Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

L.C. van der Gaag

The certainty factor model and its basis in probability theory

Computer Science/Department of Software Technology Report CS-R8816 April

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

The Certainty Factor Model and its Basis in Probability Theory

L.C. van der Gaag
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Many present-day rule-based expert systems offer some means for modelling and manipulating imprecise knowledge. The certainty factor model proposed by the authors of the MYCIN system is such a means. Since its introduction in the seventies the model has enjoyed wide-spread use in EMYCIN and similar expert system shells, not in the least because of its computational simplicity. At the same time the model has been subject to severe criticism due to its ad hoc character. Although the authors of the model have suggested a basis in Bayesian probability theory, they have not presented a theoretical treatment of the model. In this paper, the probabilistic foundation of the model is investigated. We show that some parts of the model are consistent with the basis in probability theory and some parts are not.

1980 Mathematics Subject Classification (1985): 68Txx, 60-xx.
1987 CR Categories: I.2.3. [**Deduction and Theorem Proving**]: uncertainty, I.2.5. [**Programming Languages and Software**]: expert system tools and techniques.

Key Words & Phrases: expert systems, plausible reasoning, certainty factor model, probability theory

CONTENTS

Report CS-R8816
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

1. Introduction

In real-life domains most expert knowledge is of an imprecise nature. When building an expert system for an environment in which such imprecise knowledge has to be employed, the system has to capture the uncertainties that go with the represented pieces of knowledge. For a long time Bayesian probability theory has been the only quantitative approach to manipulating uncertainty. The creators of the MYCIN system E.H. Shortliffe and B.G. Buchanan, however, have observed that Bayesian probability theory cannot be applied in rule-based top-down reasoning expert systems in a straightforward manner. They have developed an ad hoc model for manipulating uncertainty, called the *certainty factor model* [1]. Since its introduction in the seventies the model has been implemented in a large number of rule-based expert systems and expert system shells. Part of the success of the model can be accounted for by its computational simplicity. At the same time the model has been criticised severely because of its ad hoc character. Although Shortliffe and Buchanan have suggested a theoretical foundation for the model in Bayesian probability theory, they have not provided a thorough justification for their model. In this paper we show which part of the model is consistent with the probabilistic basis suggested by Shortliffe and Buchanan and which part is not. This is not the first paper examining the relationship between the certainty factor model and Bayesian probability theory. J.B. Adams has examined the probabilistic basis of the model as well, [2]. In their paper [3], B.P. Wise and M. Henrion suggest some properties that are implicitly assumed in the model. We will comment on these papers. D. Heckerman in [4] and M. Ishizuka, K.S. Fu and J.T.P. Yao in [5] have presented counterproposals for some parts of the model. As our purpose is to examine whether the original model is consistent with the probabilistic basis suggested by Shortliffe and Buchanan, we will not discuss these counterproposals.

We conclude this section with an introductory description of the notions that play an important role in the certainty factor model. The reader who is already acquainted with the model will notice that we have adopted another notational convention than the one used in [1]; it is emphasized that we have not departed from the intended meaning of the model, but merely have provided a more precise notation. A more elaborate introduction to the model and a motivation of our notational convention can be found in [6].

In a rule-based top-down reasoning expert system applying the certainty factor model for the manipulation of uncertainty, there are three major components:

(1) *Production rules* and associated *certainty factors*. Basically an expert in the domain in which the expert system is to be used models his knowledge of the field in a set of production rules of the form $e \rightarrow h$. The left-hand side e of a production rule is a positive Boolean combination of conditions, i.e. e does not contain any negations. Without loss of generality we assume that e is a conjunction of disjunctions of conditions. Throughout this paper e as well as its constituting parts will be called *(pieces of) evidence*. In general the right-hand side h of a production rule is a conjunction of conclusions. In this paper we restrict ourselves to single-conclusion production rules; notice that this restriction is not an essential one from a logical point of view. Henceforth, a conclusion will be called a *hypothesis*.
An expert associates with the hypothesis h in each production rule $e \rightarrow h$, a (real) number $CF(h,e,e \rightarrow h)$, quantifying the degree in which the observation of evidence e confirms the hypothesis h. The values $CF(x,y,z)$ of the partial function CF are called *certainty factors*; $CF(x,y,z)$ should be read as 'the certainty factor of x given y and the derivation z of x from y'. In the sequel we will use the more suggestive notation $CF(h \dashv e, e \rightarrow h)$. Certainty factors range from -1 to $+1$. A certainty factor greater than zero is associated with a hypothesis h given some evidence e if the hypothesis is confirmed in some degree by the observation of this evidence; the certainty factor $+1$ indicates that the occurrence of evidence e completely proves the hypothesis h. A negative certainty factor is suggested if the observation of evidence e disconfirms the hypothesis h. A certainty factor equal to zero is suggested by the expert, if the observation of evidence e does not influence the confidence in the hypothesis h.

(2) *User-supplied data* and associated certainty factors. During a consultation of the expert system, the user is asked to supply actual case data. The user attaches a certainty factor $CF(e \dashv u, u \rightarrow e)$ to every piece of evidence e he supplies the system with. In order to be able to treat production rules and user-supplied data uniformly we assume the set of production rules supplied by the expert to be augmented with fictitious production rules $u \rightarrow e$, where u represents the user's de facto knowledge and e a piece of user-supplied evidence.

(3) A (*top-down*) *inference engine* and a (*bottom-up*) *scheme for propagating uncertainty*. Top-down inference is a goal-directed reasoning technique in which the production rules are applied exhaustively to prove one or more goal hypotheses. Due to the application of production rules, during the inference process several intermediary hypotheses are confirmed to some degree. The certainty factor to be associated with an intermediary hypothesis h is calculated from the certainty factors associated with the production rules that were used in deriving h. For the purpose of thus propagating uncertainty, an approximation function for certainty factors is defined.

2. Elementary Probability Theory

This section presents an introduction to elementary probability theory providing a point of departure for the remaining sections of this paper. We chose [7] as a basis for our introduction, though any other introductory textbook will suffice.

Many kinds of investigations may be characterized in part by conceptually repeated experimentation under essentially the same conditions. Each experiment terminates in an *outcome* which cannot be predicted with certainty prior to the performance of the experiment. The non-empty collection of all possible outcomes is called the *sample space*, usually denoted by Ω. In the sequel we assume the sample space Ω to be a finite set. A subset e of the sample space Ω is called an *event*. If upon the performance of the experiment the outcome is in e, it is said that event e has occurred. The event that the outcome is not in e is denoted by $\bar{e}$, called the *complement* of e. So, $\bar{e} = \Omega \setminus e$. The event that occurs iff both the events e_1 and e_2 occur, is called the *intersection* of e_1 and e_2, denoted by $e_1 \cap e_2$. The event occurring if e_1 occurs, e_2 occurs or both e_1 and e_2 occur, is called the *union* of e_1 and e_2, denoted by $e_1 \cup e_2$.

Definition 2.1. *Let Ω denote a sample space. The sets $e_1,...,e_n \subseteq \Omega$, $n \geqslant 1$, are called disjoint if $e_i \cap e_j = \varnothing$, $i \neq j$, $1 \leqslant i,j \leqslant n$. The events corresponding with the disjoint sets are called mutually exclusive events.*

A set function P is defined such that if e is a subset of the sample space, then $P(e)$ is a real number indicating the 'probability' that the outcome of the experiment is an element of e. P is defined axiomatically in Definition 2.2.

Definition 2.2. *Let Ω denote a sample space. If $P(e)$ is defined for each subset $e \subseteq \Omega$, such that*

(1) $P(e) \geqslant 0$, *and*
(2) $P(\Omega) = 1$, *and*
(3) $P(e_1 \cup e_2 \cup ...) = P(e_1) + P(e_2) + ...$, *where e_i, $i = 1,2,...$, are mutually exclusive events,*

then P is called a probability set function on Ω. For each subset $e \subseteq \Omega$, the number $P(e)$ is called the probability that event e will occur.

The following propositions give some properties of a probability set function. In each proposition, Ω denotes a sample space and P a probability set function on Ω. The propositions are presented without proof: they can easily be proven using Definition 2.2.

PROPOSITION 2.1. *For each* $e \subseteq \Omega$, $P(e) = 1 - P(\bar{e})$.

PROPOSITION 2.2. $P(\emptyset) = 0$.

PROPOSITION 2.3. *Let* $e_1, e_2 \subseteq \Omega$ *such that* $e_1 \subseteq e_2$. *Then* $P(e_1) \leqslant P(e_2)$.

PROPOSITION 2.4. *Let* $e_1, e_2 \subseteq \Omega$ *such that* $e_1 \subseteq e_2$. *If* $P(e_1) = P(e_2)$ *then for each* $e_3 \subseteq \Omega$ $P(e_1 \cap e_3) = P(e_2 \cap e_3)$.

In some situations one is interested only in those outcomes which are in a given non-empty subset e of Ω, for instance when several pieces of information concerning the final outcome become known in the course of the actual performance of the experiment. These pieces of information are called pieces of *evidence*. Let h be an event, called the *hypothesis*. Given that an event e occurs, i.e. given that the evidence e is observed, we are interested in the degree to which this information influences $P(h)$, the *prior* probability of the hypothesis h. The probability of h given e is defined in the next definition.

DEFINITION 2.3. *Let* Ω *denote a sample space, and* P *a probability set function on* Ω. *For each* $h, e \subseteq \Omega$ *with* $P(e) > 0$, *the conditional probability of* h *given* e, *notation:* $P(h \mid e)$, *is*

$$P(h \mid e) = \frac{P(h \cap e)}{P(e)}.$$

PROPOSITION 2.5. *Let* Ω *denote a sample space,* P *a probability set function on* Ω *and* e *a subset of* Ω *with* $P(e) > 0$. *The conditional probabilities given* e *define a probability set function on* Ω.

PROOF. We have to show that the axioms (1), (2) and (3) of Definition 2.2 hold for the conditional probabilities given e.

(1) $P(h \mid e) = \dfrac{P(h \cap e)}{P(e)} \geqslant 0$, since $P(h \cap e) \geqslant 0$ and $P(e) > 0$.

(2) $P(\Omega \mid e) = \dfrac{P(\Omega \cap e)}{P(e)} = \dfrac{P(e)}{P(e)} = 1$.

(3) $$P(h_1 \cup h_2 \cup \ldots \mid e) = \frac{P((h_1 \cup h_2 \cup \ldots) \cap e)}{P(e)} = \frac{P((h_1 \cap e) \cup (h_2 \cap e) \cup \ldots)}{P(e)} =$$
$$= \frac{P(h_1 \cap e) + P(h_2 \cap e) + \ldots}{P(e)} =$$
$$= \frac{P(h_1 \cap e)}{P(e)} + \frac{P(h_2 \cap e)}{P(e)} + \ldots = P(h_1 \mid e) + P(h_2 \mid e) + \ldots$$

for mutually exclusive events h_i, $i = 1, 2, \ldots$ ■

Proposition 2.5 allows us to state properties of the conditional probabilities given some e, analogous to the properties stated in the Propositions 2.1 - 2.4.

It seems natural to name an event h independent of an event e when $P(h \mid e) = P(h)$: the prior probability of event h is not influenced by the knowledge that event e has occurred. However, this intuitive definition is asymmetric in its arguments and is not applicable in the case that $P(e) = 0$. Therefore a slightly modified definition is given.

Definition 2.4. *The events $e_1,...,e_n$ are called (mutually) independent if*

$$P(e_{i_1} \cap ... \cap e_{i_k}) = P(e_{i_1}) \cdot ... \cdot P(e_{i_k})$$

for each subset $\{i_1,...,i_k\} \subseteq \{1,...,n\}$, $1 \leq k \leq n$, $n \geq 1$. *The events $e_1,...,e_n$ are called conditionally independent given an hypothesis h if*

$$P(e_{i_1} \cap ... \cap e_{i_k} | h) = P(e_{i_1} | h) \cdot ... \cdot P(e_{i_k} | h)$$

for each subset $\{i_1,...,i_k\} \subseteq \{1,...,n\}$.

Note that if the events h and e are independent and if $P(e) > 0$ we have $P(h | e) = \frac{P(h \cap e)}{P(e)} = \frac{P(h)P(e)}{P(e)} = P(h)$.

Theorem 2.1. (Bayes' Theorem) *Let P denote a probability set function on a sample space Ω. For each $h,e \subseteq \Omega$ with $P(e) > 0$, $P(h) > 0$,*

$$P(h | e) = \frac{P(e | h)P(h)}{P(e)}.$$

Proof. The conditional probability of a hypothesis h given some evidence e is according to Definition 2.3, $P(h | e) = \frac{P(h \cap e)}{P(e)}$. Similarly we have $P(e | h) = \frac{P(e \cap h)}{P(h)}$, and so it follows that $P(e | h)P(h) = P(h \cap e) = P(h | e)P(e)$ from which the desired result is derived. ∎

In the remainder of this paper, we take Ω to be a fixed sample space and P to denote a fixed probability set function on Ω.

3. A Formal Description of the Certainty Factor Model

The certainty factor function we have introduced informally in Section 1 is not the basic function of the certainty factor model: this certainty factor function is defined in terms of two basic measures of uncertainty, the *measures of belief* and *disbelief*. In turn these basic measures are based on Bayesian probability theory. In Section 3.2 we provide formal definitions of these two basic measures. When applying the certainty factor model in a rule-based top-down reasoning expert system, approximation functions are used to approximate function values of the measures of belief and disbelief. These approximation functions are defined in Section 3.3. Section 3.4 provides formal definitions of the certainty factor function and its approximation function.

3.1. Preliminaries

In the foregoing we have discussed in an informal manner some basic notions that play an important role in the certainty factor model. Before presenting the formal definitions of the basic measures of uncertainty we provide some preliminary definitions of these notions.

Definition 3.1. *Let $\mathscr{A}$ denote a finite set of atomic propositions. Let $\mathscr{E}$ denote the set of conjunctions of disjunctions of elements of $\mathscr{A}$, i.e. $\mathscr{E}$ contains elements of the form $\bigwedge_{i=1}^{n} (\bigvee_{j=1}^{m_i} a_{ij})$, $a_{ij} \in \mathscr{A}$, $n, m_i \geqslant 1$, $i = 1,\ldots,n$.*

A hypothesis is an element $h \in \mathscr{A}$. A piece of evidence is an element $e \in \mathscr{E}$. Let u be a fixed element of $\mathscr{A}$, representing the user's de facto knowledge.

A production rule is an expression $e \rightarrow h$ where e is a piece of evidence and h is a hypothesis. Let $\mathscr{P}$ denote a fixed, finite set of production rules.

In Section 1 we have introduced the certainty factor function having three arguments; the third argument of a certainty factor $CF(j \dashv i, D^{i,j})$ represents a derivation of the hypothesis j from i with respect to $\mathscr{P}$. This notion is defined in the following definition.

Definition 3.2. *Let $\mathscr{P}$ be defined as above. A derivation $D^{i,j}$ of j from i with respect to $\mathscr{P}$ is defined recursively as follows.*

(1) *$e \rightarrow h$ is a derivation of h from e with respect to $\mathscr{P}$ if $e \rightarrow h \in \mathscr{P}$.*

(2) *If $D^{u,e}$ is a derivation of e from u with respect to $\mathscr{P}$ and $e \rightarrow h$ is a derivation of h from e with respect to $\mathscr{P}$, then $(D^{u,e}) \circ (e \rightarrow h)$ is a derivation of h from u with respect to $\mathscr{P}$; $(D^{u,e}) \circ (e \rightarrow h)$ is called the sequential composition of the derivations $D^{u,e}$ and $e \rightarrow h$.*

(3) *If D^{u,e_1} is a derivation of e_1 from u with respect to $\mathscr{P}$ and D^{u,e_2} is a derivation of e_2 from u with respect to $\mathscr{P}$, then (D^{u,e_1}) & (D^{u,e_2}) is a derivation of $(e_1 \wedge e_2)$ from u with respect to $\mathscr{P}$; (D^{u,e_1}) & (D^{u,e_2}) is called the conjunction of the derivations D^{u,e_1} and D^{u,e_2}.*

(4) *If D^{u,e_1} is a derivation of e_1 from u with respect to $\mathscr{P}$ and D^{u,e_2} is a derivation of e_2 from u with respect to $\mathscr{P}$, then $(D^{u,e_1}) \mid (D^{u,e_2})$ is a derivation of $(e_1 \vee e_2)$ from u with respect to $\mathscr{P}$; $(D^{u,e_1}) \mid (D^{u,e_2})$ is called the disjunction of the derivations D^{u,e_1} and D^{u,e_2}.*

(5) *If $D_1^{u,h}$ and $D_2^{u,h}$ are derivations of h from u with respect to $\mathscr{P}$, then $(D_1^{u,h}) \parallel (D_2^{u,h})$ is a derivation of h from u with respect to $\mathscr{P}$; $(D_1^{u,h}) \parallel (D_2^{u,h})$ is called the parallel composition of the derivations $D_1^{u,h}$ and $D_2^{u,h}$.*

The set of all derivations with respect to $\mathscr{P}$ is denoted by $\mathscr{D}$.

We remark that the notion of sequential composition of two derivations is defined asymmetrically. Although a symmetrical definition $(D^{u,e}) \circ (D^{e,h})$ is more natural, it goes beyond the notion of derivation in top-down inference.

In the sequel, we will omit parentheses from elements of $\mathscr{E}$ and $\mathscr{D}$ as long as ambiguity cannot occur.

The two basic measures of uncertainty that we will discuss in the sequel have a foundation in probability theory. As production rules are statements concerning atomic propositions and positive boolean combinations of atomic propositions, and probabilities are statements concerning sets we have to define a mapping from $\mathscr{E}$ to Ω. We take each atomic proposition e to identify a nonempty subset e of the sample space Ω. The logical connective $\wedge$ (conjunction) translates into the set operation $\cap$ (intersection); the logical connective $\vee$ (disjunction) translates into the set operation $\cup$ (union). In Figure 3.1 the intuitive idea of this mapping is shown. Notice that this mapping from $\mathscr{E}$ into Ω is injective. Furthermore, although we did not allow negations in production rules, each subset of Ω has a complement due to the properties of sets. We assume these complements to be nonempty.

Similarly each derivation $D^{i,j}$ with respect to $\mathscr{P}$ identifies a subset of the sample space Ω dependent upon the intermediary hypotheses that were used in deriving the hypothesis j from i:

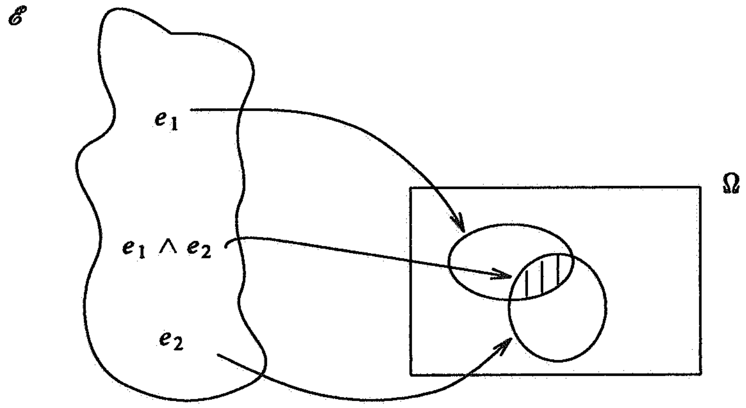

FIGURE 3.1. A mapping from $\mathscr{E}$ into Ω.

DEFINITION 3.3. *Let $\mathscr{E}$, u and $\mathscr{P}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Let the mapping from $\mathscr{E}$ into Ω be as described above. Then, an $\Omega,\mathscr{D}$-interpretation of elements of $\mathscr{D}$ is a function $\iota_\Omega: \mathscr{D} \to \Omega$ such that*

(1) *for each $u \to h \in \mathscr{P}$, $\iota_\Omega(u \to h) = \varnothing$, and*

(2) *for each $e \to h \in \mathscr{P}$ where $e \neq u$, $\iota_\Omega(e \to h) = e$, and*

(3) *for each $D_1, D_2 \in \mathscr{D}$, $\iota_\Omega(D_1 \circ D_2) = \iota_\Omega(D_1) \cup \iota_\Omega(D_2)$, and*

(4) *for each $D_1, D_2 \in \mathscr{D}$, $\iota_\Omega(D_1 \;\&\; D_2) = \iota_\Omega(D_1) \cap \iota_\Omega(D_2)$, and*

(5) *for each $D_1, D_2 \in \mathscr{D}$, $\iota_\Omega(D_1 \mid D_2) = \iota_\Omega(D_1) \cup \iota_\Omega(D_2)$, and*

(6) *for each $D_1, D_2 \in \mathscr{D}$, $\iota_\Omega(D_1 \parallel D_2) = \iota_\Omega(D_1) \cap \iota_\Omega(D_2)$.*

The basic idea of this mapping from $\mathscr{D}$ to Ω is to identify with a derivation a subset of Ω representing all information that has been concluded by the system in the course of the derivation except for the final conclusion of this derivation. So with a derivation $u \to h$ the empty set is associated since the system has not reached any conclusion during this derivation except for h. The interpretation of a conjunction of derivations and of a disjunction of derivations as the set operations intersection and union respectively are rather straightforward. The interpretation of the parallel composition of derivations as the intersection of the separate derivations, i.e. the idea of taking the intersection of all evidence that is used in deriving a hypothesis, should be intuitively appealing. The interpretation of the sequential composition of derivations as the union of the sets identified by these derivations comes forth from the idea that an expert system should have the ability to extend its focus.

EXAMPLE 3.1. Consider the derivation $D_1^{u,h} = ((u \to e) \circ (e \to h))$. Then $\iota_\Omega(D_1^{u,h}) = \varnothing \cup e = e$. Consider the derivation $D_2^{u,h} = (((u \to a) \circ (a \to h)) \parallel ((u \to e) \circ (e \to h)))$. Then $\iota_\Omega(D_2^{u,h}) = (\varnothing \cup a) \cap (\varnothing \cup e) = a \cap e$. ■

In the sequel we will see that in calculating a measure of uncertainty for a hypothesis the set corresponding with the derivation of this hypothesis is intersected with the set identified by u. From this it will be evident that the system is not allowed to focus on hypotheses contradictory to the user's de facto knowledge.

3.2. The Basic Measures of Uncertainty

In developing the certainty factor model Shortliffe and Buchanan have chosen two basic measures of uncertainty: the *measure of belief* expressing the degree to which an observed piece of evidence increases the belief in a hypothesis, and the *measure of disbelief* expressing the degree to which an observed piece of evidence decreases the belief in a hypothesis. Before stating the formal definitions of these functions, we quote their intuitive account for the measure of belief (see also Figure 3.2). We suppose a sample space Ω and a probability set function P on Ω are given.

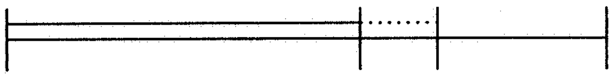

FIGURE 3.2. The degree of increased belief.

"In accordance with subjective probability theory, it may be argued that the expert's personal probability $P(h)$ reflects his or her belief in h at any given time. Thus, $1 - P(h)$ can be viewed as an estimate of the expert's disbelief regarding the truth of h. If $P(h \mid e)$ is greater than $P(h)$ the observation of e increases the expert's belief in h while decreasing his or her disbelief regarding the truth of h. In fact the proportionate decrease in disbelief is given by the following ratio:

$$\frac{P(h \mid e) - P(h)}{1 - P(h)}$$

This ratio is called the measure of increased belief in h resulting from the observation of e."
([1], pp. 247, 248)

Similarly, the measure of disbelief is accounted for. We notice that the ratio mentioned in the foregoing quotation equals $\dfrac{P(h \mid e) - P(h)}{1 - P(h)} = 1 - \dfrac{P(\bar{h} \mid e)}{P(\bar{h})}$.

In the next definition the measures of belief and disbelief are defined formally.

DEFINITION 3.4. *Let $\mathscr{E}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let ι_Ω be defined according to the Definition 3.3. Let $h, e \in \mathscr{E}$ and $D^{e,h} \in \mathscr{D}$ such that $P(e \cap \iota_\Omega(D^{e,h})) > 0$. The measure of (increased) belief MB is a function $MB: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that*

$$MB(h \dashv e, D^{e,h}) = \begin{cases} 1 & \text{if } P(h) = 1 \\ \max\left\{0, \dfrac{P(h \mid e \cap \iota_\Omega(D^{e,h})) - P(h)}{1 - P(h)}\right\} & \text{otherwise} \end{cases}$$

The measure of (increased) disbelief MD is a function $MD: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that

$$MD(h \dashv e, D^{e,h}) = \begin{cases} 1 & \text{if } P(h) = 0 \\ \max\left\{0, \dfrac{P(h) - P(h \mid e \cap \iota_\Omega(D^{e,h}))}{P(h)}\right\} & \text{otherwise} \end{cases}$$

It is noted that Shortliffe and Buchanan neither account for their choice for the measure of belief in the case where $P(h) = 1$ nor for their choice for the measure of disbelief in the case where $P(h) = 0$.

From now on we will assume a proper application of the function ι_Ω implicitly, and for instance write $P(D^{i,j})$ instead of $P(\iota_\Omega(D^{i,j}))$. For $\iota_\Omega(h)$ we will write h where appropriate.

The properties of MB and MD stated in the following proposition can easily be proven.

PROPOSITION 3.1. *Let $\mathscr{E}$ and $\mathscr{P}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4. Let $h,e \in \mathscr{E}$ and $D^{e,h} \in \mathscr{D}$. Then the following properties hold:*

(1) *If $e \rightarrow h \in \mathscr{P}$ and $P(h \mid e) = P(h)$ with $0 < P(h) < 1$ and $P(e) > 0$ then $MB(h \dashv e, e \rightarrow h) = MD(h \dashv e, e \rightarrow h) = 0$.*

(2) *If $MB(h \dashv e, D^{e,h}) > 0$ then $MD(h \dashv e, D^{e,h}) = 0$.*

(3) *If $MD(h \dashv e, D^{e,h}) > 0$ then $MB(h \dashv e, D^{e,h}) = 0$.*

Part (1) of Proposition 3.1 shows that neither the belief nor the disbelief in a hypothesis h is increased by the observation of evidence independent of h. Parts (2) and (3) show that a derivation of a hypothesis h cannot both confirm and disconfirm h.

3.3. The Approximation Functions for the Basic Measures

Leaving the notion of certainty factors aside for the moment, an expert associates with the conclusion h of a production rule $e \rightarrow h$ a measure of belief $MB(h \dashv e, e \rightarrow h)$ and a measure of disbelief $MD(h \dashv e, e \rightarrow h)$; equally a user associates with every piece of evidence e he feeds the system with a measure of belief $MB(e \dashv u, u \rightarrow e)$ and a measure of disbelief $MD(e \dashv u, u \rightarrow e)$. The objective of the application of the certainty factor model in a rule-based top-down reasoning expert system is to calculate function values $MB(h_g \dashv u, D^{u,h_g})$ and $MD(h_g \dashv u, D^{u,h_g})$ for each goal hypothesis h_g. Notice that these function values of MB and MD for goal hypotheses have u for a second argument. If the probability set function P on the sample space Ω is known (e.g. based on relative frequency) the function values $MB(h_g \dashv u, D^{u,h_g})$ and $MD(h_g \dashv u, D^{u,h_g})$ can be computed using the probabilistic definitions of MB and MD. In many of the domains in which expert systems are employed, however, the probability set function is rarely known. It is obvious that the expert and the user have supplied function values of MB and MD for only a few arguments. The certainty factor model offers approximation functions for calculating certain function values of MB and MD from the function values already known to the system. From our objective and from the notion of top-down derivation we have that only function values of MB and MD have to be approximated that have u for a second argument.

Shortliffe and Buchanan argue that the application of their approximation functions renders function values of MB and MD that are 'close enough' approximations of the real values of MB and MD. In this section we provide formal definitions of these approximation functions. In subsequent sections we examine the above mentioned claim of Shortliffe and Buchanan.

The approximation functions for the measure of belief and the measure of disbelief are defined in the following definition.

DEFINITION 3.5. *Let $\mathcal{E}$, u and $\mathcal{P}$ be defined according to Definition 3.1 and $\mathcal{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4 and let the functions $MB_\circ$, $MD_\circ$, $MB_|$, $MD_|$, $MB_\&$, $MD_\&$, $MB_{||}$ and $MD_{||}$ be as in the subsequent definitions. Let $h,e \in \mathcal{E}$ and $D^{u,h} = D_1 \odot D_2 \in \mathcal{D}$ where $\odot \in \{\circ, \&, |, ||\}$. $\overline{MB}$ is a partial function $\overline{MB}: \mathcal{E} \times \mathcal{E} \times \mathcal{D} \rightarrow [0,1]$ such that*

(1) $\overline{MB}(h \dashv e, e \rightarrow h) = MB(h \dashv e, e \rightarrow h)$ *if* $e \rightarrow h \in \mathcal{P}$, *and*

(2) $\overline{MB}(h \dashv u, D_1 \odot D_2) = MB_\odot(h \dashv u, D_1 \odot D_2)$ *otherwise.*

$\overline{MD}$ *is a partial function* $\overline{MD}: \mathcal{E} \times \mathcal{E} \times \mathcal{D} \rightarrow [0,1]$ *such that*

(1) $\overline{MD}(h \dashv e, e \rightarrow h) = MD(h \dashv e, e \rightarrow h)$ *if* $e \rightarrow h \in \mathcal{P}$, *and*

(2) $\overline{MD}(h \dashv u, D_1 \odot D_2) = MD_\odot(h \dashv u, D_1 \odot D_2)$ *otherwise.*

In Definition 3.5 we have introduced several new functions. Following the terminology of Shortliffe and Buchanan we will call these functions *combination functions*. In the remainder of this section the intended meaning of each of these combination functions is discussed in the light of rule-based top-down reasoning expert systems before the combination function is formally defined.

As has been mentioned before the function values $MB(h \dashv e, e \rightarrow h)$ and $MD(h \dashv e, e \rightarrow h)$ are associated with the conclusion h of a production rule $e \rightarrow h$. These function values express the degree to which the occurrence of evidence e influences the belief and disbelief in the hypothesis h, respectively. However, the truth of a piece of evidence e (i.e. whether e has actually occurred) may not always be determined with absolute certainty: with every piece of evidence e supplied by the user a measure of belief $MB(e \dashv u, u \rightarrow e)$ and a measure of disbelief $MD(e \dashv u, u \rightarrow e)$ are associated not necessarily equalling $+1$ and 0, respectively. Furthermore when using production rules an intermediary hypothesis e can be confirmed to some degree $MB(e \dashv u, D^{u,e})$ and disconfirmed to some degree $MD(e \dashv u, D^{u,e})$, and may in turn be used as evidence in other production rules concluding on new hypotheses. In the case of the production rule $e \rightarrow h$ described above, we are interested in the function values $MB(h \dashv u, D^{u,e} \circ (e \rightarrow h))$ and $MD(h \dashv u, D^{u,e} \circ (e \rightarrow h))$. These function values are approximated from the measures of belief and disbelief attached to the production rule and the function values associated with the intermediary hypothesis e: in the combination functions to deal with the situation that the truth of a piece of evidence is not known with certainty, the measures of belief and disbelief of the intermediary hypothesis e are used as part of a weighting factor for the measures of belief and disbelief associated with the hypothesis h in the production rule. These approximation functions are denoted by $MB_\circ$ and $MD_\circ$.

DEFINITION 3.6.[1] *Let $\mathscr{E}$, u and $\mathscr{P}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore let the functions $\overline{MB}$ and $\overline{MD}$ be defined according to Definition 3.5. Let $h,e \in \mathscr{E}$, $D^{u,e} \in \mathscr{D}$ and $e \rightarrow h \in \mathscr{P}$. $MB_\circ$ is a partial function $MB_\circ: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that*

$$MB_\circ(h \dashv u, D^{u,e} \circ (e \rightarrow h)) =$$

$$= \overline{MB}(h \dashv e, e \rightarrow h) \cdot max\left\{0, \frac{\overline{MB}(e \dashv u, D^{u,e}) - \overline{MD}(e \dashv u, D^{u,e})}{1 - min\{\overline{MB}(e \dashv u, D^{u,e}), \overline{MD}(e \dashv u, D^{u,e})\}}\right\}.$$

$MD_\circ$ is a partial function $MD_\circ: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that

$$MD_\circ(h \dashv u, D^{u,e} \circ (e \rightarrow h)) =$$

$$= \overline{MD}(h \dashv e, e \rightarrow h) \cdot max\left\{0, \frac{\overline{MB}(e \dashv u, D^{u,e}) - \overline{MD}(e \dashv u, D^{u,e})}{1 - min\{\overline{MB}(e \dashv u, D^{u,e}), \overline{MD}(e \dashv u, D^{u,e})\}}\right\}.$$

From now on, we will call the functions $MB_\circ$ and $MD_\circ$ the *combination functions for propagating uncertain evidence.* We notice that these combination functions render function values equal to zero when there is more reason to believe that e is false than there is to believe that e is true. This property of the combination functions for propagating uncertain evidence has as a consequence that a production rule $e \rightarrow h$ has no influence on the belief nor on the disbelief in h when the rule has failed during the top-down inference process.

The evidence e in a production rule $e \rightarrow h$ is a positive Boolean combination of pieces of evidence according to Definition 3.1. In order to be able to apply the combination functions $MB_\circ$ and $MD_\circ$ to approximate the measures of belief and disbelief of h after the application of this rule, the measures of belief and disbelief of e given some derivation of e from u, i.e. $MB(e \dashv u, D^{u,e})$ and $MD(e \dashv u, D^{u,e})$, have to be known. As these function values are not known in general they have to be approximated from the separate measures of belief and disbelief for each of the pieces of evidence that e comprises, viewed as hypotheses. Shortliffe and Buchanan argue

"that the measure of belief in the conjunction of two hypotheses is only as good as the belief in the hypothesis that is believed less strongly, whereas ... the measure of disbelief in such a conjunction is as strong as the disbelief in the most strongly disconfirmed."
([1], p. 256)

Complementary observations are made for disjunctions of hypotheses. In character with these contemplations Definition 3.7 formulates the combination functions $MB_\&$, $MD_\&$, $MB_|$ and $MD_|$ for approximating the measures of belief and disbelief in positive Boolean combinations of hypotheses.

1. In [1], the following combination functions are proposed:

$$MB(h,i) = MB'(h,i) \cdot max\{0, CF(i,e)\}$$
$$MD(h,i) = MD'(h,i) \cdot max\{0, CF(i,e)\}$$

where $MB'(h,i)$ and $MD'(h,i)$ are the measures of belief and disbelief which the expert associated with h given that evidence i is observed with absolute certainty. $CF(i,e)$ denotes the same expression in $MB(i,e)$ and $MD(i,e)$ we have used in our Definition 3.6. In the original formulation given above the introduction of the quoted functions MB' and MD' is necessary because in their model Shortliffe and Buchanan do not explicitly distinguish approximated function values from actual function values; as we will see, this is still more evident in their formulation of the other combination functions. We have introduced new functions $MB_\circ$ and $MD_\circ$ to emphasize the difference between the actual functions MB and MD and the approximation functions. Furthermore it is noted that in the original formulation given above the dependence of each of the left-hand sides on i as well as on e is not expressed. This observation has led to the second argument u in our formulation and the introduction of the third argument expressing a derivation with respect to the set of production rules. Notice that we have not departed from the intended meaning of the original formulation.

DEFINITION 3.7.[1] *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore let the functions $\overline{MB}$ and $\overline{MD}$ be defined according to Definition 3.5. Let $e_i \in \mathscr{E}$ and $D^{u,e_i} \in \mathscr{D}$, $i = 1,2$. $MB_{|}$ is a partial function $MB_{|}: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that*

$$MB_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) = max\{\overline{MB}(e_1 \dashv u,D^{u,e_1}),\overline{MB}(e_2 \dashv u,D^{u,e_2})\}.$$

$MD_{|}$ is a partial function $MD_{|}: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that

$$MD_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) = min\{\overline{MD}(e_1 \dashv u,D^{u,e_1}),\overline{MD}(e_2 \dashv u,D^{u,e_2})\}.$$

$MB_{\&}$ is a partial function $MB_{\&}: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that

$$MB_{\&}(e_1 \wedge e_2 \dashv u,D^{u,e_1} \,\&\, D^{u,e_2}) = min\{\overline{MB}(e_1 \dashv u,D^{u,e_1}),\overline{MB}(e_2 \dashv u,D^{u,e_2})\}.$$

$MD_{\&}$ is a partial function $MD_{\&}: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \rightarrow [0,1]$ such that

$$MD_{\&}(e_1 \wedge e_2 \dashv u,D^{u,e_1} \,\&\, D^{u,e_2}) = max\{\overline{MD}(e_1 \dashv u,D^{u,e_1}),\overline{MD}(e_2 \dashv u,D^{u,e_2})\}.$$

Henceforth the combination functions $MB_{|}$ and $MD_{|}$ will be called the *combination functions for disjunctions of hypotheses*; the combination functions $MB_{\&}$ and $MD_{\&}$ will be called the *combination functions for conjunctions of hypotheses.* When referring to the four functions together we will call them the *combination functions for composite hypotheses.*

When different successful production rules $e_i \rightarrow h$ (i.e. rules with different left-hand sides e_i) conclude on the same hypotheses h, a measure of belief $\overline{MB}(h \dashv u,D_i^{u,h})$ and a measure of disbelief $\overline{MD}(h \dashv u,D_i^{u,h})$ are calculated from each of these rules using the approximation functions $\overline{MB}$ and $\overline{MD}$. The net measure of belief and the net measure of disbelief, for example for two production rules $MB(h \dashv u,D_1^{u,h} \parallel D_2^{u,h})$ and $MD(h \dashv u,D_1^{u,h} \parallel D_2^{u,h})$, are approximated from these partial measures of belief and disbelief. Shortliffe and Buchanan justify their combination functions for combining the results of different production rules concluding on the same hypothesis as follows:

"since an *MB* (or *MD*) represents a proportionate decrease of disbelief (or belief), the *MB* (or *MD*) of a newly acquired piece of evidence should be applied proportionately to the disbelief (or belief) still remaining."
([1], p. 256)

The combination functions for combining the results of different production rules concluding on the same hypothesis are defined in the next definition.

1. In [1], the following combination functions are proposed:

$$MB(h_1 \wedge h_2,e) = min\{MB(h_1,e),MB(h_2,e)\}$$
$$MB(h_1 \vee h_2,e) = max\{MB(h_1,e),MB(h_2,e)\}$$
$$MD(h_1 \wedge h_2,e) = max\{MD(h_1,e),MD(h_2,e)\}$$
$$MD(h_1 \vee h_2,e) = min\{MD(h_1,e),MD(h_2,e)\}$$

Again, no distinction is made between approximated and actual function values: the original formulation therefore suggests properties of *MB* and *MD* that do not hold in general. Furthermore, these combination schemes can be used to combine the measures of belief and disbelief for several hypotheses given the *same* evidence. In practice however, the measures of belief and disbelief of the hypotheses to be combined are derived along different inference paths, and may differ in the second argument due to the original formulation of the combination functions for propagating uncertain evidence.

DEFINITION 3.8.[1] *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore let the functions $\overline{MB}$ and $\overline{MD}$ be defined according to Definition 3.5. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$, $i = 1,2$. $MB_{\|}$ is a partial function $MB_{\|}: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \to [0,1]$ such that*

(1) $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 0$ *if* $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 1$, *and*

(2) $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = \overline{MB}(h \dashv u, D_1^{u,h}) + \overline{MB}(h \dashv u, D_2^{u,h})(1 - \overline{MB}(h \dashv u, D_1^{u,h}))$ *otherwise.*

$MD_{\|}$ is a partial function $MD_{\|}: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \to [0,1]$ such that

(1) $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 0$ *if* $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 1$, *and*

(2) $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = \overline{MD}(h \dashv u, D_1^{u,h}) + \overline{MD}(h \dashv u, D_2^{u,h})(1 - \overline{MD}(h \dashv u, D_1^{u,h}))$ *otherwise.*

Henceforth, the functions $MB_{\|}$ and $MD_{\|}$ will be called the *combination functions for (combining the results of) co-concluding production rules* (i.e. concluding on the same hypothesis).

It is noted that the functions $\overline{MB}$ and $\overline{MD}$ are defined recursively through the eight combination functions defined in the foregoing definitions. These combination functions can only be applied in a specific order as shown in Figure 3.3.

3.4. A Derived Measure and its Approximation Function

In addition to the basic measures of uncertainty *MB* and *MD* a third measure, derived from the measures of belief and disbelief, is defined. This derived function is called the *certainty factor function*.

1. In [1] the following combination functions are proposed:

$$MB(h,e_1 \wedge e_2) = \begin{cases} 0 & \text{if } MD(h,e_1 \wedge e_2) = 1 \\ MB(h,e_1) + MB(h,e_2)(1 - MB(h,e_1)) & \text{otherwise} \end{cases}$$

$$MD(h,e_1 \wedge e_2) = \begin{cases} 0 & \text{if } MB(h,e_1 \wedge e_2) = 1 \\ MD(h,e_1) + MD(h,e_2)(1 - MD(h,e_1)) & \text{otherwise} \end{cases}$$

Once more, the original formulation suggests properties of the functions *MB* and *MD* that do not hold in general. Furthermore in describing the combination of the results of different production rules concluding on the same hypothesis, the same symbol $\wedge$ is used as in describing a conjunction of hypotheses or pieces of evidence. Shortliffe and Buchanan seem to assume that the success of a production rule $e_1 \wedge e_2 \to h$ is equivalent to the success of the two production rules $e_1 \to h$ and $e_2 \to h$. As such an equivalence is apt to be violated due to inconsistent values of *MB* and *MD* provided by the expert and the user or calculated using the approximation functions, we have felt the necessity of introducing another symbol.

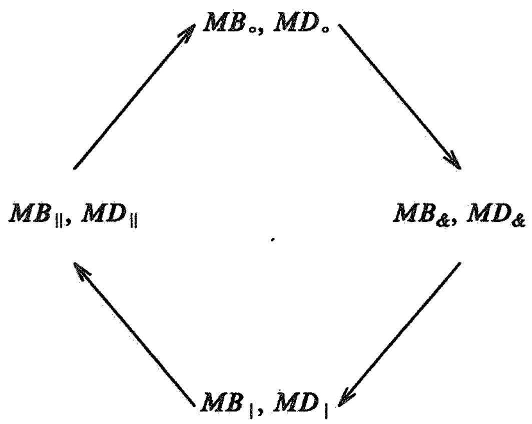

FIGURE 3.3. The order in applying the combination functions.

DEFINITION 3.9. *Let $\mathscr{E}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore let the functions MB and MD be defined according to Definition 3.4. Let $h,e \in \mathscr{E}$ and $D^{e,h} \in \mathscr{D}$. CF is a partial function $CF\colon \mathscr{E} \times \mathscr{E} \times \mathscr{D} \to [-1,1]$ such that*

$$CF(h \dashv e,D^{e,h}) = \frac{MB(h \dashv e,D^{e,h}) - MD(h \dashv e,D^{e,h})}{1 - min\{MB(h \dashv e,D^{e,h}),MD(h \dashv e,D^{e,h})\}}.$$

Using Proposition 3.1 the property in the following proposition can easily be proven.

PROPOSITION 3.2. *Let $\mathscr{E}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore let the functions MB and MD be defined according to Definition 3.4 and the function CF as above. Let $h,e \in \mathscr{E}$ and $D^{e,h} \in \mathscr{D}$. Then one of the following statements is true*

(1) $CF(h \dashv e,D^{e,h}) = MB(h \dashv e,D^{e,h})$.
(2) $CF(h \dashv e,D^{e,h}) = -MD(h \dashv e,D^{e,h})$.

Definition 3.9 describes the certainty factor function in terms of the measures of belief and disbelief as proposed in [1]: if the function values $MB(h \dashv e,D^{e,h})$ and $MD(h \dashv e,D^{e,h})$ are known, the function value $CF(h \dashv e,D^{e,h})$ can be calculated from these values. As we have discussed in the foregoing section, in general the function values of *MB* and *MD* are not known and are approximated in the model using $\overline{MB}$ and $\overline{MD}$. In practice therefore, the function values of the certainty factor function are calculated from *approximations* of the functions values of *MB* and *MD*. In the following definition the notion of certainty factor is redefined in terms of the approximation functions for *MB* and *MD*.

DEFINITION 3.10. *Let $\mathcal{E}$ be defined according to Definition 3.1 and $\mathcal{D}$ according to Definition 3.2. Furthermore, let the functions $\overline{MB}$ and $\overline{MD}$ be defined according to Definition 3.5. Let $h,e \in \mathcal{E}$ and $D^{e,h} \in \mathcal{D}$. CF' is a partial function $CF': \mathcal{E} \times \mathcal{E} \times \mathcal{D} \rightarrow [-1,1]$ such that*

$$CF'(h \dashv e,D^{e,h}) = \frac{\overline{MB}(h \dashv e,D^{e,h}) - \overline{MD}(h \dashv e,D^{e,h})}{1 - min\{\overline{MB}(h \dashv e,D^{e,h}),\overline{MD}(h \dashv e,D^{e,h})\}}.$$

It should be evident from the definitions of the functions CF and CF' that these functions (at least) coincide when production rules are considered. This property is formulated in the following corollary. We emphasize that this property does not hold for each derivation in general.

COROLLARY 3.1. *Let $\mathcal{E}$ and $\mathcal{P}$ be defined according to Definition 3.1. Furthermore, let the function CF be defined according to Definition 3.9 and the function CF' as above. Let $h,e \in \mathcal{E}$ and $e \rightarrow h \in \mathcal{P}$. Then $CF(h \dashv e,e \rightarrow h) = CF'(h \dashv e,e \rightarrow h)$.*

In the EMYCIN implementation of the certainty factor model rather than subsequently approximating the measures of belief and disbelief for each hypothesis using $\overline{MB}$ and $\overline{MD}$, and finally computing the certainty factor using Definition 3.10, only subsequently approximated certainty factors are used. For that purpose we introduce an approximation function for certainty factors.

DEFINITION 3.11. *Let $\mathcal{E}$, u and $\mathcal{P}$ be defined according to Definition 3.1 and $\mathcal{D}$ according to Definition 3.2. Furthermore, let the function CF' be defined according to Definition 3.10 and the functions $CF_\circ$, $CF_\&$, $CF_|$ and $CF_{||}$ as in the subsequent definitions. Let $h,e \in \mathcal{E}$ and $D^{u,h} = D_1 \odot D_2 \in \mathcal{D}$ where $\odot \in \{\circ, \&, |, ||\}$. $\overline{CF}$ is a partial function $\overline{CF}: \mathcal{E} \times \mathcal{E} \times \mathcal{D} \rightarrow [-1,1]$ such that*

(1) $\overline{CF}(h \dashv e,e \rightarrow h) = CF'(h \dashv e,e \rightarrow h)$ *if* $e \rightarrow h \in \mathcal{P}$, *and*

(2) $\overline{CF}(h \dashv u,D_1 \odot D_2) = CF_\odot(h \dashv u,D_1 \odot D_2)$ *otherwise.*

In Definition 3.11 several new functions are introduced. These functions $CF_\circ$, $CF_|$, $CF_\&$ and $CF_{||}$ are the combination functions for the certainty factor function. They are defined in the next three definitions.

DEFINITION 3.12. *Let $\mathcal{E}$, u and $\mathcal{P}$ be defined according to Definition 3.1 and $\mathcal{D}$ according to Definition 3.2. Furthermore, let the function $\overline{CF}$ be defined according to Definition 3.11. Let $h,e \in \mathcal{E}$, $D^{u,e} \in \mathcal{D}$ and $e \rightarrow h \in \mathcal{P}$. $CF_\circ$ is a partial function $CF_\circ: \mathcal{E} \times \mathcal{E} \times \mathcal{D} \rightarrow [-1,1]$ such that*

$$CF_\circ(h \dashv u,D^{u,e} \circ (e \rightarrow h)) = \overline{CF}(h \dashv e,e \rightarrow h) \cdot max\{0,\overline{CF}(e \dashv u,D^{u,e})\}.$$

Henceforth the approximation function $CF_\circ$ will be called the *combination function for propagating uncertain evidence* analogous to the naming of $MB_\circ$ and $MD_\circ$. This combination function shows once more that a production rule has no influence on the belief nor on the disbelief in a hypothesis when the rule has failed during the top-down inference process: the approximated certainty factor for this rule after its application equals zero.

DEFINITION 3.13. *Let $\mathscr{E}$ and $\underline{u}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the function $\overline{CF}$ be defined according to Definition 3.11. Let $e_i \in \mathscr{E}$ and $D^{u,e_i} \in \mathscr{D}$, $i = 1,2$. $CF_|$ is a partial function $CF_|: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \to [-1,1]$ such that*

$$CF_|(e_1 \vee e_2, u, D^{u,e_1} \mid D^{u,e_2}) = max\{\overline{CF}(e_1,u,D^{u,e_1}), \overline{CF}(e_2,u,D^{u,e_2})\}.$$

$CF_\&$ is a partial function $CF_\&: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \to [-1,1]$ such that

$$CF_\&(e_1 \wedge e_2, u, D^{u,e_1} \,\&\, D^{u,e_2}) = min\{\overline{CF}(e_1,u,D^{u,e_1}), \overline{CF}(e_2,u,D^{u,e_2})\}.$$

From now on, we will call the approximation functions $CF_|$ and $CF_\&$ the *combination functions for composite hypotheses.*

DEFINITION 3.14. Let $\mathscr{E}$ *and* $\underline{u}$ *be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the function $\overline{CF}$ be defined according to Definition 3.11. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$, $i = 1,2$. $CF_{||}$ is a partial function $CF_{||}: \mathscr{E} \times \mathscr{E} \times \mathscr{D} \to [-1,1]$ such that*

(1) $CF_{||}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = \overline{CF}(h \dashv u, D_1^{u,h}) + \overline{CF}(h \dashv u, D_2^{u,h})(1 - \overline{CF}(h \dashv u, D_1^{u,h}))$
if $\overline{CF}(h \dashv u, D_1^{u,h}) > 0$ *and* $\overline{CF}(h \dashv u, D_2^{u,h}) > 0$, *and*

(2) $$CF_{||}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = \frac{\overline{CF}(h \dashv u, D_1^{u,h}) + \overline{CF}(h \dashv u, D_2^{u,h})}{1 - min\{|\overline{CF}(h \dashv u, D_1^{u,h})|, |\overline{CF}(h \dashv u, D_2^{u,h})|\}}$$
if $-1 < \overline{CF}(h \dashv u, D_1^{u,h}) \cdot \overline{CF}(h \dashv u, D_2^{u,h}) \leqslant 0$, *and*

(3) $CF_{||}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = \overline{CF}(h \dashv u, D_1^{u,h}) + \overline{CF}(h \dashv u, D_2^{u,h})(1 + \overline{CF}(h \dashv u, D_1^{u,h}))$
if $\overline{CF}(h \dashv u, D_1^{u,h}) < 0$ *and* $\overline{CF}(h \dashv u, D_2^{u,h}) < 0$.

The approximation function $CF_{||}$ will be called the *combination function for (combining the results of) co-concluding production rules.*

3.5. Summary

In the foregoing sections we have defined the basic measures of uncertainty of the certainty factor model, the measure of belief MB and the measure of disbelief MD using a probability set function P on a sample space Ω (see Definition 3.4). As this probability set function is not always known in practice, not all function values of MB and MD can be computed from this probabilistic definition; the functions $\overline{MB}$ and $\overline{MD}$ are introduced to approximate function values of MB and MD respectively (see Definition 3.5). In addition in the model a third measure of uncertainty is used; the certainty factor function CF is defined using MB and MD (see Definition 3.9). As function values of MB and MD are approximated by $\overline{MB}$ and $\overline{MD}$, we have redefined the certainty factor function in terms of these approximated measures of belief and disbelief, giving CF' (see Definition 3.10). In actual implementations of the model only this certainty factor function is used for handling uncertainty. For the purpose of approximating function values of CF' the function $\overline{CF}$ is introduced (see Definition 3.11). Figure 3.4 shows the relations that have been defined between the functions employed in the certainty factor model. In Section 4 and 5 we examine these relations in detail.

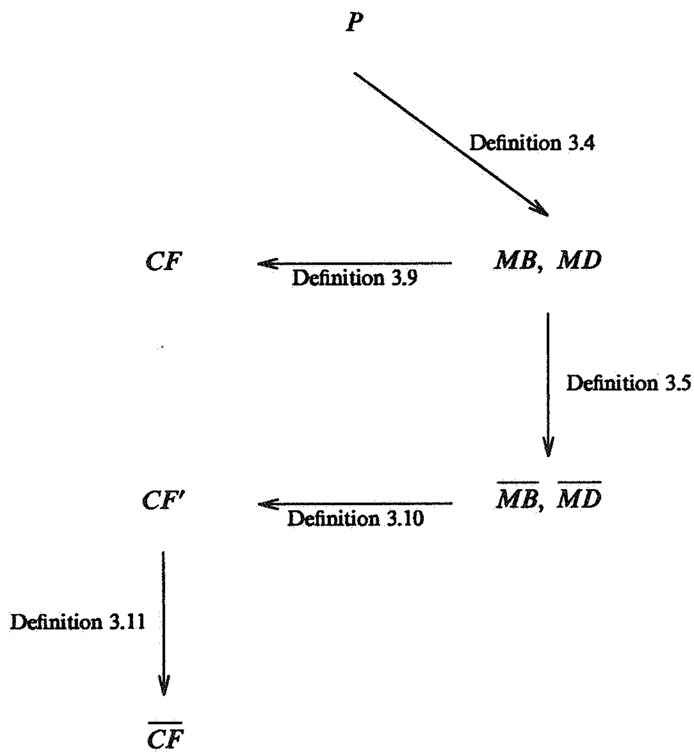

FIGURE 3.4. A diagram of functions.

4. AN ANALYSIS OF THE APPROXIMATION FUNCTIONS $\overline{MB}$ AND $\overline{MD}$

In Section 3.3 we have introduced the approximation functions $\overline{MB}$ and $\overline{MD}$ for the basic measures of uncertainty *MB* and *MD* of the certainty factor model. In this section we investigate whether these approximation functions respect the probabilistic definitions of *MB* and *MD* respectively, or more formally, we investigate whether $\overline{MB}$ is a restriction of *MB* and whether $\overline{MD}$ is a restriction of *MD*.

DEFINITION 4.1. *Let $\mathcal{U}_0$, $\mathcal{U}$ and $\mathcal{V}$ denote sets such that $\mathcal{U}_0 \subseteq \mathcal{U}$. Furthermore, let f be a function f: $\mathcal{U} \rightarrow \mathcal{V}$. A function f_0: $\mathcal{U}_0 \rightarrow \mathcal{V}$ is called a restriction of f, notation: $f_0 \simeq f$, if $f_0(u_0) = f(u_0)$ for each $u_0 \in \mathcal{U}_0$. The function f is called an extension of f_0.*

In this section, we concentrate on the right half of Figure 3.4. We recall that the approximation functions $\overline{MB}$ and $\overline{MD}$ are defined recursively through eight so-called combination functions: $MB_\circ$ and $MD_\circ$ (the combination functions for propagating uncertain evidence), $MB_|$, $MD_|$, $MB_\&$ and $MD_\&$ (the combination functions for composite hypotheses), and $MB_{||}$ and $MD_{||}$ (the combination functions for co-concluding production rules). Several authors have analysed the combination functions for combining the results of co-concluding production rules. We present our views on these functions in Section 4.1. The other combination functions have received far less attention in the literature. As the functions for composite hypotheses are applied approximately as often as the combination functions for co-concluding production rules and perhaps even more often, they can influence the resulting approximated function values of MB and MD considerably. The combination functions for composite hypotheses will be discussed in Section 4.3. Section 4.2 examines the combination functions for propagating uncertain evidence; in practice these functions are applied less often than the other combination functions.

4.1. The Combination Functions for Co-concluding Production Rules

In this section we investigate whether the combination functions for co-concluding production rules, i.e. $MB_{\|}$ and $MD_{\|}$, respect the probabilistic definitions of MB and MD respectively. We are interested in function values resulting from applying the combination functions $MB_{\|}$ and $MD_{\|}$ once. Therefore, we assume that the function values of $\overline{MB}$ and $\overline{MD}$ that are used in $MB_{\|}$ and $MD_{\|}$ are exact, i.e. we assume the properties $\overline{MB}(h \dashv u, D_i^{u,h}) = MB(h \dashv u, D_i^{u,h})$ and $\overline{MD}(h \dashv u, D_i^{u,h}) = MD(h \dashv u, D_i^{u,h})$, $i = 1,2$. We recall from Definition 3.8 that the combination function for combining the measures of belief of co-concluding production rules is defined as stated below:

(1) $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 0$ if $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 1$, and

(2) $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = MB(h,u \dashv D_1^{u,h}) + MB(h \dashv u, D_2^{u,h})(1 - MB(h \dashv u, D_1^{u,h}))$ otherwise.

Assuming the above mentioned property once more, we recall that the combination function for combining the measures of disbelief of co-concluding production rules is defined as stated below:

(1) $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 0$ if $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 1$, and

(2) $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = MD(h \dashv u, D_1^{u,h}) + MD(h \dashv u, D_2^{u,h})(1 - MD(h \dashv u, D_1^{u,h}))$ otherwise.

We will show that in some situations these combination functions respect the probabilistic definitions of MB and MD by making rather strong assumptions.

Given a hypothesis h and two derivations $D_i^{u,h}$ of h from u each not increasing the disbelief in h, Proposition 4.1 shows under which restrictions the combination functions for combining these derivations are consistent with the probabilistic definitions of MB and MD.

PROPOSITION 4.1. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4 and the functions $MB_{\|}$ and $MD_{\|}$ according to Definition 3.8. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$ such that $MD(h \dashv u, D_i^{u,h}) = 0$ and $u \cap D_i^{u,h}$ are independent and conditionally independent given $\bar{h}$, $i = 1,2$. Then*

(1) $MB(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h})$, *and*

(2) $MD(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 0$.

PROOF.

ad (1) Since $MD(h \dashv u, D_i^{u,h}) = 0$, $i = 1,2$, implies $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) \neq 1$ we have to prove $MB(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h})(1 - MB(h \dashv u, D_1^{u,h}))$.

According to Definition 3.4 we have

$$MB(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = \begin{cases} 1 & \text{if } P(h) = 1 \\ max\left\{0, \dfrac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)}\right\} & \text{otherwise} \end{cases}$$

We distinguish two cases: $P(h) = 1$ and $P(h) \neq 1$.

If $P(h) = 1$ then $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MB(h \dashv u, D_1^{u,h}) = MB(h \dashv u, D_2^{u,h}) = 1$. So $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h})(1 - MB(h \dashv u, D_1^{u,h}))$.

Now suppose $P(h) \neq 1$. By definition we have

$$MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = max\left\{0, \frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)}\right\}.$$

The fraction $\frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)}$ will be examined in detail:

$$\frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)} = 1 - \frac{1 - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{1 - P(h)} =$$

$$= 1 - \frac{P(\bar{h} \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(\bar{h})} =$$

$$= 1 - \frac{P(u \cap D_1^{u,h} \cap D_2^{u,h} \mid \bar{h})}{P(u \cap D_1^{u,h} \cap D_2^{u,h})}$$

by using Bayes' Theorem for the last equality. We recall from the conditions of the proposition that $u \cap D_1^{u,h}$ and $u \cap D_2^{u,h}$ are independent and conditionally independent given $\bar{h}$. So from Definition 2.4 we have

$$\frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)} = 1 - \frac{P(u \cap D_1^{u,h} \mid \bar{h})P(u \cap D_2^{u,h} \mid \bar{h})}{P(u \cap D_1^{u,h})P(u \cap D_2^{u,h})} =$$

$$= 1 - \frac{P(\bar{h} \mid u \cap D_1^{u,h})P(\bar{h} \mid u \cap D_2^{u,h})}{P(\bar{h})^2}$$

by using Bayes' Theorem once more for the last equality. The last term can be written as follows

$$\frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)} =$$

$$= \left[1 - \frac{1 - P(h \mid u \cap D_1^{u,h})}{1 - P(h)}\right] + \left[1 - \frac{1 - P(h \mid u \cap D_2^{u,h})}{1 - P(h)}\right] +$$

$$- \left[1 - \frac{1 - P(h \mid u \cap D_1^{u,h})}{1 - P(h)}\right] \cdot \left[1 - \frac{1 - P(h \mid u \cap D_2^{u,h})}{1 - P(h)}\right].$$

We recall from the conditions of the proposition that $MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$, so $\frac{P(h \mid u \cap D_i^{u,h}) - P(h)}{1 - P(h)} \geqslant 0$, $i = 1,2$. Using these inequalities and Definition 3.4 we have

$$\frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)} =$$

$$= MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h})(1 - MB(h \dashv u, D_1^{u,h})).$$

So,

$$MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) =$$

$$= max\left\{0, \frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)}\right\} =$$

$$= max\{0, MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h})(1 - MB(h \dashv u, D_1^{u,h}))\} =$$

$$= MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h})(1 - MB(h \dashv u, D_1^{u,h})).$$

ad (2) It suffices to show that $MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = 0$. If $MB_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = 1$ we have $MD_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = 0$ by definition; otherwise since we have $MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$ it follows that $MD_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) =$ $= MD(h \dashv u, D_1^{u,h}) + MD(h \dashv u, D_2^{u,h})(1 - MD(h \dashv u, D_1^{u,h})) = 0$.

Since $MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$ implies $P(h) \neq 0$, we have according to Definition 3.4

$$MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = max\left\{0, \frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}\right\}.$$

We distinguish two cases: $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) > 0$ and $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = 0$.

Assume $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) > 0$. The cases $P(h) = 1$ and $P(h) \neq 1$ are distinguished.

If $P(h) = 1$ then $P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) = 1$ so

$$MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = max\left\{0, \frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}\right\} = 0.$$

Now suppose $P(h) \neq 1$ then we have according to Definition 3.4

$$MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = \frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) - P(h)}{1 - P(h)} > 0.$$

From this inequality we have $P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) > P(h)$ implying

$$MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = max\left\{0, \frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}\right\} = 0.$$

Now assume $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = 0$. From $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = 0$ we have $P(h) \neq 1$ using Definition 3.4. Furthermore from the proof of part (1) it follows that $MB(h \dashv u, D_1^{u,h}) = MB(h \dashv u, D_2^{u,h}) = 0$. $MB(h \dashv u, D_1^{u,h}) = MB(h \dashv u, D_2^{u,h}) = 0$ and the conditions of the proposition, $MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$, imply $P(h \mid u \cap D_1^{u,h}) = P(h)$ and $P(h \mid u \cap D_2^{u,h}) = P(h)$. Now it can easily be shown that $P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) = P(h)$ so

$$MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = max\left\{0, \frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}\right\} = 0.$$

■

Given a hypothesis h and two derivations $D_i^{u,h}$ of h from u each not increasing the belief in h, Proposition 4.2 shows under which conditions the combination functions for combining these derivations respect the probabilistic definitions of MB and MD.

PROPOSITION 4.2. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4 and the functions $MB_{\parallel}$ and $MD_{\parallel}$ according to Definition 3.8. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$ such that $MB(h \dashv u, D_i^{u,h}) = 0$ and $u \cap D_i^{u,h}$ are independent and conditionally independent given h, i = 1,2. Then*

(1) $MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MD_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h})$, *and*

(2) $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MB_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = 0.$

PROOF. We will only prove part (1). The proof of part (2) is analogous to the proof of part (2) of the foregoing proposition.

Since $MB(h \dashv u, D_1^{u,h}) = MB(h \dashv u, D_2^{u,h}) = 0$ implies $MB(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) \neq 1$, we have to prove $MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MD(h \dashv u, D_1^{u,h}) + MD(h \dashv u, D_2^{u,h})(1 - MD(h \dashv u, D_1^{u,h}))$.

According to Definition 3.4 we have

$$MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = \begin{cases} 1 & \text{if } P(h) = 0 \\ max\left\{0, \dfrac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}\right\} & \text{otherwise} \end{cases}$$

We distinguish two cases: $P(h) = 0$ and $P(h) \neq 0$.

If $P(h) = 0$ then $MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 1$ by definition. So $MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MD(h \dashv u, D_1^{u,h}) + MD(h \dashv u, D_2^{u,h})(1 - MD(h \dashv u, D_1^{u,h}))$.

Now suppose $P(h) \neq 0$. By definition we have

$$MD(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = max\left\{0, \frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}\right\}.$$

We will examine the fraction $\dfrac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}$ in detail.

$$\frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)} = 1 - \frac{P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)} =$$
$$= 1 - \frac{P(u \cap D_1^{u,h} \cap D_2^{u,h} \mid h)}{P(u \cap D_1^{u,h} \cap D_2^{u,h})}$$

by using Bayes' Theorem for the last equation. Recall from the conditions of the proposition that $u \cap D_1^{u,h}$ and $u \cap D_2^{u,h}$ are independent and conditionally independent given h; so according to Definition 2.4 we have

$$\frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)} = 1 - \frac{P(u \cap D_1^{u,h} \mid h)P(u \cap D_2^{u,h} \mid h)}{P(u \cap D_1^{u,h})P(u \cap D_2^{u,h})} =$$
$$= 1 - \frac{P(h \mid u \cap D_1^{u,h})P(h \mid u \cap D_2^{u,h})}{P(h)^2}$$

by using Bayes' Theorem for the last equation. The last term can be written as follows

$$\frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)} = \left[1 - \frac{P(h \mid u \cap D_1^{u,h})}{P(h)}\right] + \left[1 - \frac{P(h \mid u \cap D_2^{u,h})}{P(h)}\right] +$$
$$- \left[1 - \frac{P(h \mid u \cap D_1^{u,h})}{P(h)}\right] \cdot \left[1 - \frac{P(h \mid u \cap D_2^{u,h})}{P(h)}\right].$$

From the conditions of the proposition $MB(h \dashv u,D_1^{u,h}) = MB(h \dashv u,D_2^{u,h}) = 0$, it follows that $\dfrac{P(h) - P(h \mid u \cap D_i^{u,h})}{P(h)} \geqslant 0$, $i = 1,2$. Using these inequalities and Definition 3.4 we have

$$\frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)} = MD(h \dashv u,D_1^{u,h}) + MD(h \dashv u,D_2^{u,h})(1 - MD(h \dashv u,D_1^{u,h})).$$

So,

$$MD(h \dashv u,D_1^{u,h} \parallel D_2^{u,h}) =$$
$$= max\left\{0, \frac{P(h) - P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h})}{P(h)}\right\} =$$
$$= max\{0, MD(h \dashv u,D_1^{u,h}) + MD(h \dashv u,D_2^{u,h})(1 - MD(h \dashv u,D_1^{u,h}))\} =$$
$$= MD(h \dashv u,D_1^{u,h}) + MD(h \dashv u,D_2^{u,h})(1 - MD(h \dashv u,D_1^{u,h})).$$

■

Given two derivations $D_1^{u,h}$ and $D_2^{u,h}$ of h from u with respect to the set of production rules, there are three possibilities for their relationship with the hypothesis h:

(1) both $D_1^{u,h}$ and $D_2^{u,h}$ do not increase the disbelief in h, i.e. $MD(h \dashv u,D_1^{u,h}) = MD(h \dashv u,D_2^{u,h}) = 0$ and $MB(h \dashv u,D_1^{u,h}) \geqslant 0$ and $MB(h \dashv u,D_2^{u,h}) \geqslant 0$, or

(2) both $D_1^{u,h}$ and $D_2^{u,h}$ do not increase the belief in h, i.e. $MB(h \dashv u,D_1^{u,h}) = MB(h \dashv u,D_2^{u,h}) = 0$ and $MD(h \dashv u,D_1^{u,h}) \geqslant 0$ and $MD(h \dashv u,D_2) \geqslant 0$, or

(3) one of $D_1^{u,h}$ and $D_2^{u,h}$ increases the disbelief in h while the other one increases the belief in h, i.e. $MB(h \dashv u,D_1^{u,h}) > 0$ and $MD(h \dashv u,D_2^{u,h}) > 0$, or $MD(h \dashv u,D_1^{u,h}) > 0$ and $MB(h \dashv u,D_2^{u,h}) > 0$.

Proposition 4.1 shows that in the case of part (1) the approximated function values $MB_{\parallel}(h \dashv u,D_1^{u,h} \parallel D_2^{u,h})$ and $MD_{\parallel}(h \dashv u,D_1^{u,h} \parallel D_2^{u,h})$ equal the actual function values of MB and MD provided that certain conditions hold; similarly Proposition 4.2 provides for the case that part (2) occurs. The case of part (3) is not provided for yet.

In his paper [2], Adams has stated properties similar to our Propositions 4.1 and 4.2. He however, did not recognize the restriction $MD(h \dashv u,D_i^{u,h}) = 0$ on the value of the measure of disbelief necessary to show that the combination functions $MB_{\|}$ respects the probabilistic definition of MB in the case of part (1); equally he did not recognize the restriction $MB(h \dashv u, D_i^{u,h}) = 0$ necessary to show that $MD_{\|}$ respects the definition of MD in the case of part (2). So in considering the case that part (3) occurs he seems to assume the three properties

(i) $D_1^{u,h}$ and $D_2^{u,h}$ are mutually independent, and
(ii) $D_1^{u,h}$ and $D_2^{u,h}$ are conditionally independent given $\underline{h}$, and
(iii) $D_1^{u,h}$ and $D_2^{u,h}$ are conditionally independent given h.

It can easily be shown that when these properties hold, at least one of the following statements is true

(i) $P(h) = 0$, or
(ii) $P(h) = 1$, or
(iii) $P(h \mid D_1^{u,h}) = P(h)$, or
(iv) $P(h \mid D_2^{u,h}) = P(h)$.

Therefore, taking the three assumptions together, renders the model only applicable in trivial situations.

When we consider the actual function values of MB and MD in the case that part (3) occurs, either $MB(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ or $MD(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ equals zero according to Proposition 3.1. The application of the approximation functions $MB_{\|}$ and $MD_{\|}$ however, renders function values $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ and $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ that can both be greater than zero. As the approximation functions cannot decrease the once calculated measures of belief and disbelief such an error cannot be reduced. Only if one of $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ and $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ attains the value one is the other set to zero. From this observation it is obvious that the approximation functions $MB_{\|}$ and $MD_{\|}$ are not consistent with probability theory in the case that there are 'conflicting' derivations of a hypothesis.

In [1], the case in which $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 1$ has been defined as an exceptional case for the function $MB_{\|}$; likewise, the case in which $MB_{\|}(h \dashv u, D^{u,h} \| D_2^{u,h}) = 1$ has been defined as an exceptional case for the function $MD_{\|}$. In Definition 3.8 we have followed Shortliffe and Buchanan in excepting these cases. When closely examining the definitions of $MB_{\|}$ and $MD_{\|}$ it is obvious that the function values $MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ and $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h})$ are not uniquely defined in the case where $MB(h \dashv u, D_1^{u,h}) = 1$ and $MD(h \dashv u, D_2^{u,h}) = 1$, i.e. the case where one derivation completely proves a hypothesis h and the other one completely disconfirms h. Propositions 4.3 and 4.4 show that under some of the conditions stated in the foregoing propositions, the case mentioned above cannot occur.

PROPOSITION 4.3. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$ such that $u \cap D_i^{u,h}$ are independent and conditionally independent given h, $i = 1,2$. If $MB(h \dashv u, D_1^{u,h}) = 1$ then $MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$.*

PROOF. Using the information $MB(h \dashv u, D_1^{u,h}) = 1$ we have to show that $MB(h \dashv u, D_2^{u,h}) = MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$.

From Proposition 3.1 and $MB(h \dashv u, D_1^{u,h}) = 1$ we have $MD(h \dashv u, D_1^{u,h}) = 0$. It is noted that from $MD(h \dashv u, D_1^{u,h}) = 0$ and Definition 3.4 it follows that $P(h) \neq 0$.

We distinguish two cases: $P(h) = 1$ and $P(h) \neq 1$.

If $P(h) = 1$, then also $MB(h \dashv u, D_2^{u,h}) = 1$ according to Definition 3.4. Using Proposition 3.1 once more we have $MD(h \dashv u, D_2^{u,h}) = 0$.

Now suppose $P(h) \neq 1$. According to Definition 3.4 we have

$$MB(h \dashv u, D_1^{u,h}) = max\left\{0, \frac{P(h \mid u \cap D_1^{u,h}) - P(h)}{1 - P(h)}\right\}.$$

It follows from $MB(h \dashv u, D_1^{u,h}) = 1$ and our assumption $P(h) \neq 1$ that $P(h \mid u \cap D_1^{u,h}) = 1$.

From $P(h \mid u \cap D_1^{u,h}) = \dfrac{P(h \cap u \cap D_1^{u,h})}{P(u \cap D_1^{u,h})} = 1$ and Proposition 2.4 we have

$$P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) = \frac{P(h \cap u \cap D_1^{u,h} \cap D_2^{u,h})}{P(u \cap D_1^{u,h} \cap D_2^{u,h})} = 1$$

(It is noted that from the condition of the proposition stating that $u \cap D_i^{u,h}$ are independent we have $P(u \cap D_1^{u,h} \cap D_2^{u,h}) \neq 0$).

It follows from Bayes' Theorem that $P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) = \dfrac{P(u \cap D_1^{u,h} \cap D_2^{u,h} \mid h)P(h)}{P(u \cap D_1^{u,h} \cap D_2^{u,h})}$. We recall that $u \cap D_1^{u,h}$ and $u \cap D_2^{u,h}$ are independent and conditionally independent given h, so from Definition 2.4 we have

$$P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) = \frac{P(u \cap D_1^{u,h} \mid h)P(u \cap D_2^{u,h} \mid h)P(h)}{P(u \cap D_1^{u,h})P(u \cap D_2^{u,h})}.$$

Using $P(h \mid u \cap D_1^{u,h}) = \dfrac{P(u \cap D_1^{u,h} \mid h)P(h)}{P(u \cap D_1^{u,h})} = 1$, we have

$$P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) = \frac{P(u \cap D_2^{u,h} \mid h)}{P(u \cap D_2^{u,h})} = \frac{P(h \mid u \cap D_2^{u,h})}{P(h)}.$$

From $P(h \mid u \cap D_1^{u,h} \cap D_2^{u,h}) = 1$, it follows that $P(h \mid u \cap D_2^{u,h}) = P(h)$. So $MB(h \dashv u, D_2^{u,h}) = 0$ and $MD(h \dashv u, D_2^{u,h}) = 0$ by definition. ■

PROPOSITION 4.4. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$ such that $u \cap D_i^{u,h}$ are independent and conditionally independent in $\bar{h}$, $i = 1,2$. If $MD(h \dashv u, D_1^{u,h}) = 1$ then $MB(h \dashv u, D_1^{u,h}) = MB(h \dashv u, D_2^{u,h}) = 0$.*

PROOF. Analogous to the proof of Proposition 4.3. ■

4.2. The Combination Functions for Propagating Uncertain Evidence

In this section we examine the combination functions for propagating uncertain evidence, i.e. $MB_\circ$ and $MD_\circ$. We are interested in the error introduced by applying the combination functions $MB_\circ$ and $MD_\circ$ once. Therefore, we assume that the function values of $\overline{MB}$ and $\overline{MD}$ that are used in $MB_\circ$ and $MD_\circ$ are exact, i.e. we assume the properties $\overline{MB}(e \dashv u, D^{u,e}) = MB(e \dashv u, D^{u,e})$ and $\overline{MD}(e \dashv u, D^{u,e}) = MD(e \dashv u, D^{u,e})$. We recall from Definition 3.6 that the combination functions for propagating uncertain evidence are defined as stated below:

$$MB_\circ(h \dashv u, D^{u,e} \circ (e \rightarrow h)) =$$

$$= MB(h, e, e \rightarrow h) \cdot max\left\{0, \frac{MB(e \dashv u, D^{u,e}) - MD(e \dashv u, D^{u,e})}{1 - min\{MB(e \dashv u, D^{u,e}), MD(e \dashv u, D^{u,e})\}}\right\},$$

and

$$MD_\circ(h \dashv u, D^{u,e} \circ (e \rightarrow h)) =$$

$$= MD(h,e,e \rightarrow h) \cdot max\left\{0, \frac{MB(e \dashv u,D^{u,e}) - MD(e \dashv u,D^{u,e})}{1 - min\{MB(e \dashv u,D^{u,e}),MD(e \dashv u,D^{u,e})\}}\right\}.$$

Notice the asymmetry in these functions.

Under the above mentioned assumptions the formulations of the functions $MB_\circ$ and $MD_\circ$ can be simplified using the property stated in Lemma 4.1, given below. This lemma can easily be proven using Proposition 3.1.

LEMMA 4.1. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4. Let $e \in \mathscr{E}$ and $D^{u,e} \in \mathscr{D}$. Then,*

$$max\left\{0, \frac{MB(e \dashv u,D^{u,e}) - MD(e \dashv u,D^{u,e})}{1 - min\{MB(e \dashv u,D^{u,e}),MD(e \dashv u,D^{u,e})\}}\right\} = MB(e \dashv u,D^{u,e}).$$

In his paper, Adams notices the resemblance between the function $MB_\circ$ and the probabilistic formula $P(h \mid e) = P(h \mid i)P(i \mid e)$ which holds when $h \subseteq i \subseteq e$. He states that this assumption is not strong enough to prove the combination functions for propagating uncertain evidence to be consistent with the probabilistic definitions of the measures of belief and disbelief. Proposition 4.5 however, shows that the assumption stated above is strong enough in some situations. It is noted that Proposition 4.5 uses a property of the function ι_Ω, embedding derivations in the sample space Ω: the interpretation of the operation $\circ$ as the set operation union is essential to the result stated in the proposition.

PROPOSITION 4.5. *Let $\mathscr{E}$, u and $\mathscr{P}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4 and the functions $MB_\circ$ and $MD_\circ$ according to Definition 3.6. Let $h,e \in \mathscr{E}$, $D^{u,e} \in \mathscr{D}$ and $e \rightarrow h \in \mathscr{P}$ such that $h \subseteq e \subseteq u \cap D^{u,e}$ and $P(u \cap D^{u,e}) \neq 0$. Then*

(1) $MB(h \dashv u,D^{u,e} \circ (e \rightarrow h)) = MB_\circ(h \dashv u,D^{u,e} \circ (e \rightarrow h))$, *and*
(2) $MD(h \dashv u,D^{u,e} \circ (e \rightarrow h)) = MD_\circ(h \dashv u,D^{u,e} \circ (e \rightarrow h))$.

PROOF. We will only prove part (1); part (2) follows by symmetry.

From Definition 3.6 and Lemma 4.1 it follows that we have to prove $MB(h \dashv u,D^{u,e} \circ (e \rightarrow h)) = MB(h,e,e \rightarrow h) \cdot MB(e \dashv u,D^{u,e})$. From Definition 3.4 we have

$$MB(h \dashv u,D^{u,e} \circ (e \rightarrow h)) = \begin{cases} 1 & \text{if } P(h) = 1 \\ max\left\{0, \dfrac{P(h \mid u \cap (D^{u,e} \cup e)) - P(h)}{1 - P(h)}\right\} & \text{otherwise} \end{cases}$$

We distinguish two cases: $P(h) = 1$ and $P(h) \neq 1$.

If $P(h) = 1$ then also $MB(h \dashv e,e \rightarrow h) = 1$. From the condition of the proposition $h \subseteq e$ and $P(h) = 1$ we have $P(e) = 1$, implying $MB(e \dashv u,D^{u,e}) = 1$. So we have $MB(h \dashv u,D^{u,e} \circ (e \rightarrow h)) = MB(h \dashv e,e \rightarrow h) \cdot MB(e \dashv u,D^{u,e}) = 1$.

Now suppose $P(h) \neq 1$. We have

$$MB(h \dashv u, D^{u,h} \circ (e \rightarrow h)) = max\left\{0, \frac{P(h \mid u \cap (D^{u,e} \cup e)) - P(h)}{1 - P(h)}\right\} =$$

$$= max\left\{0, \frac{P(h \mid (u \cap D^{u,e}) \cup (u \cap e)) - P(h)}{1 - P(h)}\right\} =$$

$$= max\left\{0, \frac{P(h \mid u \cap D^{u,e}) - P(h)}{1 - P(h)}\right\}.$$

Furthermore, we have by definition

$$MB(h \dashv e, e \rightarrow h) \cdot MB(e \dashv u, D^{u,e}) =$$

$$= max\left\{0, \frac{P(h \mid e) - P(h)}{1 - P(h)}\right\} \cdot max\left\{0, \frac{P(e \mid u \cap D^{u,e}) - P(e)}{1 - P(e)}\right\} =$$

$$= max\left\{0, \left[\frac{P(h \mid e) - P(h)}{1 - P(h)}\right] \cdot \left[\frac{P(e \mid u \cap D^{u,e}) - P(e)}{1 - P(e)}\right]\right\}.$$

We consider the product $\left[\frac{P(h \mid e) - P(h)}{1 - P(h)}\right] \cdot \left[\frac{P(e \mid u \cap D^{u,e}) - P(e)}{1 - P(e)}\right]$ in detail.

$$\left[\frac{P(h \mid e) - P(h)}{1 - P(h)}\right] \cdot \left[\frac{P(e \mid u \cap D^{u,e}) - P(e)}{1 - P(e)}\right] =$$

$$= \frac{P(h \mid e)P(e \mid u \cap D^{u,e}) - P(h \mid e)P(e) - P(h)P(e \mid u \cap D^{u,e}) + P(h)P(e)}{(1 - P(h))(1 - P(e))} =$$

$$= \frac{P(h \mid u \cap D^{u,e}) - P(h) - P(h)P(e \mid u \cap D^{u,e}) + P(h)P(e)}{(1 - P(h))(1 - P(e))} =$$

$$= \frac{(P(h \mid u \cap D^{u,e}) - P(h))(1 - P(e)) + P(h \mid u \cap D^{u,e})P(e) - P(h)P(e \mid u \cap D^{u,e})}{(1 - P(h))(1 - P(e))} =$$

$$= \frac{P(h \mid u \cap D^{u,e}) - P(h)}{1 - P(h)} + \frac{P(h \cap u \cap D^{u,e})P(e) - P(h)P(e \cap u \cap D^{u,e})}{P(u \cap D^{u,e})(1 - P(h))(1 - P(e))} =$$

$$= \frac{P(h \mid u \cap D^{u,e}) - P(h)}{1 - P(h)}.$$

So,

$$MB(h \dashv e, e \to h) \cdot MB(e \dashv u, D^{u,e}) = max\left\{0, \frac{P(h \mid u \cap D^{u,h}) - P(h)}{1 - P(h)}\right\} =$$

$$= MB(h \dashv u, D^{u,e} \circ (e \to h)).$$

■

4.3. The Combination Functions for Composite Hypotheses

In this section we investigate whether the combination functions for composite hypotheses, i.e. $MB_{|}$, $MD_{|}$, $MB_{\&}$ and $MD_{\&}$, respect the probabilistic definitions of MB and MD. Again we are interested in the error introduced by applying these combination functions once. We therefore assume the properties $\overline{MB}(e_i \dashv u, D^{u,e_i}) = MB(e_i \dashv u, D^{u,e_i})$ and $\overline{MD}(e_i \dashv u, D^{u,e_i}) = MD(e_i \dashv u, D^{u,e_i})$, $i = 1,2$. We recall from Definition 3.7 that the combination functions for composite hypotheses are defined as stated below; we have assumed the properties

$$MB_{|}(e_1 \vee e_2 \dashv u, D^{u,e_1} \mid D^{u,e_2}) = max\{MB(e_1 \dashv u, D^{u,e_1}), MB(e_2 \dashv u, D^{u,e_2})\}, \text{ and}$$

$$MD_{|}(e_1 \vee e_2 \dashv u, D^{u,e_1} \mid D^{u,e_2}) = min\{MD(e_1 \dashv u, D^{u,e_1}), MD(e_2 \dashv u, D^{u,e_2})\}, \text{ and}$$

$$MB_{\&}(e_1 \wedge e_2 \dashv u, D^{u,e_1} \& D^{u,e_2}) = min\{MB(e_1 \dashv u, D^{u,e_1}), MB(e_2 \dashv u, D^{u,e_2})\}, \text{ and}$$

$$MD_{\&}(e_1 \wedge e_2 \dashv u, D^{u,e_1} \& D^{u,e_2}) = max\{MD(e_1 \dashv u, D^{u,e_1}), MD(e_2 \dashv u, D^{u,e_2})\}.$$

The combination functions for composite hypotheses have received little attention in papers dealing with the certainty factor model. Adams shows that these combination functions are inconsistent with the probabilistic definitions of MB and MD by giving a counterexample ([2], p. 258). His counterexample concerning $MB_{\&}$ is shown in Example 4.1 adapted to our notational conventions.

EXAMPLE 4.1. Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the function MB be defined according to Definition 3.4 and the function $MB_{\&}$ according to Definition 3.7. Let $e_1, e_2 \in \mathscr{E}$ such that $e_1 \cap e_2 = \varnothing$, and let $D^{u,e_1}, D^{u,e_2} \in \mathscr{D}$ such that $P(u \cap D^{u,e_1} \cap D^{u,e_2}) > 0$. From Definition 3.4 we have $MB(e_1 \wedge e_2 \dashv u, D^{u,e_1} \& D^{u,e_2}) = 0$ since $P(e_1 \cap e_2 \mid u \cap D^{u,e_1} \cap D^{u,e_2}) = 0$. From Definition 3.7 however, we have $MB_{\&}(e_1 \wedge e_2 \dashv u, D^{u,e_1} \& D^{u,e_2}) = min\{MB(e_1 \dashv u, D^{u,e_1}), MB(e_2 \dashv u, D^{u,e_2})\}$, not necessarily equalling zero. ■

Similar counterexamples can be found concerning the combination functions $MD_{\&}$, $MB_{|}$ and $MD_{|}$.

Adams does not examine the combination functions for composite hypotheses in further detail, because to him "the extent or importance of the use of these (combination functions) in the employment of the model is not clear, but does not seem great" ([2], p. 258). As these combination functions are used in the application of each production rule whose left-hand side is not atomic, we feel that these combination functions might have a considerable impact on the approximated measures of belief and disbelief of the goal hypotheses.

The combination function $MB_{\&}$ bears strong resemblance to the probabilistic formula $P(a \cap b) = min\{P(a), P(b)\}$, which holds when either $a \subseteq b$ or $b \subseteq a$. Due to this similarity Wise and Henrion suggest in their paper that in the combination functions for composite hypotheses *maximum correlation* of hypotheses is assumed: "the less probable event occurs whenever the more probable event occurs" ([3], p. 73). The following example shows that the assumption of maximum correlation of hypotheses is not strong enough to derive $MB_{\&}$ and $MD_{\&}$ from the probabilistic definitions of MB and MD respectively.

EXAMPLE 4.2. Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the function MB be defined according to Definition 3.4 and the function $MB_{\&}$

according to Definition 3.7. Let $e_1, e_2 \in \mathscr{E}$ such that $e_1 \subset e_2$, $e_1 \neq \varnothing$ and $P(e_1 \cap e_2) \neq 1$, and let $D^{u,e_1}, D^{u,e_2} \in \mathscr{D}$. From Definition 3.4 we have

$$MB(e_1 \wedge e_2 \dashv u, D^{u,e_1} \,\&\, D^{u,e_2}) = max\left\{0, \frac{P(e_1 \cap e_2 \mid u \cap D^{u,e_1} \cap D^{u,e_2}) - P(e_1 \cap e_2)}{1 - P(e_1 \cap e_2)}\right\} =$$

$$= max\left\{0, \frac{P(e_1 \mid u \cap D^{u,e_1} \cap D^{u,e_2}) - P(e_1)}{1 - P(e_1)}\right\}$$

not equalling $min\{MB(e_1 \dashv u, D^{u,e_1}), MB(e_2 \dashv u, D^{u,e_2})\}$ in general. Even if we further assume $D^{u,e_1} \subset D^{u,e_2}$ we cannot show that $MB_{\&}$ respects the basis of MB in probability theory since possibly $MB(e_2 \dashv u, D^{u,e_2}) < MB(e_1 \dashv u, D^{u,e_1})$ in spite of $e_1 \subset e_2$. ■

We feel that it is not possible to identify a set of 'natural' assumptions under which the combination functions for composite hypotheses can be shown to be consistent with the probabilistic definitions of the measures of belief and disbelief.

4.4. Summary

In this section we have addressed the question whether the approximation functions $\overline{MB}$ and $\overline{MD}$ for the basic measures of uncertainty MB and MD respect the probabilistic definitions of these basic functions. As the approximation functions are defined recursively through eight combination functions, we have analysed the application of each of these combination functions in just one step in the process of approximating the actual function values of MB and MD, i.e. we have renounced errors introduced earlier during the approximation process. The analysis of some of these combination functions has helped us to formulate conditions under which the function is consistent with the probabilistic foundation of the model.

In Section 4.1 our analysis of the combination functions for co-concluding production rules, i.e. $MB_{\|}$ and $MD_{\|}$, given two derivations $D_i^{u,h}$ of the hypothesis h from the user's de facto knowledge u, $i = 1,2$, has shown that these combination functions are consistent with the probabilistic basis of the model if one of the following sets of conditions holds:

(1) Both derivations do not increase the disbelief in the hypothesis, i.e. $MD(h \dashv u, D_i^{u,h}) = 0$, and the two derivations, or to be more precise $u \cap D_i^{u,h}$, are independent and conditionally independent given the hypothesis (see Proposition 4.1).

(2) Both derivations do not increase the belief in the hypothesis, i.e. $MB(h \dashv u, D_i^{u,h}) = 0$, and the two derivations are independent and conditionally independent given the complement of the hypothesis (see Proposition 4.2).

In the case of 'conflicting' derivations the combination functions for co-concluding production rules do not respect the probabilistic definitions of the measures of belief and disbelief.

In Section 4.2 our analysis of the combination functions for propagating uncertain evidence, i.e. $MB_{\circ}$ and $MD_{\circ}$, given a production rule $e \rightarrow h$ and a derivation $D^{u,e}$ of e from the user's knowledge u, has shown that these combination functions respect the probabilistic basis of the model if $h \subseteq e \subseteq u \cap D^{u,e}$ where $P(u \cap D^{u,e}) \neq 0$ (see Proposition 4.5). This result shows that the combination functions $MB_{\circ}$ and $MD_{\circ}$ are correct in case the expert system is only able to narrow its focus and does not have the ability to turn to hypotheses slightly outside the scope of the derivation up till that moment.

In Section 4.3 our analysis of the combination functions for composite hypotheses, i.e. $MB_{|}$, $MD_{|}$, $MB_{\&}$ and $MD_{\&}$, has not enabled us to formulate conditions under which these functions can be shown to be consistent with the probabilistic basis of the model. The easy counterexamples we have given concerning these functions, however, show that any set of such conditions will be violated in most practical cases.

From these observations we have that the approximation function $\overline{MB}$ is not a restriction of the function MB. A similar statement can be made concerning MD and $\overline{MD}$.

THEOREM 4.1. *Let the functions MB and MD be defined according to Definition 3.4, and the functions* $\overline{MB}$ *and* $\overline{MD}$ *according to Definition 3.5. Then, the following statements are true:*

(1) $\overline{MB} \not\preceq MB$.
(2) $\overline{MD} \not\preceq MD$.

In Figure 4.1 this result has been added to the diagram of functions, introduced in Section 3.

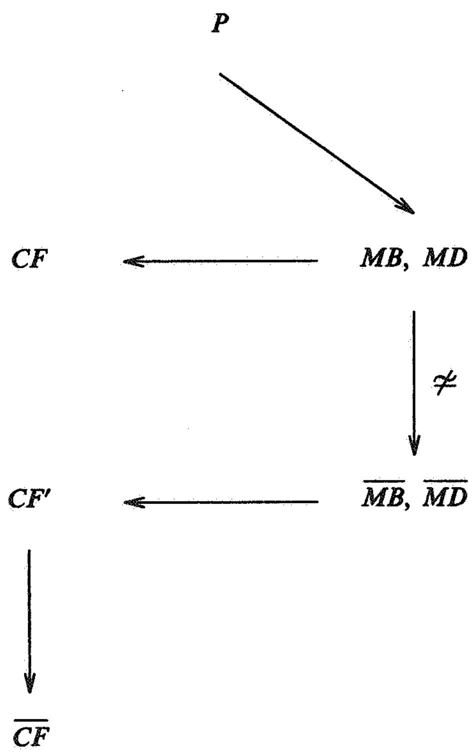

FIGURE 4.1. The diagram of functions.

5. THE CERTAINTY FACTOR FUNCTION AND ITS APPROXIMATION FUNCTION

In Section 3.4 a third measure is introduced in the certainty factor model in addition to the (basic) measures of belief and disbelief: the certainty factor. We recall from Definition 3.9 that the certainty factor function CF is defined as follows:

$$CF(h \dashv e, D^{u,e}) = \frac{MB(h \dashv e, D^{u,e}) - MD(h \dashv e, D^{u,e})}{1 - min\{MB(h \dashv e, D^{u,e}), MD(h \dashv e, D^{u,e})\}}.$$

From Proposition 3.2 we have that there is a one-to-one correspondence of the functions *MB* and *MD*, and *CF*.

We have observed that in practice another certainty factor function *CF'* is used arising from the fact that the actual function values of *MB* and *MD* are not always known and are approximated using $\overline{MB}$ and $\overline{MD}$. Definition 3.10 has redefined the certainty factor function:

$$CF'(h \dashv e, D^{e,h}) = \frac{\overline{MB}(h \dashv e, D^{e,h}) - \overline{MD}(h \dashv e, D^{e,h})}{1 - min\{\overline{MB}(h \dashv e, D^{e,h}), \overline{MD}(h \dashv e, D^{e,h})\}}.$$

Furthermore, we have described in Section 3 that in the EMYCIN implementation of the model, and in fact in all implementations since the introduction of the EMYCIN system, only subsequently approximated certainty factors are used. For that purpose we have defined an approximation function $\overline{CF}$ for certainty factors. This section focuses on the left half of Figure 3.4. In Section 5.1 we examine the question whether *CF'* is a restriction of *CF*. Section 5.2 shows that the approximation function $\overline{CF}$ is a restriction of *CF'*.

5.1. The Certainty Factor Functions CF and CF'

In this section we compare the functions *CF* defined by Shortliffe and Buchanan and *CF'*, actually employed by them in the MYCIN implementation of the model. From the definitions of these functions we have that *CF'* is a restriction of *CF* if $\overline{MB}$ is a restriction of *MB* and $\overline{MD}$ is a restriction of *MD*. So, using Theorem 4.1 we have that *CF'* is not a restriction of *CF*. This result is stated in Theorem 5.1. Before giving this theorem we state some intermediate results.

We distinguish several cases. The first case we consider is the propagation of uncertain evidence. We recall that Proposition 4.5 states that $MB_\circ(h \dashv u, D^{u,e} \circ (e \rightarrow h)) = MB(h \dashv u, D^{u,e} \circ (e \rightarrow h))$ if certain properties are assumed. So, under the same conditions as mentioned in Proposition 4.5 both certainty factor functions *CF* and $\overline{CF}$ render the same result in the case of the propagation of uncertain evidence. Again we renounce errors that were introduced earlier in the computation.

COROLLARY 5.1. *Let* $\mathscr{E}$, *u and* $\mathscr{P}$ *be defined according to Definition 3.1 and* $\mathscr{D}$ *according to Definition 3.2. Furthermore, let the function CF be defined according to Definition 3.9 and the function CF' according to Definition 3.10. Let* $h, e \in \mathscr{E}$, $D^{u,e} \in \mathscr{D}$ *and* $e \rightarrow h \in \mathscr{P}$ *such that* $h \subseteq e \subseteq u \cap D^{u,e}$ *where* $P(u \cap D^{u,e}) \neq 0$. *Then,* $CF'(h \dashv u, D^{u,e} \circ (e \rightarrow h)) = CF(h \dashv u, D^{u,e} \circ (e \rightarrow h))$.

In Section 4.3 we have argued that we feel that it is not possible to state a number of conditions under which the combination functions for composite hypotheses can be shown to be consistent with the probabilistic definitions of *MB* and *MD*. From this observation we have that in the case of composite hypotheses the certainty factor functions *CF* and *CF'* will not always render the same function values.

In the case of co-concluding production rules our observation concerning the two certainty factor functions is threefold. Corollary 5.2 follows from Proposition 4.1 and Corollary 5.3 can easily be proven using Proposition 4.2. Again errors introduced earlier in the computation are renounced.

COROLLARY 5.2. *Let* $\mathscr{E}$ *and u be defined according to Definition 3.1 and* $\mathscr{D}$ *according to Definition 3.2. Let the function MD be defined according to Definition 3.4. Furthermore, let the function CF be defined according to Definition 3.9 and the function CF' according to Definition 3.10. Let* $h \in \mathscr{E}$ *and* $D_i^{u,h} \in \mathscr{D}$ *such that* $MD(h \dashv u, D_i^{u,h}) = 0$ *and* $u \cap D_i^{u,h}$ *are mutually independent and conditionally independent given* $\bar{h}$, $i = 1,2$. *Then* $CF'(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = CF(h \dashv u, D_1^{u,h} \parallel D_2^{u,h})$.

COROLLARY 5.3. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Let the function MB be defined according to Definition 3.4. Furthermore, let the function CF be defined according to Definition 3.9 and the function CF′ according to Definition 3.10. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$ such that $MB(h \dashv u, D_i^{u,h}) = 0$ and $u \cap D_i^{u,h}$ are mutually independent and conditionally independent given h, $i = 1,2$. Then $CF'(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = CF(h \dashv u, D_1^{u,h} \parallel D_2^{u,h})$.*

The case that remains to be considered in our examination of the behaviour of the two certainty factor functions with respect to co-concluding production rules, is the case in which there is a derivation of h from u confirming h and a derivation of h from u disconfirming h. In his paper Adams observes that the model combines separately all derivations favouring a hypothesis and all derivations not favoring the hypothesis when calculating the corresponding certainty factor. Using Propositions 4.1 and 4.2 the property stated in Proposition 5.1 can easily be generalised to confirm his observation.

PROPOSITION 5.1. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the functions MB and MD be defined according to Definition 3.4 and the functions $MB_{\parallel}$ and $MD_{\parallel}$ according to Definition 3.8. Let the function CF′ be defined according to Definition 3.10. Let $h \in \mathscr{E}$ and $D_i^{u,h} \in \mathscr{D}$, $i = 1,2$, such that $u \cap D_i^{u,h}$ are mutually independent and $MB(h \dashv u, D_1^{u,h}) > 0$ and $MD(h \dashv u, D_2^{u,h}) > 0$. Then,*

$$CF'(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = \frac{MB(h \dashv u, D_1^{u,h}) - MD(h \dashv u, D_2^{u,h})}{1 - min\{MB(h \dashv u, D_1^{u,h}), MD(h \dashv u, D_2^{u,h})\}}.$$

PROOF. From Definition 3.10 we have

$$CF'(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = \frac{MB_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) - MD_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h})}{1 - min\{MB_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}), MD_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h})\}}$$

It will be evident that it suffices to show under the conditions mentioned above that

(1) $MB_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MB(h \dashv u, D_1^{u,h})$, and

(2) $MD_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) = MD(h \dashv u, D_2^{u,h})$.

We will only prove part (1); part (2) follows by symmetry.

From the condition of the proposition $MD(h \dashv u, D_2^{u,h}) > 0$ and Proposition 3.1 it follows that $MB(h \dashv u, D_2^{u,h}) = 0$. From Definition 3.8 we have

$$\begin{aligned} MB_{\parallel}(h \dashv u, D_1^{u,h} \parallel D_2^{u,h}) &= MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h})(1 - MB(h \dashv u, D_1^{u,h})) = \\ &= MB(h \dashv u, D_1^{u,h}). \end{aligned}$$

■

It should be obvious that in the case of conflicting evidence the certainty factor functions *CF* and *CF′* do not always render the same result.

Theorem 5.1 states the final result of this section:

THEOREM 5.1. *Let the function CF be defined according to Definition 3.9 and the function CF′ according to Definition 3.10. Then, $CF' \not\equiv CF$.*

5.2. The Approximation Function for Certainty Factors

We have already remarked before that in the EMYCIN implementation of the model only subsequently approximated certainty factors are used. For that purpose we have introduced the approximation function $\overline{CF}$ for certainty factors. This section investigates the question whether $\overline{CF}$ is a restriction of CF'. We recall that the approximation function $\overline{CF}$ is defined recursively through eight combination functions: $CF_\circ$ (the combination functions for propagating uncertain evidence), $CF_|$ and $CF_\&$ (the combination functions for composite hypotheses), and $CF_{||}$ (the combination function for co-concluding production rules). We will examine these combination functions separately.

We recall from Definition 3.12 that the combination function for propagating uncertain evidence is defined as stated below; we have assumed the property $\overline{CF}(e \dashv u,D^{u,e}) = CF(e \dashv u,D^{u,e})$:

$$CF_\circ(h \dashv u,D^{u,e} \circ (e \to h)) = CF(h \dashv e,e \to h) \cdot max\{0,CF(e \dashv u,D^{u,e})\}$$

Proposition 5.2 shows that the function $CF_\circ$ respects the definition of the function CF'.

PROPOSITION 5.2. *Let $\mathscr{E}$, u and $\mathscr{P}$ be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the function CF' be defined according to Definition 3.10 and the function $CF_\circ$ according to Definition 3.12. Let $h,e \in \mathscr{E}$, $D^{u,e} \in \mathscr{D}$ and $e \to h \in \mathscr{P}$. Then,*

$$CF'(h \dashv u,D^{u,e} \circ (e \to h)) = CF_\circ(h \dashv u,D^{u,e} \circ (e \to h)).$$

PROOF. From Definition 3.12 it follows that we have to show $CF'(h \dashv u,D^{u,e} \circ (e \to h)) = CF(h \dashv e,e \to h) \cdot max\{0,CF(e \dashv u,D^{u,e})\}$.

From Definition 3.10 we have

$$CF'(h \dashv u,D^{u,e} \circ (e \to h)) = \frac{MB_\circ(h \dashv u,D^{u,e} \circ (e \to h)) - MD_\circ(h \dashv u,D^{u,e} \circ (e \to h))}{1 - min\{MB_\circ(h \dashv u,D^{u,e} \circ (e \to h)),MD_\circ(h \dashv u,D^{u,e} \circ (e \to h))\}}.$$

Using Definition 3.6 and Lemma 4.1 it follows that

$$CF'(h \dashv u,D^{u,e} \circ (e \to h)) =$$

$$= \frac{MB(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e}) - MD(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e})}{1 - min\{MB(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e}),MD(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e})\}} =$$

$$= \frac{(MB(h \dashv e,e \to h) - MD(h \dashv e,e \to h)) \cdot MB(e \dashv u,D^{u,e})}{1 - min\{MB(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e}),MD(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e})\}} =$$

$$= \frac{(MB(h \dashv e,e \to h) - MD(h \dashv e,e \to h)) \cdot max\{0,MB(e \dashv u,D^{u,e}) - MD(e \dashv u,D^{u,e})\}}{1 - min\{MB(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e}),MD(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e})\}}.$$

Using Proposition 3.1 it can easily be shown that $1 - \min\{MB(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e}),MD(h \dashv e,e \to h) \cdot MB(e \dashv u,D^{u,e})\} = 1$, so

$$CF'(h \dashv u,D^{u,e} \circ (e \to h)) =$$

$$= (MB(h \dashv e,e \to h) - MD(h \dashv e,e \to h)) \cdot max\{0,MB(e \dashv u,D^{u,e}) - MD(e \dashv u,D^{u,e})\}$$

Furthermore we have $1 - \min\{MB(h \dashv e,e \to h),MD(h \dashv e,e \to h)\} = 1$ and $1 - \min\{MB(e \dashv u,D^{u,e}),MD(e \dashv u,D^{u,e})\} = 1$. It follows that

$$MB(h \dashv e,e \to h) - MD(h \dashv e,e \to h) = \frac{MB(h \dashv e,e \to h) - MD(h \dashv e,e \to h)}{1 - min\{MB(h \dashv e,e \to h),MD(h \dashv e,e \to h)\}} =$$

$$= CF(h \dashv e,e \to h),$$

and

$$MB(e \dashv u,D^{u,e}) - MD(e \dashv u,D^{u,e}) = \frac{MB(e \dashv u,D^{u,e}) - MD(e \dashv u,D^{u,e})}{1 - min\{MB(e \dashv u,D^{u,e}),MD(e \dashv u,D^{u,e})\}} =$$

$$= CF(e \dashv u,D^{u,e}).$$

So,

$$CF'(h \dashv u,D^{u,e} \circ (e \to h)) = CF(h \dashv e,e \to h) \cdot \max\{0,CF(e \dashv u,D^{u,e})\}.$$

■

We recall from Definition 3.13 that the combination function for disjunctions of hypotheses is defined as stated below; we have assumed the property $\overline{CF}(e_i,u,D^{u,e_i}) = CF(e_i,u,D^{u,e_i})$, $i = 1,2$:

$$CF_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) = max\{CF(e_1 \dashv u,D^{u,e_1}),CF(e_2 \dashv u,D^{u,e_2})\}$$

Proposition 5.3 shows that the function $CF_{|}$ respects the definition of the function CF'.

PROPOSITION 5.3. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the function CF' be defined according to Definition 3.10 and the function $CF_{|}$ according to Definition 3.13. Let $e_i \in \mathscr{E}$ and $D^{u,e_i} \in \mathscr{D}$, $i = 1,2$. Then,*

$$CF'(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) = CF_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}).$$

PROOF. From Definition 3.13 it follows that we have to show that $CF'(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) = max\{CF(e_1 \dashv u,D^{u,e_1}),CF(e_2 \dashv u,D^{u,e_2})\}$.
From Definition 3.10 and Definition 3.7 we have

$$CF'(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) =$$

$$= \frac{MB_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) - MD_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2})}{1 - min\{MB_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}),MD_{|}(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2})\}} =$$

$$= \frac{max\{MB(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2})\} - min\{MD(e_1 \dashv u,D^{u,e_1}),MD(e_2 \dashv u,D^{u,e_2})\}}{1 - min\{max\{MB(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2})\},min\{MD(e_1 \dashv u,D^{u,e_1}),MD(e_2 \dashv u,D^{u,e_2})\}\}}.$$

Using Proposition 3.1 it can easily be shown that the denumerator of the fraction equals 1. So,

$$CF'(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) =$$

$$= max\{MB(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2})\} - min\{MD(e_1 \dashv u,D^{u,e_1}),MD(e_2 \dashv u,D^{u,e_2})\}$$

(1) We assume $MB(e_1 \dashv u,D^{u,e_1}) = 0$ and $MB(e_2 \dashv u,D^{u,e_2}) = 0$. The other case

$MD(e_1 \dashv u,D^{u,e_1}) = MD(e_2 \dashv u,D^{u,e_2}) = 0$ follows by symmetry.

From this assumption and Proposition 3.1 we have $MD(e_1 \dashv u,D^{u,e_1}) \geqslant 0$ and $MD(e_2 \dashv u,D^{u,e_2}) \geqslant 0$.

Now suppose $MD(e_1 \dashv u,D^{u,e_1}) \leqslant MD(e_2 \dashv u,D^{u,e_2})$. The other case $MD(e_1 \dashv u,D^{u,e_1}) \geqslant MD(e_2 \dashv u,D^{u,e_2})$ follows by symmetry. Our assumptions imply

$$max\{MB(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2})\} - min\{MD(e_1 \dashv u,D^{u,e_1}),MD(e_2 \dashv u,D^{u,e_2})\} =$$

$$= - MD(e_1 \dashv u,D^{u,e_1}) =$$

$$= MB(e_1 \dashv u,D^{u,e_1}) - MD(e_1 \dashv u,D^{u,e_1}) =$$

$$= max\{MB(e_1 \dashv u,D^{u,e_1}) - MD(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2}) - MD(e_2 \dashv u,D^{u,e_2})\}.$$

(2) Assume $MB(e_1 \dashv u,D^{u,e_1}) > 0$ and $MD(e_2 \dashv u,D^{u,e_2}) > 0$. The case $MD(e_1 \dashv u,D^{u,e_1}) > 0$ and $MB(e_2 \dashv u,D^{u,e_2}) > 0$ follows by symmetry.

From this assumption and Proposition 3.1 we have $MD(e_1 \dashv u,D^{u,e_1}) = 0$ and $MB(e_2 \dashv u,D^{u,e_2}) = 0$. So,

$$max\{MB(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2})\} - min\{MD(e_1 \dashv u,D^{u,e_1}),MD(e_2 \dashv u,D^{u,e_2})\} =$$

$$= MB(e_1 \dashv u,D^{u,e_1}) =$$

$$= MB(e_1 \dashv u,D^{u,e_1}) - MD(e_1 \dashv u,D^{u,e_1}) =$$

$$= max\{MB(e_1 \dashv u,D^{u,e_1}) - MD(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2}) - MD(e_2 \dashv u,D^{u,e_2})\}.$$

From (1) and (2), we have

$$CF'(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) =$$

$$= max\{MB(e_1 \dashv u,D^{u,e_1}) - MD(e_1 \dashv u,D^{u,e_1}),MB(e_2 \dashv u,D^{u,e_2}) - MD(e_2 \dashv u,D^{u,e_2})\}.$$

Using Proposition 3.1 we can show that $1 - min\{MB(e_i \dashv u,D^{u,e_i}),MD(e_i \dashv u,D^{u,e_i})\} = 1,\ i = 1,2$, from which we have

$$MB(e_i \dashv u,D^{u,e_i}) - MD(e_i \dashv u,D^{u,e_i}) = \frac{MB(e_i \dashv u,D^{u,e_i}) - MD(e_i \dashv u,D^{u,e_i})}{1 - min\{MB(e_i \dashv u,D^{u,e_i}),MD(e_i \dashv u,D^{u,e_i})\}} =$$

$$= CF(e_i \dashv u,D^{u,e_i}).$$

Therefore, we have

$$CF'(e_1 \vee e_2 \dashv u,D^{u,e_1} \mid D^{u,e_2}) = max\{CF(e_1 \dashv u,D^{u,e_1}),CF(e_2 \dashv u,D^{u,e_2})\}.$$

■

We recall from Definition 3.13 that the combination function for conjunctions of hypotheses is defined as stated below; we have assumed the property $\overline{CF}(e_i,u,D^{u,e_i}) = CF(e_i,u,D^{u,e_i}),\ i = 1,2$:

$$CF_{\&}(e_1 \wedge e_2 \dashv u,D^{u,e_1}\ \&\ D^{u,e_2}) = min\{CF(e_1 \dashv u,D^{u,e_1}),CF(e_2 \dashv u,D^{u,e_2})\}$$

The proof of Proposition 5.4 is analogous to the proof of the foregoing proposition.

PROPOSITION 5.4. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to Definition 3.2. Furthermore, let the function CF′ be defined according to Definition 3.10 and the function $CF_{\&}$ according to Definition 3.13. Let $e_i \in \mathscr{E}$ and $D^{u,e_i} \in \mathscr{D}$, $i = 1,2$. Then,*

$$CF'(e_1 \wedge e_2 \dashv u, D^{u,e_1} \,\&\, D^{u,e_2}) = CF_{\&}(e_1 \wedge e_2 \dashv u, D^{u,e_1} \,\&\, D^{u,e_2}).$$

The combination function that remains to be examined is the combination function for co-concluding production rules. We recall from Definition 3.14 that this combination function is defined as stated below. Once more we have renounced errors that were introduced earlier in the approximation of the actual certainty factors, i.e. we have assumed the property $\overline{CF}(h \dashv u, D_i^{u,h}) = CF(h \dashv u, D_i^{u,h})$, $i = 1,2$:

(1) $CF_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}) = CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h})(1 - CF(h \dashv u, D_1^{u,h}))$
if $CF(h \dashv u, D_1^{u,h}) > 0$ and $CF(h \dashv u, D_2^{u,h}) > 0$, and

(2) $$CF_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}) = \frac{CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h})}{1 - min\{|CF(h \dashv u, D_1^{u,h})|, |CF(h \dashv u, D_2^{u,h})|\}}$$
if $-1 < CF_{\|}(h \dashv u, D_1^{u,h}) \cdot CF_{\|}(h \dashv u, D_2^{u,h}) \leqslant 0$, and

(3) $CF_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}) = CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h})(1 + CF(h \dashv u, D_1^{u,h}))$
if $CF(h \dashv u, D_1^{u,h}) < 0$ and $CF(h \dashv u, D_2^{u,h}) < 0$.

In Proposition 5.5 it is shown that the combination function $CF_{\|}$ respects the definition of the function CF'.

PROPOSITION 5.5. *Let $\mathscr{E}$ and u be defined according to Definition 3.1 and $\mathscr{D}$ according to definition 3.2. Furthermore, let the function CF′ be defined according to Definition 3.10 and the function $CF_{\|}$ according to Definition 3.14. Let $e_i \in \mathscr{E}$ and $D^{u,e_i} \in \mathscr{D}$, $i = 1,2$. Then,*

$$CF'(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}) = CF_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}).$$

PROOF. It follows from Definition 3.14 that we have to show

$CF'(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}) =$

$$= \begin{cases} CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h})\,(1 - CF(h \dashv u, D_1^{u,h})) & \text{if } CF(h \dashv u, D_1^{u,h}),\ CF(h \dashv u, D_2^{u,h}) > 0 \\ \dfrac{CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h})}{1 - min\{|CF(h \dashv u, D_1^{u,h})|,\ |CF(h \dashv u, D_2^{u,h})|\}} & \text{if } -1 < CF(h \dashv u, D_1^{u,h}) \cdot CF(h \dashv u, D_2^{u,h}) \leqslant 0 \\ CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h})\,(1 + CF(h \dashv u, D_1^{u,h})) & \text{if } CF(h \dashv u, D_1^{u,h}),\ CF(h \dashv u, D_2^{u,h}) < 0 \end{cases}$$

From Definition 3.10 we have

$$CF'(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}) = \frac{MB_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}) - MD_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h})}{1 - min\{MB_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h}), MD_{\|}(h \dashv u, D_1^{u,h} \,\|\, D_2^{u,h})\}}.$$

We will consider this fraction in detail.

(1) Assume $MB(h \dashv u, D_1^{u,h}) > 0$ and $MB(h \dashv u, D_2^{u,h}) > 0$. The case $MD(h \dashv u, D_1^{u,h}) > 0$ and $MD(h \dashv u, D_2^{u,h}) > 0$ follows by symmetry.

From Proposition 3.1 we have $MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$. Hence, from our assumptions it follows that $CF(h \dashv u, D_1^{u,h}) > 0$ and $CF(h \dashv u, D_2^{u,h}) > 0$.

From $MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$ we have $MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = 0$. Therefore, the denumerator for the fraction equals 1. It follows that

$$
\begin{aligned}
&CF'(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = \\
&\quad = MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) - MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = \\
&\quad = MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h}) - MB(h \dashv u, D_1^{u,h}) \cdot MB(h \dashv u, D_2^{u,h}) = \\
&\quad = (MB(h \dashv u, D_1^{u,h}) - MD(h \dashv u, D_1^{u,h})) + (MB(h \dashv u, D_2^{u,h}) - MD(h \dashv u, D_2^{u,h})) + \\
&\qquad - (MB(h \dashv u, D_1^{u,h}) - MD(h \dashv u, D_1^{u,h})) \cdot (MB(h \dashv u, D_2^{u,h}) - MD(h \dashv u, D_2^{u,h})).
\end{aligned}
$$

From $1 - \min\{MB(h \dashv u, D_i^{u,h}), MD(h \dashv u, D_i^{u,h})\} = 1$, $i = 1,2$, we have

$$
\begin{aligned}
MB(h \dashv u, D_i^{u,h}) - MD(h \dashv u, D_i^{u,h}) &= \frac{MB(h \dashv u, D_i^{u,h}) - MD(h \dashv u, D_i^{u,h})}{1 - \min\{MB(h \dashv u, D_i^{u,h}), MD(h \dashv u, D_i^{u,h})\}} = \\
&= CF(h \dashv u, D_i^{u,h}).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&CF'(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = \\
&\quad = CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h}) - CF(h \dashv u, D_1^{u,h}) \cdot CF(h \dashv u, D_2^{u,h}) = \\
&\quad = CF(h \dashv u, D_1^{u,h}) + CF(h \dashv u, D_2^{u,h})(1 - CF(h \dashv u, D_1^{u,h})).
\end{aligned}
$$

(2) Now assume $MB(h \dashv u, D_1^{u,h}) = 0$ and $MB(h \dashv u, D_2^{u,h}) > 0$. The case $MB(h \dashv u, D_1^{u,h}) > 0$ and $MB(h \dashv u, D_2^{u,h}) = 0$, similar cases for MD and the case where $MB(h \dashv u, D_1^{u,h}) = MB(h \dashv u, D_2^{u,h}) = MD(h \dashv u, D_1^{u,h}) = MD(h \dashv u, D_2^{u,h}) = 0$ follow by symmetry.

From Proposition 3.1 we have $MD(h \dashv u, D_1^{u,h}) \geqslant 0$ and $MD(h \dashv u, D_2^{u,h}) = 0$. Hence, from our assumptions we have $CF(h \dashv u, D_1^{u,h}) \leqslant 0$ and $CF(h \dashv u, D_2^{u,h}) > 0$. From now on we assume $CF(h \dashv u, D_1^{u,h}) \cdot CF(h \dashv u, D_2^{u,h}) > -1$. So, the numerator of the fraction can be written as follows

$$
\begin{aligned}
&MB_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) - MD_{\|}(h \dashv u, D_1^{u,h} \| D_2^{u,h}) = \\
&\quad = MB(h \dashv u, D_1^{u,h}) + MB(h \dashv u, D_2^{u,h}) - MB(h \dashv u, D_1^{u,h}) \cdot MB(h \dashv u, D_2^{u,h}) + \\
&\qquad - MD(h \dashv \dashv u, D_1^{u,h}) - MD(h \dashv u, D_2^{u,h}) + MD(h \dashv u, D_1^{u,h}) \cdot MD(h \dashv u, D_2^{u,h}) = \\
&\quad = MB(h \dashv u, D_2^{u,h}) - MD(h \dashv u, D_1^{u,h}) = \\
&\quad = (MB(h \dashv u, D_1^{u,h}) - MD(h \dashv u, D_1^{u,h})) + (MB(h \dashv u, D_2^{u,h}) - MD(h \dashv u, D_2^{u,h})).
\end{aligned}
$$

From $1 - min\{MB(h,e_1),MD(h,e_1)\} = 1 - min\{MB(h,e_2),MD(h,e_2)\} = 1$ we have

$$MB_{||}(h \dashv u,D_1^{u,h} \parallel D_2^{u,h}) - MD_{||}(h \dashv u,D_1^{u,h} \parallel D_2^{u,h}) =$$

$$= \frac{MB(h \dashv u,D_1^{u,h}) - MD(h \dashv u,D_1^{u,h})}{1 - min\{MB(h \dashv u,D_1^{u,h}),MD(h \dashv u,D_1^{u,h})\}} +$$

$$+ \frac{MB(h \dashv u,D_2^{u,h}) - MD(h \dashv u,D_2^{u,h})}{1 - min\{MB(h \dashv u,D_2^{u,h}),MD(h \dashv u,D_2^{u,h})\}} =$$

$$= CF(h \dashv u,D_1^{u,h}) + CF(h \dashv u,D_2^{u,h}).$$

The denumerator of the fraction equals

$$1 - min\{MB_{||}(h \dashv u,D_1^{u,h} \parallel D_2^{u,h}),MD_{||}(h \dashv u,D_1^{u,h} \parallel D_2^{u,h})\} =$$

$$= 1 - min\{MB(h \dashv u,D_1^{u,h}) + MB(h \dashv u,D_2^{u,h}) - MB(h \dashv u,D_1^{u,h}) \cdot MB(h \dashv u,D_1^{u,h}),$$
$$MD(h \dashv u,D_1^{u,h}) + MD(h \dashv u,e_2^{u,h}) - MD(h \dashv u,D_1^{u,h}) \cdot MD(h \dashv u,D_2^{u,h})\} =$$
$$= 1 - min\{MB(h \dashv u,D_2^{u,h}),MD(h \dashv u,D_1^{u,h})\}.$$

It can easily be shown that

$$MB(h \dashv u,D_2^{u,h}) = \frac{MB(h \dashv u,D_2^{u,h}) - MD(h \dashv u,D_2^{u,h})}{1 - min\{MB(h \dashv u,D_2^{u,h}),MD(h \dashv u,D_2^{u,h})\}} =$$

$$= |CF(h \dashv u,D_2^{u,h})|.$$

Furthermore, we can easily show that

$$MD(h \dashv u,D_1^{u,h}) = - \frac{MB(h \dashv u,D_1^{u,h}) - MD(h \dashv u,D_1^{u,h})}{1 - min\{MB(h \dashv u,D_1^{u,h}),MD(h \dashv u,D_1^{u,h})\}} =$$

$$= |CF(h \dashv u,D_1^{u,h})|.$$

So, we have

$$CF'(h \dashv u,D_1^{u,h} \parallel D_2^{u,h}) = \frac{CF(h \dashv u,D_1^{u,h}) + CF(h \dashv u,D_2^{u,h})}{1 - min\{|CF(h \dashv u,D_1^{u,h})|,|CF(h \dashv u,D_2^{u,h})|\}}.$$

■

From the Proposition 5.2, 5.3, 5.4 and 5.5 we have that $\overline{CF} \simeq CF'$. The propositions even prove the stronger result stated in Theorem 5.2.

THEOREM 5.2. *Let the function CF' be defined according to Definition 3.10 and the function $\overline{CF}$ according to Definition 3.11. Then, $CF' = \overline{CF}$.*

5.3. Summary

In this section we have investigated the relations between the functions CF, CF' and $\overline{CF}$. In Section 5.1 we have shown that the certainty factor function CF defined by Shortliffe and Buchanan, and the function CF' actually used in the MYCIN implementation of the model, do not always render the same function values for the arguments of interest. In Section 5.2 we have shown that the approximation function $\overline{CF}$ used in the EMYCIN implementation respects the definition of CF'. In fact, CF' and $\overline{CF}$ coincide. In Figure 5.1 the results from this section have been added to the diagram of functions introduced in Section 3.

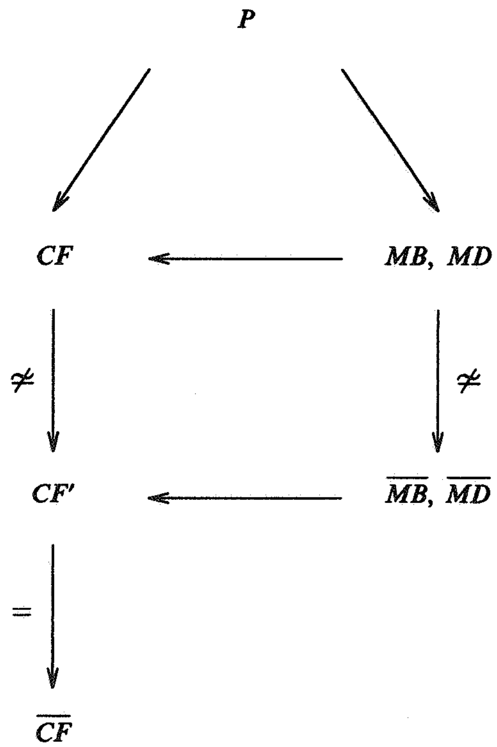

FIGURE 5.1. The diagram of functions.

6. Conclusion

The certainty factor model has been proposed by E.H. Shortliffe and B.G. Buchanan as a model for handling uncertainty in rule-based top-down reasoning expert systems. The model has been employed in a wide variety of expert systems. In this paper we have not discussed the issues that arise when actually using the model, for instance we have not looked at the model from a knowledge engineering point of view.

The presentation of the model by Shortliffe and Buchanan lacks a proper formal description. We have introduced definitions of the basic notions of rule-based top-down reasoning expert systems that are used in the model such as productions rules and derivations. From these preliminaries we have presented a formal description of the model in Section 3. In this formalisation we have extended the functions used in the model with the notion of derivation. In [6] the reaser can find our motivation for doing so.

In the model two basic measures of uncertainty are defined: the measures of belief and disbelief. These measures are defined in terms of a probability set function. We have not addressed the question whether these measures of belief and disbelief actually model the type of uncertainty that is encountered in real-life domains. Shortliffe and Buchanan argue that such a probability set function is rarely known and that the actual function values of the measures of belief and disbelief should be approximated. For that purpose they have introduced approximation functions. We have investigated in Section 4 whether these approximation functions respect the probabilistic definitions of the measures of belief and disbelief. Section 4.4 summarises the results of our analysis of these approximation functions: we have shown that in some cases these functions respect the probabilistic basis by making rather strong assumptions. We have not paid attention to the question whether it is expected that these conditions are met in practice. Nor have we discussed the impact of the application of the model in situations in which the assumed properties do not hold. In other cases the approximation functions cannot be shown to be consistent with the foundation in probability theory suggested in [1]. We have not investigated the error introduced by applying the functions in these cases.

In actual implementations of the certainty factor model the measures of belief and disbelief are not used. A third function derived from these two basic measures is used: the (redefined) certainty factor function. This function is defined in terms of the measures of belief and disbelief. For the purpose of subsequently computing certainty factors again an approximation function is defined. Section 5 shows that this approximation function respects the definition of the certainty factor function.

Figure 5.1 summarizes the main results of this paper.

7. References

[1] E.H. Shortliffe & B.G. Buchanan (1984). A model of inexact reasoning in medicine, in: B.G. Buchanan & E.H. Shortliffe (eds.), *Rule-Based Expert Systems. The MYCIN Experiments of the Stanford Heuristic Programming Project*, Addison-Wesley, Reading, Massachusetts, pp. 233 - 262.

[2] J.B. Adams (1984). Probabilistic reasoning and certainty factors, in: B.G. Buchanan & E.H. Shortliffe (eds.), *Rule-Based Expert Systems. The MYCIN Experiments of the Stanford Heuristic Programming Project*, Addison-Wesley, Reading, Massachusetts, pp. 263 - 271.

[3] B.P. Wise & M. Henrion (1986). A framework for comparing uncertain inference systems to probability, in: L.N. Kanal & J.F. Lemmer (eds.), *Uncertainty in Artificial Intelligence*, North-Holland, Amsterdam.

[4] D. Heckerman (1986). Probabilistic interpretations for MYCIN's certainty factors, in: L.N. Kanal & J.F. Lemmer (eds.), *Uncertainty in Artificial Intelligence*, North-Holland, Amsterdam.

[5] M. Ishizuka, K.-S. Fu & J.T.P. Yao (1981). *A Theoretical Treatment of Certainty Factor in Production Systems*, Report no. CE-STR-81-6, School of Civil Engineering, Purdue university,

West Lafayette.

[6] L.C. VAN DER GAAG (1987). *A Network Approach to the Certainty Factor Model,* Report no. CS-R8757, Centre for Mathematics and Computer Science, Amsterdam.

[7] R.V. HOGG & A.T. CRAIG (1978). *Introduction to Mathematical Statistics,* MacMillan Publishing Company, New York.