects $\alpha-1$ of the α lines L_i, $i \neq \alpha$. Every point L_iM which is collinear with x is on a line K_j, $j \neq \alpha$. If $L_iM \sim x$ for these $\alpha-1$ i's, we find that M meets at least α of the lines $K_1, \ldots, K_{\alpha+1}$, hence at least $\alpha-1$ of the lines $K_2, \ldots, K_{\alpha+1}$, a contradiction. Let $t = L_iM$ be a point not collinear with x. Considering $x \not\sim t$ we see that M intersects $\alpha-1$ of the α lines in $[x,y] \backslash \{K_i\}$. This shows that $i = 2$ or 3, so there are at most two such points t, and that M meets $K_1, K_4, K_5, \ldots, K_{\alpha+1}$. Let $V = \{K_4M, K_5M, \ldots, K_\alpha M\}$ and count pairs (y,v), $y \in L$, $y \not\sim x$, $v \in V$, $v \sim y$. The number of such pairs is at least $(s-\alpha+1)(\alpha-5)$ (first choose $y, s-\alpha+1$ possibilities, then given y we can find $\alpha-3$ points $L_iM \sim x$ as above, possibly one on $K_1(y)$, and one is $z_{\alpha+1}$), and at most $(\alpha-3)(\alpha-2)$ (first choose v, then y). It follows that for $\alpha > 5$, $s \leq 2\alpha - 1 + \lfloor \frac{6}{\alpha-5} \rfloor$. Let $W = V \cup \{q,q'\} = \{w \in M \mid w \sim x\}$ and count pairs (y,w), $y \in L$, $y \not\sim x$, $w \in W$, $w \sim y$. This yields $(s-\alpha+1)(\alpha-4) \leq (\alpha-3)(\alpha-2) + 2(\alpha-1)$, hence $s \leq 2\alpha + \lfloor \frac{8}{\alpha-4} \rfloor$ if $\alpha > 4$. Above we saw that for any $y \in L$ with $x \not\sim y \not\sim u$, $K_1 = K_1(y)$ meets M. But if $s+1 > \alpha + (\alpha-2) + 2(\alpha-1) = 4\alpha-4$, we can find $y \in L$ such that $y \not\sim x$, u,q and q', a contradiction. Therefore we have $s < 4\alpha-4$. We now have obtained a contradiction for all $\alpha \geq 4$ and the lemma is proved. □

LEMMA 10. *Some hypotheses as in Lemma 9. Then* M *intersects exactly* $\alpha-1$ *lines of* $[x,L]$.

PROOF. Take $y \in L$, $y \not\sim x$ and let K_i and L_i be defined as before. Put $K := K_{\alpha+1}$ and let $A(x,L)$ be the set of lines $\neq K,L$ through $z_{\alpha+1}$ intersecting at least $\alpha-1$ lines of $[x,L]$, $A(y,K)$ the set of lines $\neq K,L$ through $z_{\alpha+1}$ intersecting at least $\alpha-1$ lines of $[y,K]$. Suppose a lines of $A(x,L)$ intersects $\alpha-1$ lines of $[x,L]$ and b lines of $A(x,L)$ intersect α lines of $[x,L]$. Counting the points $u \sim z_{\alpha+1}$ on $K_2, K_3, \ldots, K_\alpha$, such that $u \neq x, z_2, \ldots, z_\alpha$ yields $a(\alpha-2) + b(\alpha-1) = (\alpha-1)(\alpha-2)$. Hence $a = 0$ and $b = \alpha-2$ or $a = \alpha-1$ and $b = 0$. Thus $|A(x,L)| = \alpha-2$ or $\alpha-1$ according as every line in $A(x,L)$ intersects all lines or all but one line in $[x,L]$. A similar result holds for $A(y,K)$. Now $A(x,L) = A(y,K)$, for suppose $N \in A(x,L)$ then by Lemma 8, N intersects at least $\alpha-1$ lines of $[y,x]$, so at least $\alpha-2 \geq 2$ lines of $[y,K]$. Hence $N \in A(y,K)$ by Lemma 9. Similarly, $N \in A(y,k)$ implies $N \in A(x,L)$. Suppose $|A(x,L)| = \alpha-2$, i.e. there are $\alpha-2$ lines through $z_{\alpha+1}$ intersecting all lines of $[x,L] \cup [y,K]$. It follows that $K_2L_{\alpha+1} \not\sim z_{\alpha+1}$ so we can apply (**) to $K_2L_{\alpha+1}$ and $z_{\alpha+1}$. This shows that $L_{\alpha+1} \in [K_2L_{\alpha+1}, z_{\alpha+1}]$ intersects all $N \in A(y,K) \subseteq [z_{\alpha+1}, K_2L_{\alpha+1}]$, a contradiction, for $L_{\alpha+1} \sim N$ implies $|[y,N]| \geq$ $\geq \alpha+1$. □

<u>LEMMA 11</u>. *Let* $x \in X$, $L \in L$ *such that* x *is collinear with* α *points* $z_2, \ldots, z_{\alpha+1}$ *on* L. *Let* M *be a line through* $z_{\alpha+1}$ *intersecting* $\alpha-1$ *lines of* $[x,L]$ *and let* $y \in L$, $y \not\sim x$. *Then, if* $[x,y] = \{K_1(y), K_2 = xz_2, \ldots, K_{\alpha+1} = xz_{\alpha+1}\}$, M *intersects* $K_1(y)$.

<u>PROOF</u>. Suppose M does not intersect K_2, say. As shown in Lemma 10, M also intersects $\alpha-1$ lines of $[y, K_{\alpha+1}] = \{L_1 = L, L_2, \ldots, L_\alpha\}$. So M intersects at least one of $L_{\alpha-1}$ and L_α and since $\alpha \geq 4$, $L_2 \neq L_{\alpha-1}, L_\alpha$. Suppose M intersects $L_{\alpha-1}$ (L_α) in a point v. If $v \not\sim x$ then apply (**) to v and x. It follows that $M \in [v,x]$ intersects $K_1(y) \in [x,v]$ for M misses $K_2 \in [x,v]$. If $v \sim x$ then $v = L_{\alpha-1}K_i$ $(v = L_\alpha K_i)$ for some i. By Lemma 10 applied to x and $L_{\alpha-1}(L_\alpha)$ it follows that M intersects $K_1(y) \in [x, L_{\alpha-1}]$ $(K_1(y) \in [x, L_\alpha])$, for M does not intersect $K_2 \in [x, L_{\alpha-1}]$ $(K_2 \in [x, L_\alpha])$. □

<u>COROLLARY</u>. *The line* $K_1(y)$ *is the same for all* $y \in L$, $y \not\sim x$.

<u>LEMMA 12</u>. *Let* $x \in X$, $L \in L$ *such that* x *is collinear with* α *points* $z_2, z_3, \ldots, z_{\alpha+1}$ *on* L. *Put* $K_i = xz_i$, $i = 2, \ldots, \alpha+1$ *and let* K_1 *be defined by* $\{K_1, K_2, \ldots, K_{\alpha+1}\} = [x,y]$ *for any* $y \in L$, $y \not\sim x$. *Then every line which intersects* K_1 *and a* $K_i (i \neq 1)$ *not in* x, *intersects* L *and therefore exactly* α *lines of* $\{K_1, \ldots, K_{\alpha+1}\}$.

<u>PROOF</u>. Fix $i \in \{2, \ldots, \alpha+1\}$. The number of pairs (u,v) such that

$u \in K_1\backslash\{x\}$, $v \in K_i\backslash\{x\}$, $u\sim v$ equals $s(\alpha-1)$. If $y \in L$, $y \not\sim x$ and $[y,x] = \{L_1=L, L_2, \ldots, L_{\alpha+1}\}$, then each of the $\alpha-1$ lines $L_2, L_3, \ldots, L_{i-1}, L_{i+1}, \ldots, L_{\alpha+1}$ gives rise to such a pair (u,v). Each point z_j, $j = 2,3,\ldots,i-1,i+1,\ldots,\alpha+1$ is on $\alpha-1$ lines $\neq K_j, L$ which intersect α lines of $\{K_1,\ldots,K_{\alpha+1}\}$. They all intersect K_1 by Lemma 11 and no two miss the same K_k since otherwise some K_ℓ would be hit $\alpha+1$ times. Thus each point z_j, $j=2,3,\ldots,i-1,i+1,\ldots,\alpha+1$ gives rise to $(\alpha-2)$ pairs (u,v). Finally there are $(\alpha-1)$ pairs (u,v) with $v = z_i$. In all, the lines intersecting L contain $(s+1-\alpha)(\alpha-1) + (\alpha-1)(\alpha-2) + (\alpha-1) = s(\alpha-1)$, i.e. all, pairs (u,v). □

If in Lemma 12 we replace $L = L_1$ by a line L_j missing K_j, then it follows that every line intersecting two lines of $\{K_1,\ldots,K_{\alpha+1}\}$ not in x, intersects exactly α lines of $\{K_1,\ldots,K_{\alpha+1}\}$. Using this result and the foregoing lemmas we can now proceed as in the case $\mu = \alpha^2$. For any two intersecting lines L_1, L_2 we can define in an obvious way a partial geometry $\langle L_1,L_2\rangle = (X(L_1,L_2), \mathcal{L}(L_1,L_2))$, now with parameters $\tilde{s} = s$, $\tilde{t} = \alpha$, $\tilde{\alpha} = \alpha$ (so $\langle L_1,L_2\rangle$ is an $(\alpha+1)$-net of order $s+1$). Again $\tilde{\mu} = \tilde{\alpha}(\tilde{t}+1) = \alpha(\alpha+1) = \mu$, so with the same proof as the proof of Theorem 4 we have the following theorem.

THEOREM 8. *Let* $(X,\mathcal{L})$ *be a s.p.g. with parameters* $s,t,\alpha,\mu = \alpha(\alpha+1)$. *If* $\alpha \geq 4$, $s \geq f(\alpha)$ (f as in Lemma 9) *and* $t \geq \alpha+1$ (i.e. *if* $(X,\mathcal{L})$ *is not a p.g.*), *then* $(X,\mathcal{L})$ *satisfies* (D).

Fix a $(d-2)$-dimensional subspace S of PG(d,q), q a prime power, $d \in \mathbb{N}$. Then with the lines of PG(d,q) which have no point with S in common as "points" and with the planes of PG(d,q) intersecting S in exactly one point as "lines" and with the natural incidence relation, one obtains a s.p.g. with parameters $s = q^2-1$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q$, $\mu = q(q+1)$.

DEBROEY [1] showed that a s.p.g. with parameters $s,t,\alpha \geq 2$, $\mu = \alpha(\alpha+1)$ and satisfying (D) is of this type. Combining this result with Theorems 7 and 8 we arrive at the following theorem.

THEOREM 9. *Let* $(X,\mathcal{L})$ *be a s.p.g. with parameters* $s,t,\alpha,\mu = \alpha(\alpha+1)$ *which is not a p.g.. If* $\alpha = 2$ *and not* $s = t = 28$ *or if* $\alpha \geq 4$ *and* $s \geq f(\alpha)$, *then* $(X,\mathcal{L})$ *is isomorphic to a s.p.g. consisting of the lines in* PG(d,q) *missing a given* $(d-2)$*-dimensional subspace of* PG(d,q) *and the planes inter-*

secting this subspace in one point. In particular $s = q^2-1$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q$, $\mu = q(q+1)$ *for some prime power* q *and* $d \in \mathbb{N}$ *and any s.p.g. with these parameters with* $q \neq 3$ *and* $d \geq 4$ *is of this type.*

REFERENCES

[1] DEBROEY, I., *Semi partial geometries satisfying the diagonal axiom,* J. Geometry, vol 13/2 (1979) 171-190.

[2] DEBROEY, I. & J.A. THAS, *On semi partial geometries,* J. Comb. Th. (A) 25 (1978) 242-250.

MC NR

35221