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ABSOLUTENESS OF INTUITIONISTIC LOGIC

Preliminary report

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Absoluteness of Intuitionistic Logic

by

Daniel Leivant

ABSTRACT

We say that a logical calculus L is <u>absolute</u> for a class C of number-theoretic sentences, if for every schema F in the languages of L, $F \notin L \Rightarrow F^* \notin C$ for some arithmetical instance F^* of F. So when C is the class of all true sentences (according to a given truth-definition), then "L is absolute for C" reads "L is (weakly) complete (for the given notion of truth)".

We deal here with intuitionistic propositional and predicate logics (L_0 and L_1 resp.), for which we prove absoluteness for intuitionistic (Heyting's) arithmetic A, and for certain extensions of A.
(The term "absoluteness" is used here rather than "completeness" because "completeness" refers traditionally to a notion of semantic truth. "L is absolute for C" is sometimes expressed as "C is faithful to L".)

KEY WORDS & PHRASES: *Predicate logic, absoluteness, de Jongh's theorem, regular number theories.*

1. INTRODUCTION

1.1. STATEMENT OF THE RESULTS

When $F[p_1,\ldots,p_k]$ is a scheme of L_0 with (at most) the k propositional letters shown, and when $A_1,\ldots,A_k$ are arithmetical sentences, write $F[A_1,\ldots,A_k]$ for the sentence which comes from $F[p_1,\ldots,p_k]$ by substituting A_i for every occurrence of p_i $(i=1,\ldots,k)$. When $F[P_1^{n_1},\ldots,P_k^{n_k}]$ is a scheme of L_1 with (at most) the k predicate letters shown, where $P_i^{n_i}$ is n_i-place, and $A_i^{n_i}$ is an arithmetical formula with n_i free variables $(i=1,\ldots,k)$, write $F[A_1,\ldots,A_k]$ for the formula which comes from $F[P_1,\ldots,P_k]$ by replacing every atomic subformula $P_i(x_1,\ldots,x_{n_i})$ by $A_i(x_1,\ldots,x_{n_i})$.

Regular and strongly regular number theories are defined in 1.4 below. It will be shown elsewhere that the class of strongly regular number theories embraces the arithmetical fragments of the majority of intuitionistic formal systems.

<u>THEOREM I</u> (Locally uniform Σ_1^0 absoluteness for L_0).
Let A^* be a regular number theory. For every $k < \omega$ there are Σ_1^0 sentences $A_1,\ldots,A_k$ s.t.

$$L_0 \nvdash F[p_1,\ldots,p_k] \quad \Rightarrow \quad A^* \nvdash F[A_1,\ldots,A_k].$$

Or more precisely: there is a quantifier-free (q.f.) formula $E_0(x)$ s.t.

$$\vdash_A \forall k \forall x_{\underline{L_0\text{-Fml}}(x)} \left[\neg \underline{Pr}_{L_0}(x) \;\;\&\;\; \nu(x) \le k \;\rightarrow\; \neg \underline{Pr}_{A^*}(\underline{sub}^k_{\Sigma_1^0}(x, \ulcorner E_0 \urcorner) \right]$$

where

$\underline{L_0\text{-Fml}}(x)$:≡ "x *is the g.n. of a schema in the language of* L_0";

$\underline{\text{Pr}}_{L_0}$ *is a (fixed) provability predicate for* L_0;

$\nu(\ulcorner F \urcorner)$:= "*the number of propositional letters occurring in* F",

and $\underline{\text{sub}}^k_{\Sigma^0_1}$ *is a prim. rec. function which satisfies*

$$\underline{\text{sub}}^k_{\Sigma^0_1}(\ulcorner F[p_1,\dots,p_k]\urcorner, \ulcorner E_0\urcorner) = \ulcorner F[\exists x E_0\langle k,x\rangle,\dots,\exists x E_0\langle k,x\rangle]\urcorner.$$

THEOREM II (Globally uniform Π^0_2 absoluteness for L_1).
Let A^* *be a strongly regular number theory. There are* Π^0_2 *predicates* $\{A^j_i\}_{i,j<\omega}$ *s.t.*

$$L_1 \nvdash F[P^{n_{i_1}}_{i_1},\dots,P^{n_{i_k}}_{i_k}] \quad\Rightarrow\quad A^* \nvdash F[A^{n_1}_{i_1},\dots,A^{n_k}_{i_k}].$$

Or more precisely: there is a q.f. formula $E_1(x)$ *s.t.*

$$\vdash_A \forall x_{\underline{L_1\text{-Fml}}(x)}\left[\neg\underline{\text{Pr}}_{L_1}(x) \rightarrow \neg\underline{\text{Pr}}_{A^*}(\text{sub}_{\Pi^0_2}(x,\ulcorner E_1\urcorner))\right]$$

where $\underline{\text{sub}}_{\Pi^0_2}$ *is a prim. rec. function which satisfies*

$$\underline{\text{sub}}_{\Pi^0_2}(\ulcorner F[P^{n_1}_1,\dots,P^{n_k}_k]\urcorner, \ulcorner E_1\urcorner) = \ulcorner F[Q^{n_1}_1,\dots,Q^{n_k}_k]\urcorner.$$

where

$$Q^{n_i}_i(\vec{z}) :\equiv \forall x \exists y\ E_1\langle x,y,i,n_i,\langle\vec{z}\rangle\rangle.$$

1.2. HISTORICAL NOTE

D.H.J. DE JONGH has proved already in 1969 the absoluteness of L_0 for A (and extensions $A^{\prec}$ of A with transfinite induction over some prim.rec. well-ordering $\prec$). C. SMORYNSKI ([72]) proved that the substitution may be chosen to be Σ^0_1, but not uniformly in the logical schemata. H. FRIEDMAN ([72]) proved that there is a globally uniform Π^0_2 substitution for the absoluteness of L_0. This last result is essentially a corollary of our theorem II.

All the results just mentioned were obtained in classical metamathematics. It seems, however, that they can be reformulated in intuitionistic metamathematics, in particular in view of the recent discovery by W. FELDMAN and H. DE SWART of intuitionistic completeness proofs for Kripke's semantics.

So the main novelty of theorem I is the locally uniform Σ_1^0 substitution. Nevertheless, we present this result in some detail, for two reasons. Firstly, it may be used as an expository introduction to the proof of theorem II; secondly, the method employed might turn out te be helpful in solving a number of other problems concerning the relation between L_0 and A.

As to predicate logic, DE JONGH proved (unpublished) the (local) absoluteness (for A) of the disjunction-free fragment of L_1; he also proved the absoluteness of full L_1, but where in each formula all quantifiers are restricted to a fixed unary predicate. These two restrictions allow a model theoretic treatment using Kripe models with a constant universe, and a special notion of "forced realizability" which utilizes results from the theory of Turing degrees.

1.3. DESCRIPTION OF A^∞.

By a <u>sentence</u> we mean a closed formula of A built up from 0, f_j^i (i,j=0,1,...), =, ⊥, &, v, →, ∀, ∃ and bounded variables. A <u>sequent</u> is a syntactical object of the form $\underline{a} \Rightarrow F$ where $\underline{a}$ is a finite set of sentences and F is a sentence.

Propositional rules of A^∞:

[T] $\qquad \underline{a} \Rightarrow F \qquad$ where $F \in \underline{a}$

[&I] $\qquad \dfrac{\underline{a} \Rightarrow F_0 \qquad \underline{a} \Rightarrow F_1}{\underline{a} \Rightarrow F_0 \& F_1}$; $\qquad$ [&E$_i$] $\qquad \dfrac{\underline{a} \Rightarrow F_0 \& F_1}{\underline{a} \Rightarrow F_i}$ $\qquad$ (i=0,1)

[→I] $\qquad \dfrac{\underline{a}, F \Rightarrow G}{\underline{a} \Rightarrow F \to G}$; $\qquad$ [→E] $\qquad \dfrac{\underline{a} \Rightarrow F \to G \qquad \underline{a} \Rightarrow F}{\underline{a} \Rightarrow G}$

(where $\underline{a}, F$ stands for $\underline{a} \cup \{F\}$)

[vI$_i$] $\qquad \dfrac{\underline{a} \Rightarrow F_i}{\underline{a} \Rightarrow F_0 v F_1}$ (i=0,1); $\qquad$ [vE] $\qquad \dfrac{\underline{a} \Rightarrow F_0 v F_1 \quad \underline{a}, F_0 \Rightarrow G \quad \underline{a}, F_1 \Rightarrow G}{\underline{a} \Rightarrow G}$

[⊥] $\qquad \dfrac{\underline{a} \Rightarrow \bot}{\underline{a} \Rightarrow F}$

Quantification and arithmetical rules of A^∞:

[TE] $\underline{a} \Rightarrow E$ where E is a true equation when every function-symbol f^i_j is interpreted as the j'th i-place prim. rec. function.

[FE] $$\frac{\underline{a} \Rightarrow E}{\underline{a} \Rightarrow \bot}$$ where E is a false equation.

[∀I] $$\frac{\langle \underline{a} \Rightarrow F(\bar{n}) \rangle_{n<\omega}}{\underline{a} \Rightarrow \forall x F(x)}$$

[∀E] $$\frac{\underline{a} \Rightarrow \forall x F(x)}{\underline{a} \Rightarrow F(t)}$$ (t a term); [∃I] $$\frac{\underline{a} \Rightarrow F(t)}{\underline{a} \Rightarrow \exists x F(x)}$$

[∃E] $$\frac{\underline{a} \Rightarrow \exists x F(x) \quad \langle \underline{a}, F(\bar{n}) \Rightarrow G \rangle_{n<\omega}}{\underline{a} \Rightarrow G}$$

A function ϕ is a derivation of A^∞ (notation: $\underline{\text{Der}}^\infty(\phi)$) if

(1) ϕ describes a tree: $\phi u = 0 \longrightarrow \phi(u * \langle n \rangle) = 0$,
$\phi(u * \langle n \rangle) = 0 \longrightarrow \phi(u * \langle n+1 \rangle) = 0$;
(where * denotes concatenation of sequent numbers).

(2) For every u (= the code of a node in the universal spread) $(\phi u)_0$ is the code of one of the inference rules ρ above (under some fixed encodement), while $(\phi u)_1$ and $(\phi(u * \langle n \rangle))_1$ $(n<\omega)$ are codes of sequents which relate as the conclusion and the premise sequents of ρ (and when no n'th premise is required, $(\phi(u * \langle n \rangle))_1 = 0$).

(3) ϕ is well-founded: $\forall X \exists x\, \phi(\bar{X}(x)) = 0$.

EXAMPLE. The ("informal") derivation

$$\frac{[T] \quad \{A\} \Rightarrow A \qquad\qquad [TE] \quad \{A\} \Rightarrow \bar{0} = \bar{0}}{[\&I] \quad \{A\} \Rightarrow A \;\&\; \bar{0} = \bar{0}}$$

is formalized by the function ϕ defined by

$\phi\langle\ \rangle := \langle \ulcorner \&I \urcorner, \ulcorner \{A\} \Rightarrow A \;\&\; \bar{0} = \bar{0} \urcorner \rangle$

$\phi\langle 0 \rangle := \langle \ulcorner T \urcorner, \ulcorner \{A\} \Rightarrow A \urcorner \rangle$

$\phi\langle 1 \rangle := \langle \ulcorner TE \urcorner, \ulcorner \{A\} \Rightarrow \bar{0} = \bar{0} \urcorner \rangle$

$\phi u := 0$ for every $u \notin \{\langle\ \rangle, \langle 0 \rangle, \langle 1 \rangle\}$.

A number d is a *recursive derivation of* A^∞ (notation: $\underline{\text{Der}}^\infty_{\text{rec}}(d)$) if {d} is a total function (i.e. - $\forall x \exists y T(d,x,y)$) and clauses (1)-(3) above hold when ϕ and = are replaced by {d} and $\simeq$ respectively.

A derivation ϕ is *normal* (notation: $\underline{\text{NDer}}^\infty(\phi)$) if:

(1) No major (i.e. - leftmost) premise of an elimination rule in ϕ is derived by an instance of an introduction rule;

(2) No major premise of an elimination rule nor a premise of an instance of [∃I] or [FE] is derived by an instance of [vE], [∃E] or [⊥].

The central property of normal derivations is the *subformula property*: every formula occurring in a normal derivation is a subformula of the derived sequent. We shall assume this property of normal derivations without proof.

$\underline{\text{Prf}}^\infty(\phi, \ulcorner F \urcorner) :\equiv \underline{\text{Der}}^\infty(\phi)\ \&\ \phi\langle\ \rangle = \ulcorner \Rightarrow F \urcorner$. Predicates like $\underline{\text{NDer}}^\infty_{\text{rec}}(d)$, $\underline{\text{NPrf}}^\infty_{\text{rec}}(d, \ulcorner F \urcorner)$ etc. are defined analogously.

1.4. REGULAR NUMBER THEORIES

Let T be a theory in the language of analysis. Write

$$A^\infty[T] \quad :\equiv \quad \{F \mid T \vdash \exists \phi\ \underline{\text{NPrf}}^\infty(\phi, \ulcorner F \urcorner)\}$$

$$A^\infty_{\text{rec}}[T] :\equiv \quad \{F \mid \exists d\ [T \vdash \text{NPrf}^\infty_{\text{rec}}(d, \ulcorner F \urcorner)]\}$$

or, otherwise stated,

$$\underline{\text{Pr}}_{A^\infty[T]}(\ulcorner F \urcorner) \quad :\equiv \underline{\text{Pr}}_T \ulcorner \exists \phi\ \underline{\text{Nprf}}^\infty(\phi, \ulcorner F \urcorner) \urcorner$$

$$\underline{\text{Pr}}_{A^\infty_{\text{rec}}[T]}(\ulcorner F \urcorner) :\equiv \exists d\ \underline{\text{Pr}}_T \ulcorner \underline{\text{NPrf}}^\infty_{\text{rec}}(d, \ulcorner F \urcorner) \urcorner.$$

An r.e. set A^* of arithmetical sentences, closed under Modus Ponens, is a *regular number theory* when for some consistent r.e. $T \supseteq \mathcal{V}_0 + \text{BI}$, $A^* \subseteq A^\infty_{\text{rec}}[T]$. Here $\mathcal{V}_0$ stands for intuitionistic elementary analysis, and can be identified with the theory H of HOWARD-KREISEL [66]; BI stands for the schema BI_D of bar-induction for decidable predicates on p.336 there.

For T as above, let

$$T^* :\equiv T^C + AC_{00} + \Pi^0_1$$

where T^{c} is the classical completion of T, AC_{00} is the schema

$$\forall x \exists y A(x,y) \longrightarrow \exists \alpha \forall x A(x,\alpha x),$$

and Π_1^0 is the set of all true Π_1^0 sentences. Formally, we define the proof predicate $\underline{\mathrm{Prf}}_T$ by

$$\underline{\mathrm{Prf}}_{T^*}(p,\ulcorner F \urcorner) :\equiv \exists x < p \text{ "x encodes a conjunction of instances of the rule of excluded third, of instances of } AC_{00} \text{ and of true } \Pi_1^0 \text{ sentences" } \& \ \underline{\mathrm{Prf}}_T(p,\underline{\mathrm{imp}}(x,\ulcorner F \urcorner))$$

where $\underline{\mathrm{imp}}$ is a prim. rec. function which satisfies

$$\underline{\mathrm{imp}}(\ulcorner F \urcorner,\ulcorner G \urcorner) = \ulcorner F \rightarrow G \urcorner.$$

A theory A^* as above is defined now to be <u>strongly regular</u> if there is an r.e. theory T s.t. T^* is consistent and $A^* \subseteq A^{\infty}_{\mathrm{rec}}[T^*]$.

2. RECURSION THEORETIC SOLUTION OF A REDUCED FORM OF THEOREM I.

2.0. We wish to find Σ_1^0 sentences $A_1,\dots,A_k$ s.t.

$$(*) \qquad \not\vdash_{L_0} F[p_1,\dots,p_k] \quad \Rightarrow \quad \not\vdash_{A^*} F[A_1,\dots,A_k].$$

If the theories L_0 and A^* are replaced by their classical completions, the solution could be based on truth-values arguments, using recursion-theoretic methods only, as was done (independently) by KRIPKE [63] and MYHILL [72]. The complication for the intuitionistic case depends mainly on the presence of implications in the schema F, or more precisely - on negative nestings of implications. It is in such cases that the usual intuitionistic interpretation of connectives uses a notion of impredicativity ("for every construction.... there is a construction....").

Let us count the negative nestings of implications by a measure μ, i.e. -

$$\begin{aligned}
&\mu\ulcorner F\urcorner := \text{for atomic } F,\\
&\mu\ulcorner F\&G\urcorner := \mu\ulcorner F\vee G\urcorner := \underline{\max}[\mu\ulcorner F\urcorner,\mu\ulcorner G\urcorner],\\
&\mu\ulcorner F\to G\urcorner := \underline{\max}[\mu\ulcorner F\urcorner+1,\mu\ulcorner G\urcorner]; \text{ and for the full language of } L_1,\\
&\mu\ulcorner \forall xF\urcorner := \mu\ulcorner \exists xF\urcorner := \mu\ulcorner F\urcorner
\end{aligned}$$

We shall see that for schemata F s.t. $\mu\ulcorner F\urcorner \le 1$ the classical recursion-theoretic methods work. The complexity involved in the growth of the μ-measure is further illustrated by the fact (cf. LEIVANT [74]) that the consistency of A_k is provable in A_{k+1} for every k, where

$$A_k :\equiv A \text{ restricted to formulae F s.t. } \mu\ulcorner F\urcorner \le k.$$

2.1. STATEMENT OF THE REDUCED SOLUTION

We define a sequence U_k of propositional schemata, where $U_k \equiv U_k[p_1,\ldots,p_k]$ and $\mu\ulcorner U_k \urcorner \leq 1$ as follows:

$$U_0 \quad :\equiv \bot$$
$$U_1[p] :\equiv p \vee \neg p.$$

Assuming U_k to be defined, let

$$U_k^i[p_1,\ldots,p_{k+1}] \quad :\equiv U_k[p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_{k+1}]$$
$$U_{k+1}[p_1,\ldots,p_{k+1}] :\equiv \bigvee_{i=1,\ldots,k+1} [p_i \to U_k^i].$$

We shall solve in this section (*) for the schemata U_k, i.e. -

PROPOSITION. *We can uniformly construct* Σ_1^0 *sentences* $A_1^k,\ldots,A_k^k$ *s.t.*

$$\nvdash_{A^*} U_k[A_1^k,\ldots,A_k^k] \qquad (k<\omega).$$

Here A^* *may be taken to be any consistent r.e. extension of* A *which satisfies disjunction instantiation (the so-called "disjucntion property"), i.e.* -

$$\vdash_{A^*} A \vee B \quad \Rightarrow \quad [\vdash_{A^*} A \text{ or } \vdash_{A^*} B].$$

2.2. Actually proposition 2.1 gives a solution of (*) for *all* schemata F s.t. $\mu\ulcorner F \urcorner \leq 1$, on account of the following

PROPOSITION. *For any schema* F *of* L_0 *s.t.* $\mu\ulcorner F \urcorner \leq 1$,

$$\nvdash_{L_0} F[p_1,\ldots,p_k] \quad \Rightarrow \quad \nvdash_{L_0} F \to U_k.$$

SKETCH OF PROOF. Use a primary induction on k (= the number of propositional letters occurring in F), secondary induction on the length of F, and ternary induction on the length of the left main subformula of F. □

2.3. LEMMA. (propositional logic. Compare KLEENE [52] §33).

[a1] *If* G *is a positive occurrence of a subformula of* F *(see e.g.* PRAWITZ [65] p.43 *for the definition of positive and negative occurrences) then*

$$E \vdash_{L_0} G \to H \quad \Rightarrow \quad E \vdash_{L_0} F \to F[H/\underline{G}]$$

(where F[H/G] *comes from* F *by replacing the occurrence* G *by* H*)*

[a2] *If* G *is a negative occurrence in* F*, then*

$$E \vdash_{L_0} G \to H \quad \Rightarrow \quad E \vdash_{L_0} F[H/\underline{G}] \to F$$

[b] *Let* F^q *be the propositional schema which comes from* F *by replacing (simultaneously) every occurrence of some (fixed) propositional letter* p *in* F *by* pvq, *where* q *is a fixed propositional letter. Then*

$$\neg q \vdash_{L_0} F^q \to F.$$

PROOF.

[a]: Straightforward by induction on the length of F (simultaneously for [a1] and [a2]).

[b]: Since $q \vdash_{L_0} p \vee q$ we get by repeated application of [a1] (*) $\vdash_{L_0} F^{q-} \to F$, where F^{q-} comes from F by replacing only negative occurrences p in F by pvq. But $\neg q \vdash_{L_0} p \vee q \to p$, so we get by iterated application of [a2]: (**) $\neg q \vdash_{L_0} F^q \to F^{q-}$. (*) and (**) yield [b]. □

2.4. SIMPLIFIED DEFINITION OF EFFECTIVELY INSEPARABLE R.E. SETS

It is just to smoothen the exposition that we use the following

LEMMA. *Two disjoint r.e. sets* A,B *are effectively inseparable iff there is a recursive function* f *s.t.*

$$\left.\begin{matrix} W_i \cap A = \emptyset \\ W_j \cap B = \emptyset \end{matrix}\right\} \Rightarrow f(i,j) \notin W_i \cup W_j$$

PROOF.

I. The "if" direction is trivial, since the function f satisfies more than what is required from a function of effective inseparability (cf. e.g. ROGERS [67] p.94).

II. Let, on the other hand, f_1 be a function of effective inseparability for A,B and let i,j satisfy

(1) $A \cap W_i = \emptyset, \quad B \cap W_j = \emptyset.$

By the reduction principle (cf. ROGERS [67], p.72) there are functions g,h s.t.

(2) $W_{g(i)} \subseteq W_i; \quad W_{h(j)} \subseteq W_j,$

(3) $W_{g(i)} \cup W_{h(j)} = W_i \cup W_j$ and

(4) $W_{g(i)} \cap W_{h(j)} = \emptyset.$

Take now

(5) $W_{g'(i)} := W_{g(i)} \cup B; \quad W_{h'(j)} := W_{h(j)} \cup A.$

Then

(6) $W_{g'(i)} \supseteq B; \quad W_{h'(j)} \supseteq A$

while by (4), (2), (1) and the assumed $A \cap B = \emptyset$,

(7) $$W_{g'(i)} \cap W_{h'(j)} = [W_{g(i)} \cap W_{h(j)}] \cup [W_{g(i)} \cap A] \cup$$
$$\cup [W_{h(j)} \cap B] \cup [A \cap B] \quad = \emptyset.$$

For the f defined by

$$f(i,j) := f_1(g'(i),h'(j))$$

we have now, by (6) (7) and the choice of f_1 that $f(i,j) \notin W_i \cup W_j$ as required. □

2.5. DEFINITION OF THE DESIRED Σ_1^0 SENTENCES

The following construction generalizes the method of MYHILL [72]. Let A,B be r.e. sets, effectively inseparable (in the sense of 2.4) through the function f, and let A^* be any consistent r.e. extension of A. Following

SHEPHERDSON [60] we may define (explicitly) a Σ_1^0 formula $F(a) \equiv \exists x F_0(x,a)$ s.t.

(1) $$A = \{m \mid \vdash_{A^*} F(\bar{m})\}; \quad B = \{m \mid \vdash_{A^*} \neg F(\bar{m})\}$$

(To see that this holds also intuitiouistically, either inspect Shepherdson's proof, or observe that the equations above are formalizable as Π_2^0 statements.)

We construct now, by recursion on k, an infinite sequence $\{A_i^k\}_{i<\omega}$ s.t.

(2) $$\nvdash_{A^*} U_k[A_{i_1}^k, \ldots, A_{i_k}^k] \text{ for every distinct } i_1, \ldots, i_k.$$

<u>Basis</u>: By the assumed properties of A^* there is a Rosser sentence R for A^*; set $A_i :\equiv R$ for every i.

<u>Recursion step</u>: Assume A_i^k, $i < \omega$ to be defined and to satisfy (2). We define a sequence of sentences $\{G_j^k\}_{j<\omega}$ s.t. no finite boolean combination of the G_j^k's implies in A^* $U_k[A_{i_1}^k, \ldots, A_{i_k}^k]$ for some distinct $i_1, \ldots, i_k$. (By a boolean combination we mean here a set $\{H_j\}_j$ where H_j is either G_j^k or $\neg G_j^k$.)

<u>Sub-basis</u>: Let

(3) $$W_{g(1,k)} := \{m \mid \exists \text{ distinct } i_1, \ldots, i_k \text{ for which } F(\bar{m}) \vdash_{A^*} U_k[A_{i_1}^k, \ldots, A_{i_k}^k]\}$$

(4) $$W_{h(1,k)} := \{m \mid \exists \text{ distinct } i_1, \ldots, i_k \text{ for which } \neg F(\bar{m}) \vdash_{A^*} U_k[A_{i_1}^k, \ldots, A_{i_k}^k]\}$$

Now $W_{g(1,k)} \cap A = \emptyset$ and $W_{h(1,k)} \cap B = \emptyset$ by (1) and (2). Hence

$$f(g(1,k),h(1,k)) \notin W_{g(1,k)} \cup W_{h(1,k)}.$$

Define

$$G_1^k :\equiv F(f(g(1,k),h(1,k)));$$

then

(5) $$\left.\begin{matrix} G_1^k \\ \neg G_1^k \end{matrix}\right\} \nvdash_{A^*} U_k[A_{i_1}^k, \ldots, A_{i_k}^k] \qquad \text{as required.}$$

Sub-recursion step: Assume that $G^k_1,\dots,G^k_l$ are defined, and satisfy

(6) $G^* \nvdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}]$ for every boolean combination G^* of $G^k_1,\dots,G^k_l$ and every distinct $i_1,\dots,i_k$.

Define

$$W_{g(l+1,k)} := \{m \mid \exists \text{ distinct } i_1,\dots,i_k \text{ s.t. } F(\bar{m}),\ G^* \vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \text{ for some boolean combination } G^* \text{ of } G^k_1,\dots,G^k_l\}$$

$$W_{h(l+1,k)} := \{m \mid \dots \neg F(\bar{m}),\ G^* \vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \dots\}.$$

As in the treatment of the sub-basis we have here

$$W_{g(l+1,k)} \cap A = \emptyset; \quad W_{h(l+1,k)} \cap B = \emptyset.$$

So, defining

$$G^k_{l+1} :\equiv F(f(g(l+1,k),h(l+1,k))),$$

we have

$$G^* \nvdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \text{ for every boolean combination } G^* \text{ of } G^k_1,\dots,G^k_{l+1}.$$

Main recursion-step continued: Define now A^{k+1}_i to be (the purely Σ^0_1 equivalent of) $A^k_i \vee G^k_i$. To conclude the proof, assume

$$\vdash_{A^*} U_{k+1}[A^{k+1}_{i_1},\dots,A^{k+1}_{i_{k+1}}] \text{ for some distinct } i_1,\dots,i_{k+1}.$$

By the disjunction instantiation property of A^* we get, w.l.g.,

$$\vdash_{A^*} A^{k+1}_{i_1} \to U_k[A^{k+1}_{i_2},\dots,A^{k+1}_{i_{k+1}}].$$

But recalling the definition of A^{k+1}_j, this implies

$$G^k_{i_1} \vdash_{A^*} U_k[A^k_{i_2} \vee G^k_{i_2},\ldots,A^k_{i_{k+1}} \vee G^k_{i_{k+1}}]$$

which by 2.3 [b] implies

$$G^k_{i_1}, \neg G^k_{i_2},\ldots,\neg G^k_{i_{k+1}} \vdash_{A^*} U_k[A^k_{i_2},\ldots,A^k_{i_{k+1}}] \text{ ,}$$

contradicting the construction of the sequence G^k_j. Hence

$$\nvdash_{A^*} U_{k+1}[A^{k+1}_{i_1},\ldots,A^{k+1}_{i_{k+1}}]$$

as required. □

Note, finally, that the above construction can be rendered totally uniform. That is - every A^k_i can be presented as $\exists x F_0(f'(i,k),x)$ for a suitable total recursive function f'. This formula does not belong, strictly speaking, to the formalism of A. But it is equivalent to the following formula of prim. rec. arithmetic:

$$\exists z\ T(e,\langle i,k\rangle,(z)_0)\ \&\ F_0(U((z)_0),(z)_1),$$

where e is the g.n. of the function f', T and U are Kleene's computation-predicate and result-extracting function respectively. We thus obtain from the above construction the full power of theorem I for schemata F s.t. $\mu\ulcorner F\urcorner \leq 1$.

3. PROOF THEORETIC REDUCTION OF THEOREM I

3.0. Here we prove, for a regular number theory $A^* \subseteq A^\infty[T]$,

<u>PROPOSITION</u>. *If* $\nvdash_{L_0} F[p_1,\ldots,p_k]$ *and* $\vdash_{A^*} F[A_1,\ldots,A_k]$, *then*

$$\vdash_{A^\infty_{rec}} U_k[A_1,\ldots,A_k]$$

for any Σ^0_1 *sentences* $A_1,\ldots,A_k$.

Combined with the solution given in section 2 for the schemata U_k, this implies theorem I.

The proposition is proved as follows. In 3.1 - 3.7 below we prove (for some prim.rec. f)

$$(1)\quad \vdash_{V_0+BI} \neg\underline{Pr}_{L_0}(\ulcorner F\urcorner) \;\&\; \underline{Nprf}^\infty_{rec}(d,\ulcorner F[A_1,\ldots,A_k]\urcorner) \rightarrow \underline{Nprf}^\infty_{rec}(fd,\ulcorner U_k[A_1,\ldots,A_k]\urcorner).$$

So, for a theory $T \supseteq V_0$ + BI and a proof-predicate $\underline{Pr}_T$ for it which is proved in A to be closed under Modus Ponens,

$$(2)\quad \vdash_A \underline{Pr}_T\ulcorner\neg\underline{Pr}_{L_0}(\ulcorner F\urcorner)\urcorner \;\&\; \underline{Pr}_T\ulcorner\underline{NPrf}^\infty_{rec}(d,\ulcorner F[A_1,\ldots,A_k]\urcorner)\urcorner \rightarrow \underline{Pr}_T\ulcorner\underline{NPrf}^\infty_{rec}(fd,\ulcorner U_k[A_1,\ldots,A_k]\urcorner)\urcorner.$$

But $\underline{Pr}_{L_0}$ is a prim.rec. predicate, so (2) implies

$$(3)\quad \vdash_A \neg\underline{Pr}_{L_0}(\ulcorner F\urcorner) \;\&\; \underline{Pr}_{A^*}(\ulcorner F[A_1,\ldots,A_k]\urcorner) \rightarrow \underline{Pr}_{A^\infty_{rec}[T]}(\ulcorner U_k[A_1,\ldots,A_k]\urcorner)\urcorner$$

for any $A^* \subseteq A^\infty_{rec}[T]$.

3.1. HEURISTICAL CONSIDERATION LEADING TO THE REDUCTION

3.1.1. Assume the premise of 3.0(1). It means that a normal derivation d of F in A^{∞}_{rec} is given where some quantification or arithmetical rule must occur, because $\neg\underline{Pr}_{L_0}\ulcorner F\urcorner$. We "climb up" in the proof-tree d in search for such an occurrence, starting at the root $\langle\,\rangle$.

To allow a smoother semi-formal exposition. let us write - for a node u - $\rho^{d,u}$ for the inference rule encoded by $(\{d\}u)_0$, and

$$s^{d,u} \equiv \underline{a}^{d,u} \Rightarrow F^{d,u}$$

for the sequent encoded by $(\{d\}u)_1$.

At every stage of our search in d we arrive to some node u where the sentence $F^{d,u}$ is a Σ^0_1 substitution of a schema of L_0, and where $\neg\underline{Pr}_{L_0}(\ulcorner s^{d,u}\urcorner)$, i.e. $\underline{a}^{d,u} \Rightarrow F^{d,u}$ cannot be proven using the rules of L_0 only.

Suppose now that a node u is "selected" at a given stage of the search. If $\rho^{d,u}$ is a propositional rule, then at least one of the premises $u*\langle n\rangle$, $n \le 2$ of u in d must satisfy $\neg\underline{Pr}_{L_0}(\ulcorner s^{d,u*\langle n\rangle}\urcorner)$, because $\neg\underline{Pr}_{L_0}(\ulcorner s^{d,u}\urcorner)$ since u is "selected". We "climb up" to the leftmost of these premises.

$\rho^{d,u}$ cannot be [$\forall$I] or [$\forall$E], by the subformula property of d, because $\forall$ does not occur in $F[A_1,\ldots,A_k]$.

If $\rho^{d,u}$ is [$\exists$E], and $\neg\underline{Pr}_0(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$ (i.e. - the major premise is not provable using propositional rules only), then we climb up to $u*\langle 0\rangle$. Else, we proceed simultanuously to all minor premises $u*\langle n+1\rangle$, $n \le \omega$. The major premise $F^{d,u*\langle 0\rangle} \equiv: \exists zCz$ must be a Σ^0_1 sentence, by the subformula property. So, for every n,

$$s^{d,u*\langle n\rangle} \equiv \underline{a}^{d,u},C\bar{n} \Rightarrow F^{d,u}$$

where $C\bar{n}$ is an equation, and $F^{d,u}$ is a Σ^0_1 substitution of a propositional schema. It is easy to see (3.3 below) that if $\underline{Pr}_{L_0}(\ulcorner \underline{a}^{d,u},C\bar{n}\Rightarrow F^{d,u}\urcorner)$ for some n, then $\underline{Pr}_{L_0}(\ulcorner \underline{a}^{d,u}\Rightarrow F^{d,u}\urcorner)$, which contradicts our assumption that the node u is selected. It follows that all nodes $u*\langle n+1\rangle$ corresponding to the minor premises of $\rho^{d,u}$ satisfy our conditions on "selected" nodes.

Now since d is a well founded tree, the search described above must terminate along every branch of the universal spread. It cannot stop at a top node of d, because

(i) if $\rho^{d,u}$ = [TE] then $F^{d,u}$ is an equation, and so u is not selected;

(ii) if $\rho^{d,u}$ = [T] then $\underline{\mathrm{Pr}}_{L_0}(\ulcorner s^{d,u} \urcorner)$.

Hence the search determined by any successive choice of minor (or major) premises of instances of [∃E] must stop at some node u s.t. $\rho^{d,u}$ is either [∃I] or [FE].

3.1.2. Let us now consider how this information on the "search" described above may be used to construct a proof in A^{∞}_{rec} for $U_k[A_1,\ldots,A_k]$. To start with, take the simplest case, where k = 1, F ≡ F[∃xEx] , and let u be some terminating node of the search.

Case 1. $\rho^{d,u}$ = [∃I]

$$\begin{array}{lll} & \rho^{d,u*\langle 0\rangle} & \underline{a} \Rightarrow Et \\ \text{the node } (u) \rightarrow & [\exists I] & \underline{a} \Rightarrow \exists x Ex \end{array}$$

Obviously, the inference rule $\rho^{d,u*\langle 0\rangle}$ cannot be an introduction rule. If $\rho^{d,u*\langle 0\rangle}$ is [→E], then we have the configuration

$$\text{the node } (u*0) \rightarrow \quad \frac{\underline{a} \Rightarrow G \rightarrow Et \qquad \underline{a} \Rightarrow G}{\underline{a} \Rightarrow Et}$$

But no subformula of $F[A_1,\ldots,A_k]$ has the form G→Et where Et is an equation. So $\rho^{d,u*\langle 0\rangle}$ cannot be [→E], and the cases [&E] and [∀E] are ruled out likewise. $\rho^{d,u*\langle 0\rangle}$ cannot be one of [⊥], [vE], [∃E], by our definition of normality. We are thus left with the case that u*⟨0⟩ is a top node of d, and $\rho^{d,u*\langle 0\rangle}$ is [TE] or [T]. In the first case we may construct

$$\begin{array}{ll} [TE] & \Rightarrow Et \\ [\exists I] & \Rightarrow \exists x Ex \\ [vI_0] & \Rightarrow \exists x Ex \vee \neg \exists x Ex \end{array}$$

So we have obtained a derivation for U_1[∃xEx].

On the other hand, the case $\rho^{d,u*\langle 0\rangle}$ = [T] is ruled out as follows. Assume that $\rho^{d,u*\langle 0\rangle}$ = [T]. Then Et ∈ $\underline{a}$, and since d derives a sequent ⇒F with an empty precedent, Et must be "discharged" in d somewhere below the node u. Again by the subformula property of d, this discharge cannot be at an instance of [→I] or of [vE], and so it must be at an instance of [∃E], and we should have the following configuration (where $t = \bar{n}$).

$$
\begin{array}{lcc}
 & \underline{a} \Rightarrow E\bar{n} & \\
\textcircled{u}\!\rightarrow & \underline{a} \Rightarrow \exists x Ex & \\
\quad\cdots & & \cdots \\
\underline{b} \Rightarrow \exists x Ex & \underline{b}, E\bar{n} \Rightarrow B & \\
\hline
\textcircled{v}\!\rightarrow \quad [\exists E] & \underline{b} \Rightarrow B &
\end{array}
$$

Here the two indicated occurrences of Σ_1^0 formulae must be identical for the case considered. Since the node u is selected, so must be v, but <u>not</u> $v*\langle 0\rangle$. This means that $\neg Pr_{L_0}(\ulcorner \underline{a} \Rightarrow \exists x Ex \urcorner)$, but $\underline{Pr}_{L_0}(\ulcorner \underline{b} \Rightarrow \exists x Ex \urcorner)$. From the configuration just shown we must have, however, $\underline{b} \subseteq \underline{a}$, and this is a contradiction.

Case 2. $\rho^{d,u} = [FE]$,

$$
\begin{array}{ll}
 & \underline{a} \Rightarrow E \\
\hline
[FE] & \underline{a} \Rightarrow \bot
\end{array}
\qquad \text{say.}
$$

As in case 1, we find that $u*\langle 0\rangle$ must be a top node of d, and since E here is a false equation, we are left with the case that $\rho^{d,u*\langle 0\rangle}$ is [T]; so we must find in d the following configuration:

$$
\begin{array}{lcccc}
 & [T] & \underline{a} \Rightarrow E & & \\
\textcircled{u}\!\rightarrow & & \underline{a} \Rightarrow \bot & & \\
\quad\cdots & & \Sigma_n & \Sigma_{n+1} & \cdots \\
\underline{b} \Rightarrow \exists x Ex & & \underline{b}, E\bar{n} \Rightarrow B & & \\
\hline
\textcircled{v}\!\rightarrow \quad [\exists E] & \underline{b} \Rightarrow B & & &
\end{array}
$$

and we may assume w.l.g. (by the well-foundedness of d) that the configuration of the type shown does not repeat itself within any of the subderivations Σ_m. Since u is selected, so must be v, and hence $v*\langle m+1\rangle$ for every $m < \omega$. Each search in a subderivation Σ_m must come to an end at some node u_m, and the argument of case 1 (about ruling out $\rho^{d,u*\langle 0\rangle} = [T]$) shows that since $v*\langle 0\rangle$ is not selected, ρ^{d,u_m} is not [∃I], and must therefore be [FE].

Hence we can extract from the configuration above the derivation:

$$\frac{[T]\quad \exists xEx \Rightarrow \exists xEx \qquad \left\{\begin{array}{ll}[T] & \exists xEx, E\bar{n} \Rightarrow E\bar{n} \\ [FE] & \exists xEx, E\bar{n} \Rightarrow \bot\end{array}\right\}_{n<\omega}}{\begin{array}{lr}[\exists E] & \exists xEx \Rightarrow \bot \\ [\to I] & \Rightarrow \neg\exists xEx \\ [vI_1] & \Rightarrow \exists xEx \vee \neg\exists xEx\end{array}}$$

and again we found a derivation in A^{∞}_{rec} for $U_1[\exists xEx]$. This concludes our observation on the case that $k = 1$, $F = F[\exists xEx]$.

3.1.3. Consider now the case $k = 2$, i.e. - $F \equiv F[\exists xE_0x, \exists xE_1x]$. Here the following configuration may occur

$$\frac{\underline{a} \Rightarrow \exists xE_0x \qquad \{\Sigma_n\}_{n<\omega}}{(u) \longrightarrow\ [\exists E]\quad \underline{a} \Rightarrow B}$$

where the node u is selected, and the search continues to the minor subderivations Σ_n (i.e. - $\underline{Pr}_{L_0}(\ulcorner \underline{a} \Rightarrow \exists xE_0x\urcorner)$). But now, from our argument for the case $k = 1$ it is clear that, for the node u_m at which the search in the minor subderivation Σ_m terminates $F^{u_m} \nvDash \exists xE_0x$ $(m<\omega)$. So we may apply the argument for the case $k = 1$ to each of the minor subderivations separately, and extract from each of these a derivation Σ^*_m for $\exists xE_1x \vee \neg\exists xE_1x$. Since the method of doing this is uniform, we can actually collect the derivations Σ^*_m to yield the following derivation of A^{∞}_{rec}.

$$\frac{[T]\quad \exists xE_0x \Rightarrow \exists xE_0x \qquad \left\{\begin{array}{c}\Sigma^*_m \\ \exists xE_0x \Rightarrow \exists xE_1x \vee \neg\exists xE_1x\end{array}\right\}_{m<\omega}}{\begin{array}{lr}[\exists E] & \exists xE_0x \Rightarrow \exists xE_1x \vee \neg\exists xE_1x \\ [\to I] & \Rightarrow \exists xE_0x \to \exists xE_1x \vee \neg\exists xE_1x \\ [vI_0] & \Rightarrow U_2[\exists xE_0x, \exists xE_1x]\end{array}}$$

Iterating this process, with some technical symmetrization arguments, we obtain 3.0.(1).

3.2. NOTATIONS

$\underline{\text{Subordinated}}\ (d,u,v) :\equiv\ \exists w,n<v \left[v=w*\langle 0\rangle \ \&\ w*\langle n+1\rangle \prec u \ \&\ \rho^{d,w} = [\exists E] \right]$

$\equiv$ "v is a major premise node of an instance of [∃E] in d, and u is a node in one of the minor subderivations of this instance".

Here $\prec$ stands for the initial-segment relation (between sequent-numbers).

$\underline{\text{Selected}}\ (d,u) :\equiv\ "F^{d,u}$ is not q.f." $\&\ \neg\underline{\text{Pr}}_{L_0}(\ulcorner s^{d,u}\urcorner)\ \&$
$\forall w<u \left[\underline{\text{Subordinated}}(d,u,w*\langle 0\rangle) \rightarrow \underline{\text{Pr}}_{L_0}(\ulcorner s^{d,w*\langle 0\rangle}\urcorner) \right].$

When $\underline{\text{NPrf}}^{\infty}_{\text{rec}}(d,\ulcorner F[A_1,\ldots,A_k]\urcorner)$ $(A_1,\ldots,A_k\ \Sigma^0_1$ sentences) write

$\underline{b}^{d,u} \ := \ \{F^{d,v} \mid \underline{\text{Subordinated}}(d,u,v)\}$

$a_0^{d,u} \ := \ \{E \in \underline{a}^{d,u} \mid E \text{ an equation}\}$

$U^{d,u} \ :\equiv \ U_m[A_{i_1},\ldots,A_{i_m}]$ where $\{A_{i_1},\ldots,A_{i_m}\} := \{A_1,\ldots,A_k\} \setminus \underline{b}^{d,u}$

(set-theoretic difference)

3.3. LEMMA. *Let* $A_1,\ldots,A_k$ *be* Σ^0_1 *sentences, let* $\underline{a} \Rightarrow G$ *be formed of subformula of* $F[A_1,\ldots,A_k]$ *only, and let* E *be an equation. Then*

$$\underline{\text{Pr}}_{L_0}(\ulcorner a,E\Rightarrow G\urcorner) \ \Rightarrow\ \underline{\text{Pr}}_{L_0}(\ulcorner \underline{a}\Rightarrow G\urcorner).$$

PROOF. Let Π be a normal proof for $\underline{a}$,E⇒G which uses propositional inference-rules only, and let Π^* come from Π be eliminating E from all precedents of sequents in Π. Check by inspection on cases for inference rules that Π^* is a correct derivation. (Note that by normality no formula of the form $E \rightarrow H$ may occur in Π). □

3.4. LEMMA. (in A) *Assume* $\underline{\text{NPrf}}^{\infty}_{\text{rec}}(d,\ulcorner F[A_1,\ldots,A_k]\urcorner)$;

(a) $\underline{\text{Selected}}(d,u) \longrightarrow \left[\rho^{d,u} = [\exists I] \ \vee\ \rho^{d,u} = [FE] \ \vee\ \exists n\leq 2\ \underline{\text{Selected}}(d,u*\langle n\rangle) \right]$

(b) $\underline{\text{Selected}}(d,u)\ \&\ \rho^{d,u} = [\exists E]\ \&\ \underline{\text{Pr}}_{L_0}(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$
$\longrightarrow \forall n>0\ \underline{\text{Selected}}(d,u*\langle n\rangle)$.

PROOF. Assume $\rho^{d,u} \neq$ [∃I],[FE] and the premise of (a), and consider cases for $\rho^{d,u}$. $\rho^{d,u}$ cannot be [T] or [TE], because Selected(d,u). $\rho^{d,u}$ is not [∀I] or [∀E] by the subformula property of d. If $\rho^{d,u}$ is a propositional inference-rule, the proof is immediate. We are left with the case that $\rho^{d,u}$ is [∃E]. If $\neg \underline{Pr}_{\perp_0}(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$ then we are done (for part (a)). Else, then Selected(d,u*⟨0⟩) for every n > 0 by 3.3. □

3.5. ASSIGNMENT OF DERIVATIONS TO THE SELECTED NODES

Assume, as above, $\underline{Nprf}^{\infty}_{rec}(d,\ulcorner F[A_1,\ldots,A_k]\urcorner)$. We define a function {a(d,u)} recursively in {d} and u by the conditions given below. By the s.m.n.-theorem a(d,u) is then a prim.rec. function.

(i) If ¬Selected(d,u), then {a(d,u)} ≡ 0.

(ii) Else, and $\rho^{d,u}$ = [∃I], then {a(d,u)} describes the finite derivation

$$
\begin{array}{rl}
[\rho^{d,u*\langle 0\rangle}] & \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow F^{d,u*\langle 0\rangle} \\
[\exists I] & \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow F^{d,u} \\
\left.\begin{array}{l}\text{instances of } [\to I] \\ \text{and of } \quad [\vee I]\end{array}\right\} & \vdots \\
 & \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}
\end{array}
$$

Note that, by the argument of 3.1.2, $F^{d,u} \notin \underline{b}^{d,u}$ and that $\rho^{d,u*\langle 0\rangle}$ is either [T] or [TE].

(iii) Else, and $\rho^{d,u}$ = [FE]. Let {a(d,u)} describe formally

$$
\begin{array}{rl}
[T] & \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow F^{d,u*\langle 0\rangle} \\
[FE] & \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow \perp \\
[\perp] & \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}
\end{array}
$$

(iv) Else, and $\rho^{d,u}$ is a propositional inference rule. Let u*⟨n⟩ be the leftmost premise of u in d s.t. Selected(d,u*⟨n⟩) (cf. 3.4.(a)), and let {a(d,u)} := {a(d,u*⟨n⟩)}.

(v) Else, and $\rho^{d,u} = [\exists E]$;

Subcase A: If $\neg \underline{Pr}_{L_0}(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$, let $a(d,u) := a(d,u*\langle 0\rangle)$..

Subcase B: Else, and $\exists xEx :\equiv F^{d,u*\langle 0\rangle} \notin \underline{b}^{d,u}$, then let $\{a(d,u)\}$ describe

$$\frac{[T]\quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow \exists xEx \qquad \overset{\Sigma'_n}{\left\{ \underline{a}_0^{d,u*\langle n\rangle} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow U^{d,u*\langle n\rangle} \right\}_{0<n<\omega}}}{[\exists E]\quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow U^{d,u*\langle 1\rangle}}$$

$$[\to I]\qquad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow \exists xEx \to U^{d,u*\langle 1\rangle}$$

$$\left.\begin{array}{r}\text{instances of}\\ [\vee I]\end{array}\right\}\qquad \vdots$$

$$\underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}$$

Here, if Σ_n is described by $\{a(d,u*\langle n\rangle)\}$, then Σ'_n comes from Σ_n by joining the formula $\exists xEx$ to all precedents. Note that by the case's conditions

$$\left.\begin{array}{l} \underline{b}^{d,u*\langle n\rangle} = \underline{b}^{d,u} \cup \{\exists xEx\} \\ \underline{a}_0^{d,u*\langle n\rangle} = \underline{a}_0^{d,u} \cup \{E\bar{n}\} \\ U^{d,u*\langle n\rangle} \equiv U^{d,u*\langle 1\rangle} \end{array}\right\} \quad \text{for } n > 0.$$

Subcase C. As subcase B, but $\exists xEx \in \underline{b}^{d,u}$. Then let $\{a(d,u)\}$ describes

$$\frac{\underline{a}_0^{d,u} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow \exists xEx \qquad \overset{\Sigma'_n}{\left\{ \underline{a}_0^{d,u*\langle n\rangle} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow U^{d,u} \right\}_{0<n<\omega}}}{[\exists E]\quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}}$$

Note that here $U^{d,u*\langle n\rangle} \equiv U^{d,u}$ for every n.

3.6.1. PROPOSITION.

$$\vdash_{Y_0 + \mathrm{BI}} \underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(d, \ulcorner F[A_1,\ldots,A_k] \urcorner) \ \& \ \underline{\mathrm{Selected}}(d,u) \longrightarrow$$
$$\underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(a(d,u), \ulcorner \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u} \urcorner). \qquad \square$$

3.6.2. COROLLARY.

$$\underline{\mathrm{NPrf}}_{\mathrm{rec}}(d, \ulcorner F[A_1,\ldots,A_k] \urcorner) \ \& \ \neg \underline{\mathrm{Pr}}_{L_0}(\ulcorner F \urcorner) \longrightarrow$$
$$\underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(a(d,\langle\,\rangle), \ulcorner \Rightarrow U_k[A_1,\ldots,A_k] \urcorner) .$$

PROOF. Use 3.6.1, for $u = \langle\,\rangle$. $\square$

4. STRUCTURE OF THE PROOF OF THEOREM II

4.1. PRELIMINARIES

4.1.1. Fix a q.f. formula $E(x) :\equiv f(x)=0$ (where f is a fixed prim.rec. function). Write

$$E_i^n(z_1,\dots,z_n) :\equiv \forall x \exists y\ E\langle x,y,i,n,\langle z_1,\dots,z_n\rangle\rangle$$

where $\langle\ ,\dots,\ \rangle$ is a fixed encodement for finite sequences (cf. e.g. TROELSTRA [73], p.24).

$$E^* :\equiv \forall i,u,z \forall x \exists y\ E\langle x,y,i,u,z\rangle$$

$$B^E[w] :\equiv \forall \langle i,n,z\rangle\ [\ \underline{\text{Ineq}}(\langle i,n,z\rangle,w) \ \longrightarrow\ \forall x \exists y\ E\langle x,y,i,n,z\rangle\ ]$$

where $\underline{\text{Ineq}}(a,b)$ is an equation which expresses the inequality $a \neq b$. More intuitively,

$$B^E[\langle j,m,\langle \vec{s}\rangle\rangle] :\equiv \forall \langle i,n,\langle \vec{t}\rangle\rangle_{\langle i,n,\langle \vec{t}\rangle\rangle \neq \langle j,m,\langle \vec{s}\rangle\rangle}\ E_i^n(\vec{t})$$

$$s^E[w] :\equiv B[w] \Rightarrow \forall x \exists y\ E\langle x,y,(w)_0,(w)_1,(w)_2\rangle$$

$$\equiv B[w] \Rightarrow E_{(w)_0}^{(w)_1}((w)_{2,0},\dots,(w)_{2,(w)_1})$$

i.e.

$$s^E[\langle i,n,\langle \vec{z}\rangle\rangle] \equiv \forall j,m,\vec{w}_{\langle j,m,\vec{w}\rangle \neq \langle i,n,\vec{z}\rangle}\ E_j^m(\vec{w}) \Rightarrow E_i^n(\vec{z}).$$

The sequents $s[w]$ play here the same role as the schemata U_k in the treatment of L_0 above.

PROOF. Assume $\underline{a} \cup \underline{b} \vdash_{L_1A} F$, and let Δ be a normal derivation of L_1A for $\underline{a} \cup \underline{b} \vdash F$ (cf. PRAWITZ [65]). By induction on the length of Δ, using the subformula property and the definition of E-atoms. one prove easily that every formula occurring in Δ is either an E-sentence or an open Σ^0_1 or q.f. formula. Hence formulae in $\underline{b}$ are actually not used in Δ, and so $\underline{a} \vdash_{L_1A} F$. □

4.2.2. LEMMA. *Let* $\underline{a}$,F *be closed formulae of* L_1. *Then*

$$\underline{a}^E \vdash_{L_1A} F^E \quad\Rightarrow\quad \underline{a} \vdash_{L_1} F$$

where

$$\underline{a}^E := \{G^E \mid G \in \underline{a}\}.$$

PROOF. Let Δ be a normal derivation (in the sense of 1.3) of L_1A for $\underline{a}^E \vdash F^E$ and let $\underline{G}$ be an occurrence of a formula in Δ, G not an E-sentence. By the subformula property of Δ, G must then have one of the forms [a] $E\langle u,v,i,n,z\rangle$ or [b] $\exists y\, E\langle u,y,i,n,z\rangle$.

By the normality of Δ (defined as in 1.3) this instance must either

(i) be a top formula of Δ (in case [a]), or

(ii) occur immediately below a top formula, or

(iii) occur as a premise of $\exists$E derived by $\forall$E (in case [b]).

Note now that E^n_i is defined so that the order of variables in each E-atom is fixed, so that the two first variables of the matrix are bounded by the $\forall\exists$ quantifiers preceding it. Furthermore, two E-atoms formed from distinct E^n_i are syntactically distinct. Hence every occurrence G as above must occur in a configuration of the form

$$\Pi \equiv \dfrac{\dfrac{\begin{matrix}\Sigma\\ \forall x\exists y E\langle x,y,i,n,z\rangle\end{matrix}}{\exists y E\langle u,y,i,n,z\rangle}\,\forall E \qquad \begin{matrix}(1)\\ \left[\dfrac{\dfrac{E\langle u,v,i,n,z\rangle}{\exists y E\langle u,y,i,n,z\rangle}\,\exists I}{\forall x\exists y E\langle x,y,i,n,z\rangle}\,\forall I\right]_j \\ \Gamma \\ H\end{matrix}}{H}\,(1)\ \exists E$$

4.1.2. An E-*sentence* is a sentence built up using the formation rules of L_1 only, with E^n_i taken in place of the predicate letter P^n_i (i,n=0,1,...). An E-*atom* is an E-sentence of the form $E^n_i(t_1,\ldots,t_n)$. We call the implicitly indicated occurrences of t_i in the E-atom above (i=1,...,n) the *formal occurrences* in $E^n_i(\vec{t})$. Since the order of formally-occurring terms in each E-atom is fixed, by the very definition of E^n_1, it is uniformly decidable whether two E-atoms are instances of the same E^n_i.

Let d be a normal derivation of A^{∞}_{rec} for an E-sentence. By the subformula property of d, every formula occurring in d is either an E-sentence, an E-atom or a Σ^0_1 sentence with an E-atom as a matrix. It is easily seen that if we replace every formal occurrence of each term t (in some formula in d) by the numeral $\bar{n}$ s.t. $\bar{n}$=t, we get a correct and normal derivation of the same E-sentence. We call such a normal derivation an E-derivation.

Notation: E-Der(d); E-Prf(d,$\ulcorner F \urcorner$). Since we deal with E-derivations only, we shall assume that each E-atom has the form $E^n_i(\bar{m}_1,\ldots,\bar{m}_n)$. If $F[P^{n_1}_{i_1},\ldots,P^{n_q}_{i_q}]$ is a schema of L_1 whose predicate-letters are among those shown, we write F^E for $F[E^{n_1}_{i_1},\ldots,E^{n_q}_{i_q}]$. So $\ulcorner F^E \urcorner = \underline{\text{sub}}_{\Pi^0_2}(\ulcorner F \urcorner, \ulcorner E \urcorner)$.

4.1.3. We write $[\exists E^1]$ for an instance of $[\exists E]$ whose major premise (i.e. the antecendent of the leftmost premise sequent) has a q.f. matrix. For an instance of $[\exists E]$ which does not satisfy this we write $[\exists E^*]$.

4.2. DERIVATION OF E-SENTENCES IN L_1A

Let intuitionistic predicate logic L_1 be formally generated by Gentzen's system of natural deduction (cf. PRAWITZ [65]). The system L_1A is defined as follows. The language of L_1A is the language of A extended with letters for parameters (i.e. - free variables). The rules of inference of L_1A are exactly those of L_1.

4.2.1. LEMMA. *Let every formula in* $\underline{a}$,F *be either an* E-*sentence, an open* Σ^0_1 *formula or an open equation. Let* $\underline{b}$ *be a set of closed equations. Then*

$$\underline{a} \cup \underline{b} \vdash_{L_1A} F \quad \Rightarrow \quad \underline{a} \vdash_{L_1A} F\ .$$

Replace the subderivation Π of Δ by

$$\Pi^* \quad :\equiv \quad \begin{array}{c} \left[\begin{array}{c} \Sigma \\ \forall x \exists y E\langle x,y,i,n,z\rangle \end{array} \right]_j \\ \Gamma \\ H \end{array}$$

Note that Π^* is normal. Repeating this operation we get, by induction on the number of occurrences of Σ^0_1 formulae in Δ, a derivation Δ^* where all occurrences are of E-formulae. Replace in Δ^* every occurrence $E^n_i(\vec{v})$ of an E-atom (including occurrences as a subformula) by $P^n_i(\vec{v})$, and the result is a correct derivation of L_1 for $\underline{a} \vdash$ F. □

4.3. We wish to prove theorem II, which may be restated in the following form.

<u>THEOREM II</u> (restated). *For any* $T \supseteq V_0 + BI$ *there is a q.f.* E(x) *s.t.*

$$A^T \vdash \neg\underline{\Pr}_{L_1}(\ulcorner F \urcorner) \;\rightarrow\; \neg\underline{\Pr}_{A^\infty_{rec}[T]}(\underline{\mathrm{sub}}_{\Pi^0_2}(\ulcorner F\urcorner, \ulcorner E\urcorner))$$

where

$$A^T \;:=\; A + \underline{\mathrm{CMP}}(T) + \underline{\mathrm{Rfn}}_{C_0}(T) + \underline{\mathrm{Con}}(T^*),$$

$$\underline{\mathrm{CMP}}(T) \;:\equiv\; \forall x,y\,[\,\underline{\Pr}_T(\underline{\mathrm{imp}}(x,y)) \;\rightarrow\; (\underline{\Pr}_T(x) \rightarrow \underline{\Pr}_T(y))\,],$$

$$\underline{\mathrm{Rfn}}_{C_0}(T) \;:\equiv\; \forall x\,[\,\underline{\Pr}_T(x) \;\&\; \text{"x encodes a formula in } C_0\text{"} \;\rightarrow\; \underline{\mathrm{Tr}}_{C_0}(x)\,],$$

and where C_0 *is the class of formulae of the form* $\Pi^0_2 \rightarrow \neg\neg\Sigma^0_2$, *and* $\underline{\mathrm{Tr}}_{C_0}$ *is a truth definition for* C_0.

$$\underline{\mathrm{Con}}(T^*) \;:\equiv\; \forall x\,[\,\text{"x encodes a conjunction of instances of excluded-third, of instance of } AC_{00} \text{ and of true } \Pi^0_1 \text{ sentences"} \;\rightarrow\; \neg\underline{\Pr}_{T^*}(\underline{\mathrm{neg}}(x))\,].$$

4.4. THE PROOF THEORETIC REDUCTION

Fix E as above. We define a (classically) Π^0_1 predicate $\underline{Crit}(d,u)$ for which we prove

(1) $\vdash_{\mathcal{Y}_0+BI} \underline{E\text{-}Prf}(d,\ulcorner F\urcorner) \rightarrow [\ E^* \ \&\ \neg\underline{Pr}_{L_1A}(\ulcorner F\urcorner) \rightarrow \neg\neg\exists u\ \underline{Crit}(d,u)\ ].$

Since $\mathcal{T} \supseteq \mathcal{Y}_0$+BI we get from (1)

(2) $\vdash_{A+\underline{CMP}(\mathcal{T})} \underline{Pr}_{\mathcal{T}}\ulcorner\underline{E\text{-}Prf}(d,\ulcorner F\urcorner)\urcorner \longrightarrow Pr_{\mathcal{T}}\ulcorner[E^*\ \&\ \neg\underline{Pr}_{L_1A}\ulcorner F\urcorner \rightarrow \neg\neg\exists u\ \underline{Crit}(d,u)]\urcorner$

and so

(3) $\vdash_{A+\underline{CMP}(\mathcal{T})+\underline{Rfn}_{C_0}(\mathcal{T})} \underline{Pr}_{\mathcal{T}}\ulcorner\underline{E\text{-}Prf}(d,\ulcorner F\urcorner)\urcorner\ \&\ E^*\ \&\ \neg\underline{Pr}_{L_1A}\ulcorner F\urcorner \longrightarrow$

$\neg\neg\exists u\ \underline{Crit}(d,u).$

On the other hand we prove

(4) $\vdash_{\mathcal{Y}^C_0+AC_{00}} \underline{E\text{-}Der}(d)\ \&\ \underline{Crit}(d,u)\ \&\ \underline{Res}(d,u,x) \longrightarrow \exists\phi\ \underline{NPrf}^\infty(\phi,\ulcorner s^E[x]\urcorner)$

where

$\underline{Res}(d,u,x) :\equiv \forall y\ [\ T(d,u,y) \longrightarrow$

"if $\underline{antecedent}((Uy)_i)$ encodes $E^n_i(\vec{t})$ then $x=\langle i,n,\langle\vec{t}\rangle\rangle$"].

Since $\mathcal{T}^* \supseteq \mathcal{Y}^C_0+AC_{00}$ and $\underline{CMP}(\mathcal{T}) \longrightarrow \underline{CMP}(\mathcal{T}^*)$ trivially, we get from (4)

(5) $\vdash_{A+\underline{CMP}(\mathcal{T})} \underline{Pr}_{\mathcal{T}}\ulcorner E\text{-}Der(d)\urcorner \longrightarrow$

$\underline{Pr}_{\mathcal{T}^*}(\ulcorner\underline{Crit}(d,u)\ \&\ \underline{Res}(d,u,x) \rightarrow \underline{NPr}^\infty(\ulcorner s^E[x]\urcorner)\urcorner).$

But $\underline{Crit}(d,u)$ and $\underline{Res}(d,u,x)$ are classically Π^0_1, so

(6) $\vdash_{A+\underline{CMP}(\mathcal{T})} \underline{Pr}_{\mathcal{T}}\ulcorner E\text{-}\underline{Der}(d)\urcorner\ \&\ \underline{Crit}(d,u)\ \&\ \underline{Res}(d,u,x) \longrightarrow$

$\underline{Pr}_{\mathcal{T}^*}\ulcorner\underline{NPr}^{\infty}\ulcorner s^E[x]\urcorner\urcorner.$

We have however, trivially,

$$\vdash_A \text{"\{d\} is total"} \;\rightarrow\; \exists x\; \underline{\text{Res}}(d,u,x)$$

and so

$$\vdash_{A+\underline{\text{CMP}}(T)+\underline{\text{Rfn}}_{C_0}(T)} \underline{\text{Pr}}_T \ulcorner \underline{\text{E-Der}}(d) \urcorner \;\rightarrow\; \exists x\; \underline{\text{Res}}(d,u,x).$$

Hence we get from (6)

$$(7) \quad \vdash_{A+\underline{\text{CMP}}(T)+\underline{\text{Rfn}}_{C_0}(T)} \underline{\text{Pr}}_T \ulcorner \underline{\text{E-Der}}(d) \urcorner \;\&\; \exists u\; \underline{\text{Crit}}(d,u) \;\rightarrow\; \exists x\; \underline{\text{Pr}}_{T^*} \ulcorner \underline{\text{NPr}}^{\infty} \ulcorner s^E[x] \urcorner \urcorner .$$

Combining (3) and (7) yields

$$(8) \quad \vdash_{A+\underline{\text{CMP}}(T)+\underline{\text{Rfn}}_{C_0}(T)} \underline{\text{Pr}}_T \ulcorner \underline{\text{E-Prf}}(d, \ulcorner F \urcorner) \urcorner \;\&\; E^* \;\&\; \neg \underline{\text{Pr}}_{L_1 A} \ulcorner F \urcorner \;\rightarrow\; \neg\neg \exists x\; \underline{\text{Pr}}_{T^*} \ulcorner \underline{\text{NPr}}^{\infty} \ulcorner s^E[x] \urcorner \urcorner .$$

But from 4.2.2 we have

$$\vdash_A \neg \underline{\text{Pr}}_{L_1} \ulcorner F \urcorner \;\rightarrow\; \neg \underline{\text{Pr}}_{L_1 A} \ulcorner F^E \urcorner \qquad (F \text{ a schema of } L_1)$$

so

$$(9) \quad \vdash_{A+\underline{\text{CMP}}(T)+\underline{\text{Rfn}}_{C_0}(T)} \neg \underline{\text{Pr}}_{L_1} \ulcorner F \urcorner \;\&\; \underline{\text{Pr}}_{A^{\infty}_{rec}[T]} \ulcorner F^E \urcorner \;\&\; E^* \;\rightarrow\; \neg\neg \exists x\; \underline{\text{Pr}}_{A^{\infty}[T^*]} \ulcorner s^E[x] \urcorner$$

This completes the proof theoretic reduction. Note that for any predicate Crit (not necessarily Π^0_1) for which (1) and (4) hold, we could prove a statement (7^+) similar to (7), but with $\underline{\text{Pr}}_{T^*} \ulcorner \exists x\; \underline{\text{NPr}}^{\infty} \ulcorner s^E[x] \urcorner \urcorner$ as the antecedant. T^* is however a highly non-constructive theory, so there is no way to pull the existential quantifier out of the provability symbol here.

4.5. SOLUTION OF THE REDUCED PROBLEM

In this part we prove for every Σ^0_2 theory S the existence of a q.f. E(x) s.t.

(10) $$\vdash_{A+\underline{\mathrm{Con}}(S)+\underline{\mathrm{Comp}}_{\Sigma^0_2}(S)} \forall x\ \neg\underline{\mathrm{Pr}}_S\ulcorner s^E[x]\urcorner \quad \& \quad \neg\neg E^*$$

where $\underline{\mathrm{Pr}}_S$ is a fixed Σ^0_2 provability predicate for S, and where

(11) $$\underline{\mathrm{Comp}}_{\Sigma^0_2}(S) :\equiv \forall x\ [\ \underline{\mathrm{Tr}}_{\Sigma^0_2}(x) \rightarrow \underline{\mathrm{Pr}}_S(x)\ ].$$

Here $\underline{\mathrm{Tr}}_{\Sigma^0_2}(x)$ is a (canonical) truth definition for Σ^0_2 sentence. We wish to apply (10) to $S \equiv A^\infty[T^*]$, where T and T^* are as in 1.4. First, note

(12) $$\vdash_{Y_0} \neg\underline{\mathrm{NPr}}^\infty(\ulcorner\bot\urcorner),$$

so

(13) $$\vdash_A \underline{\mathrm{Con}}(T^*) \rightarrow \underline{\mathrm{Con}}(A^\infty[T^*]).$$

Also, for Σ^0_2 sentences F we have directly

(14) $$\vdash_{Y_0} F \rightarrow \underline{\mathrm{Pr}}^\infty\ulcorner F\urcorner$$

and since $T^* \supseteq Y_0$, and quite trivially $\underline{\mathrm{CMP}}(T) \rightarrow \underline{\mathrm{CMP}}(T^*)$, this implies

(15) $$\vdash_{A+\underline{\mathrm{CMP}}(T)} \underline{\mathrm{Pr}}_{T^*}\ulcorner F\urcorner \rightarrow \underline{\mathrm{Pr}}_{A^\infty[T^*]}\ulcorner F\urcorner.$$

By the very definition of $\underline{\mathrm{Pr}}_{T^*}$ however

$$\vdash_A F \rightarrow \underline{\mathrm{Pr}}_{T^*}\ulcorner F\urcorner \qquad \text{for every } \Pi^0_1 \text{ F},$$

and so

(16) $$\vdash_{A+\underline{\mathrm{CMP}}(T)} F \rightarrow \underline{\mathrm{Pr}}_{T^*}\ulcorner F\urcorner \qquad \text{for every } \Sigma^0_2 \text{ F}.$$

Hence we get from (15) and (16)

(17) $$\vdash_{A+\underline{CMP}(T)} F \rightarrow \underline{Pr}_{A^{\infty}[T^{*}]} \ulcorner F \urcorner$$ for every Σ_2^0 formula F.

Now observe that steps (15)-(18) can be uniformly formalized (within A), i.e. - (11) holds for $S \equiv A^{\infty}[T^{*}]$, as wanted.

We now proceed to prove (1) and (4) (the proof theoretic reduction), and (1) (the recursion theoretic solution) which together imply as we have just seen theorem II.

5. THE PROOF THEORETIC REDUCTION FOR THEOREM II

<u>5.1. LEMMA</u>. *Let the numeral* $\bar{n}$ *not occur in* $\underline{a}$, F, $\exists x G x$.

(i) *If* (1) $\underline{a}, G\bar{n} \vdash_{L_1 A} F$ *then*

(2) $\underline{a}, Gv \vdash_{L_1 A} F$ *where* v *is a parameter which does not occur in* $\underline{a}$, $G\bar{n}$, F.

(ii) *If* $\underline{a} \vdash_{L_1 A} G\bar{n}$ *then* $\underline{a} \vdash_{L_1 A} Gv$ *(for* v *as above).*

<u>PROOF</u>. Given a normal derivation of $L_1 A$ for (1) replace every occurrence of $\bar{n}$ by v, and observe, by inspection on cases for the inference rules, that the result is a correct derivation. The proof of (ii) is similar. □

5.2. SEMI FORMAL HEURISTIC OUTLINE OF THE REDUCTION

5.2.1. *Preliminary notations.*

$R_1(d,u) :\equiv \neg \underline{Pr}_{L_1 A} \ulcorner s^{d,u} \urcorner$.

$R_2(d,u) :\equiv$ "all equations in $\underline{a}^{d,u}$ are true".

$R_3(d,u) :\equiv$ "$F^{d,u}$ is an E-sentence".

$R_4(d,u) :\equiv$ "$F^{d,u}$ is an E-atom, and $\rho^{d,u}$ is [$\forall$I]".

$R_5(d,u) :\equiv$ "$F^{d,u}$ is a Σ^0_1-sentence".

Note that each $R_j(d,u)$ may be formally defined as a Π^0_1 predicate. Example:

$R_3(d,u) :\equiv \forall y$ [$T(d,u,y) \longrightarrow$ "$\underline{\text{antecedent}}((Uy)_1)$ is the g.n. of an E-sentence"].

$\underline{\text{Start}}(d,u) :\equiv \bigwedge_{i=1,2,3} R_i(d,u)$.

$\underline{\text{Crit}}_1(d,u) :\equiv \bigwedge_{i=1,2,4} R_i(d,u)$.

5.2.2. *Locating an arithmetical inference in* E-*derivations (the predicate* Crit*).*

We want to define a predicate Crit and to prove for it 4.3(1),(4). The idea is that when E-Der(d) and Crit(d,u) ("u is a critical node in the proof-tree described by d") then the subderivation d^u of d (where $\{d^u\} := \lambda x.\{d\}(u*x)$) has sufficiently nice properties so as to enable the extraction from it of a derivation for s[w] for some w.

As a first attempt to define such a predicate we try, as in the proof of theorem I, to look, when $\underline{\text{E-Prf}}(d,\ulcorner F\urcorner)$ and $\neg\underline{\text{Pr}}_{L_1 A}(\ulcorner F\urcorner)$, for a "genuine" use of an arithmetical inference in d. A starting node for such a search up may be any node v of d s.t. Start(d,v). When Start(d,v) we can weakly find (i.e. - $\neg\neg\exists$) a node $v*\langle n\rangle$ s.t. $\underline{\text{Start}}(d,v*\langle n\rangle)$, using lemma 5.1 when $\rho^{d,v}$ is [$\forall$I] or [$\exists E^*$], and E^* and 4.2.1 when $\rho^{d,v}$ is [$\exists E^1$] (lemma 5.4 below). Thus the search up in d may continue. The only cases where this process stops are when $R_4(d,v)$ or when $\rho^{d,v}$ is [FE]. In the last case, the definition of normality of 1.3 implies (as in 3.1.2) that a false equation occurs in $\underline{a}^{d,v}$, contradicting $R_2(d,v)$. Thus, by the well-foundedness of the proof-tree d, we find a node $u\succ v$ s.t. $\underline{\text{Crit}}_1(d,u)$.

When $\underline{\text{Crit}}_1(d,u)$ we can actually find in each subderivation $d^{u*\langle m\rangle}$ an inference of the form

$$(*)\qquad \frac{G}{\boxed{u*w}\rightarrow\ [\exists I]\quad F^{d,u*w}}$$

(G is a true equation and $F^{d,u*w} \equiv F^{d,u*\langle m\rangle}$). So these can be collected to yield a derivation of the form:

$$\frac{\langle\Sigma_m\rangle_{m<\omega}}{[\forall I]\quad B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow E_i^n(\vec{t})}$$

where $F^{d,u} \equiv: E_i^n(\vec{t})$, and each Σ_m is (schematically) of the form (*).

Unfortunately, the crude statement that the situation above occurs is not Π^0_1, essentially because there is no bound on the length of the w corresponding to each $m<\omega$. A certain refinement of the argument is therefore necessary.

5.2.3. *Heuristic for the disjunction-free fragment*

Assume, again, E-Der(d) and $\underline{\text{Crit}}_1$(d,u). The subderivation d^u of d takes then the form

$$(1) \qquad \frac{\left\{\begin{array}{c}\Sigma_m \\ \underline{a} \Rightarrow \exists y E\langle \bar{m},y,i,n,\langle \vec{t}\rangle\rangle\end{array}\right\}_{m<\omega}}{[\forall I] \quad \underline{a} \Rightarrow \forall x \exists y E\langle x,y,i,n,\langle \vec{t}\rangle\rangle}$$

where each Σ_m is formally described by $d^{u*\langle m\rangle}$.

From each Σ_m we wish to extract a derivation of A^∞ for

$$(2) \qquad B[\langle i,n,\langle \vec{t}\rangle\rangle] \Rightarrow \exists y E\langle \bar{m},y,i,n,\langle \vec{t}\rangle\rangle.$$

Fix some m, and let us analyse the structure of Σ_m.

We assume first that d is a derivation for a disjunction-free E-sentence; this implies, by the subformula property, that disjunction does not occur in the derivation d, and in particular - in the subderivation Σ_m we are looking at.

In addition we may assume

$$(3) \qquad \forall w \succ u*\langle m\rangle \ \neg\underline{\text{Start}}(d,w).$$

Because if Start(d,w), $w \succ u$ then we could start our initial search afresh; this could not be iterated indefinitely, because d is well-founded.

Consider now the main inference rule of Σ_m, $\rho^{d,u*\langle m\rangle}$. By the subformula property of d we have to consider the following cases only.

(i) $\rho^{d,u*\langle m\rangle} = [\bot]$; then $s^{d,u*\langle m,0\rangle} = \underline{a} \Rightarrow \bot$ and so Start(d,u*⟨m,0⟩) contradicting (3).

(ii) [∀E];

$$(4) \qquad \frac{\underline{a} \Rightarrow E^k_j(\vec{s})}{[\forall E] \quad \underline{a} \Rightarrow \exists y E\langle \bar{m},y,i,n,\langle \vec{t}\rangle\rangle} \qquad \text{say.}$$

Recall that $E^k_j(\vec{s}) \equiv \forall x \exists y E\langle x,y,j,k,\langle\vec{s}\rangle\rangle$, and so necessarily $\langle i,n,\langle\vec{t}\rangle\rangle \equiv \langle j,k,\langle\vec{s}\rangle\rangle$ (syntactical identity). Therefore $s^{d,u*\langle m,0\rangle} \equiv s^{d,u}$ and so Start(d,u*⟨m,0⟩), contradicting (3) once again.

(iii) $[\exists E^1]$; since d is normal, Σ_m must then have the form

$$
(5)\quad \dfrac{\begin{matrix}\Delta \\ \underline{a} \Rightarrow E^k_j(\vec{s}) \\ [\forall E]\ \ \underline{a} \Rightarrow \exists z Cz\end{matrix} \qquad \left\{\begin{matrix}\Gamma_p \\ \underline{a}, C\bar{p} \Rightarrow \exists y E\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle\end{matrix}\right\}_{p<\omega}}{\underline{a} \Rightarrow \exists y E\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle}
$$

First, if $\langle j,k,\langle\vec{s}\rangle\rangle = \langle i,n,\langle\vec{t}\rangle\rangle$ then Start(d,u*⟨m,0,0⟩) as in (ii), contradicting (3).

(iv) If, in (iii), $\exists z Cz$ is true, let $p := \mu z.Cz$, and consider - in place of Σ_m - its subderivation Γ_p (formally described by $d^{u*\langle m,p+1\rangle}$).

Before concluding the case $\rho^{d,u*\langle m\rangle} = [\exists E^1]$ let us turn first to case

(v) If $\rho^{d,u*\langle m\rangle}$ is $[\exists E^*]$, let $\bar{p}$ be the first numeral which does not occur in the sequents $s^{d,u*\langle m\rangle}$, $s^{d,u*\langle m,0\rangle}$ and consider (as in case (iv)) the subderivation $d^{u*\langle m,p+1\rangle}$.

(vi) If $\rho^{d,u*\langle m\rangle}$ is $[\exists E^1]$, and (iii) and (iv) do not apply, then in (5) $\langle j,k,\langle\vec{s}\rangle\rangle \neq \langle i,n,\langle\vec{t}\rangle\rangle$ and $\exists z Cz$ is false; so we can extract from (5) the following derivation of A^∞ for (2):

$$
(6)\quad \dfrac{\begin{matrix}B[\langle i,n,\langle\vec{t}\rangle\rangle] \\ [\forall E]\ \ E^k_j(\vec{s}) \\ [\forall E]\ \ \exists z Cz\end{matrix} \qquad \left\{\begin{matrix} C\bar{p} \\ [FE]\ \ \bot \\ [\bot]\ \ \exists y E\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle\end{matrix}\right\}_{p<\omega}}{[\exists E^1] \qquad \exists y E\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle}
$$

(here we dropped the precedents of sequents).

Finally we have the case

(vii) $[\exists I]$; then since every equation in $\underline{a}$ is true, we get as in 3.1.2 that the equation $F^{d,u*\langle m,0\rangle}$ is true, and we have (2) for the m considered.

These are all the cases in the absence of disjunction. Cases (i)-(iii) rule out possible failures of the construction; cases (iv),(v) allow the search to continue, while cases (vi) and (vii) yield the required derivation for (2).

Note that if E^* is true, then $\exists zCz$ in (6) is also true, and so case (vi) is excluded. Our argument here must however be independent of E^* (cf. 4.4(4)-(6)), and so case (vi) is considered throughout.

In order to clarify a bit the form of a search which proceeds through (iv),(v), let us consider by example the outcome of case (v), and suppose that now case (ii) applies to Γ_p (:≡ the derivation formally described by $d^{u*\langle m,p+1\rangle}$). I.e.- the following configuration occurs:

$$
\begin{array}{lcccc}
 & & & \Gamma_p & \\
 & & & \underline{a}, C\bar{p} \Rightarrow E^n_i(\vec{t}) & \Gamma_{p+1} \\
 & \cdots & & & \cdots \\
\underline{a} \Rightarrow \exists zCz & & [\forall E] & \underline{a}, C\bar{p} \Rightarrow \exists yE\langle \bar{m},y,i,n,\langle\vec{t}\rangle\rangle & \\
\hline
\text{the node } \textcircled{u*\langle m\rangle} \longrightarrow & & [\exists E^*] & \underline{a} \Rightarrow \exists yE\langle \bar{m},y,i,n,\langle\vec{t}\rangle\rangle &
\end{array}
$$

Here (3) implies, as in (i)-(iii)

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a} \Rightarrow \exists zCz \urcorner) \qquad \text{and} \qquad \neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}, C\bar{p} \Rightarrow E^n_i(\vec{t}) \urcorner)$$

which by 5.1(i) and the choice of $\bar{p}$ give

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a} \Rightarrow E^n_i(\vec{t}) \urcorner)$$

contradicting $\underline{Crit}_1(d,u)$. So we have adapted the argument of (ii) to the case that a search for a proof of (2) proceeds via case (v). Other arguments are adapted in about the same way, and this allows the iteration of the search through (iv)-(v) above.

By the well-foundedness of d the process must terminate, that is - one of cases (vi),(vii) ultimately appears, and we obtain a proof for (2), as wished.

5.2.4. *Disjunction reconsidered*

When disjunction does occur in the derivation d above, we must add to (i)-(vii) above another case:

(viii) $\rho^{d,u*\langle m\rangle}$ is $|\vee E|$. We then consider simultanuously *both* minor premises of $\rho^{d,u*\langle m\rangle}$, i.e. - the nodes $u*\langle m+1\rangle$ and $u*\langle m+2\rangle$.

As in the last paragraph of 5.2.3, let us see what happens if case (ii) applies now to both $u*\langle m,1\rangle$ and $u*\langle m,2\rangle$. We have then the following configuration:

$$
\begin{array}{llllll}
 & & \Gamma_1 & & \Gamma_2 \\
\Delta & & \underline{a},G_1 \Rightarrow E^n_i(\vec{t}) & & \underline{a},G_2 \Rightarrow E^n_i(\vec{t}) \\
\underline{a} \Rightarrow G_1 \vee G_2 & [\forall E] & \underline{a},G_1 \Rightarrow \exists yE\langle\ldots\rangle & [\forall E] & \underline{a},G_2 \Rightarrow \exists yE\langle\ldots\rangle \\
\hline
\bigcirc\!\!\!\!\!\!\; u*\langle m\rangle \longrightarrow & [\vee E] & \underline{a} \Rightarrow \exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle & &
\end{array}
$$

As in the last paragraph of 5.2.3

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}\Rightarrow G_1 \vee G_2\urcorner),\quad \neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a},G_1\Rightarrow E^n_i(\vec{t})\urcorner),\quad \neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a},G_2\Rightarrow E^n_i(\vec{t})\urcorner),$$

and so

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}\Rightarrow E^n_i(\vec{t})\urcorner), \quad \text{contradicting} \quad \underline{Crit}_1(d,u).$$

This argument may be generalized to conclude that, at least for one successive choice of minors of [vE] in the search described by (iv),(v),(viii) the construction leads to a node falling under one of the cases (vi),(vii) thus allowing a construction of a proof of A^∞ (incidently - of A^∞_{rec}) for (2).

The assertion that this is the case is now seen quite easily to be formalizable as a Π^0_1 predicate (over d,u).

5.2.5. *Remark: why does the presence of disjunction necessitate an additional argument*

We have seen in 5.2.4 that the presence of disjunction in d requires an extra argument which is not needed for the treatment of the existential quantifier. It might be in place to note here that v is, in L_1, in a way indeed more complex than ∃; or - roughly - v implies the presence of "plurality" in ways that are not implied by ∃. This is illustrated by the following facts.

[A] For a schema $\forall xF(x)$ of L_1

$$\vdash_{L_1} \forall xF(x) \quad \Longleftrightarrow \quad \vdash_{L_1} \exists xF(x).$$

This is of course not the case with & and v.

[B] Kripke models with a constant domain are complete for the disjunction-free fragment of L_1, but not for the existential-free fragment.

5.3. FORMALIZATION OF THE PREDICATE Crit

$$\underline{Step}(d,w,p) \;:\equiv\; \bigvee_{i=1,2,3} \underline{Step}_i(d,w,p)$$

where

$\underline{Step}_1(d,w,p)$:≡ "$\rho^{d,w} = [\exists E^1]$, and if $F^{d,w*\langle 0\rangle} \equiv: \exists zCz$ then $p = \mu z.Cz+1$ ".

$\underline{Step}_2(d,w,p)$:≡ "$\rho^{d,w} = [\exists E^*]$, and if $F^{d,w*\langle 0\rangle} \equiv: \exists zCz$ then p is 1 + "the numeric value of the first numeral which does not occur in $s^{d,w}, s^{d,w*\langle 0\rangle}$" ".

$\underline{Step}_3(d,w,p)$:≡ "$\rho^{d,w} = [\vee E]$ and $1 \le p \le 2$".

These three predicates correspond to cases (iv), (v) and (viii) in 5.2.3/4, where the search described there proceeds to the p'th premise of the node w. It should be noted that Step is a Δ_1^0 predicate. For example

$$\underline{Step}_1(d,w,p) \;\equiv\; \forall x,y\,[\,T(d,w,x) \;\&\; T(d,w*\langle 0\rangle,y) \;\longrightarrow\; A(x,y,p)\,]$$

$$\equiv\; \exists x,y\,[\,T(d,w,x) \;\&\; T(d,w*\langle 0\rangle,y) \;\&\; A(x,y,p)\,]$$

where

$$A(x,y,p) \;:\equiv\; (Ux)_0 = \ulcorner\exists E^1\urcorner \;\&\; \underline{Tr}_{QF}(\underline{inst}(antecedent((Uy)_1),p\dot{-}1))$$
$$\&\; \forall q<p\; \neg\underline{Tr}_{QF}(\underline{inst}(\underline{antecedent}((Uy)_1),q\dot{-}1)).$$

$\underline{Tr}_{QF}$ is a (Δ_1^0) truth predicate for equations, and inst is a prim.rec. function which satisfies $\underline{inst}(\ulcorner\exists xGx\urcorner,n) = \ulcorner G\bar{n}\urcorner$.

$$\underline{Selected}(d,v) \;:\equiv\; \forall i<\underline{lth}(v)\; \underline{Step}(d,(v|i),(v)_i)$$

where

$$(v|i) := \langle (v)_0,\ldots,(v)_{i\dot{-}1}\rangle \qquad (\text{for } i \le \underline{lth}(v))$$

$\underline{\text{Final}}(d,v) \quad :\equiv \quad \bigvee_{i=1,2,3} \underline{\text{Final}_i}(d,v)$

where

$\underline{\text{Final}_1}(d,v) \quad :\equiv \quad \underline{\text{Selected}}(d,v) \quad \& \quad \rho^{d,v} = [\bot] \text{ or } [\forall E]$

$\underline{\text{Final}_2}(d,v) \quad :\equiv \quad \underline{\text{Selected}}(d,v) \quad \& \quad \rho^{d,v} = [\exists E^1]$

$\underline{\text{Final}_3}(d,v) \quad :\equiv \quad \underline{\text{Selected}}(d,v) \quad \& \quad \rho^{d,v} = [\exists I].$

These predicates correspond to the cases in 5.2.3 where the construction may stop, whether successfully or not.

$\underline{\text{Final}}^+(d,v,\ulcorner A \urcorner) \quad :\equiv \quad \underline{\text{Final}_2^+}(d,v,\ulcorner A \urcorner) \quad \vee \quad \underline{\text{Final}_3}(d,v)$

where

$\underline{\text{Final}_2^+}(d,v,\ulcorner A \urcorner) \quad :\equiv \quad \underline{\text{Final}_2}(d,v) \quad \& \quad F^{d,v*\langle 0,0 \rangle} \not\equiv A.$

When for 5.2.3 $A \equiv E_i^n(\vec{t}) \equiv F^{d,u}$ then $\underline{\text{Final}}^+(d,v,\ulcorner A \urcorner)$ expresses the conclusion of the construction by one of (vi), (vii), or possibly its continuation through (iv). In any case, a "failure" through one of (i)-(iii) is excluded. It is important to note that Final and $\underline{\text{Final}}^+$ are both Δ_1^0 predicates.

Let us use the binary encodement of finite sets of numbers. The predicates $n \in x$, $x = \emptyset$ etc. are then just prim.rec. numeric expressions.

$$\underline{\text{Bar}}(d,x) \quad :\equiv \quad x \neq \emptyset \quad \&$$
$$\forall w \in x \; \{ \; \underline{\text{Final}}(d,w) \quad \& \quad \forall u,y<x \; [\; \rho^{d,u} = [\vee E]$$
$$\& \quad w = u*\langle {1 \atop 2} \rangle *y \quad \longrightarrow \quad \exists w' \in x \exists z<x \; w' = u*\langle {2 \atop 1} \rangle *z \;] \; \}.$$

I.e. - a "bar" for d is a finite non-empty set of "final" nodes, which intersects both minor subderivations of each instance of vE if it intersects one of them.

$$\underline{\text{Crit}_2}(d,u) \quad :\equiv \quad \forall m,x \; [\; \underline{\text{Bar}}(d^{u*\langle m \rangle},x) \quad \longrightarrow \quad \exists w \in x \; \underline{\text{Final}}^+(d^{u*\langle m \rangle},w,\ulcorner F^{d,u} \urcorner) \;]$$

$$\underline{\text{Crit}}(d,u) \quad :\equiv \quad \underline{\text{Crit}_1}(d,u) \quad \& \quad \underline{\text{Crit}_2}(d,u).$$

Note that Crit is intuitionistically equivalent to a Π^0_1 predicate.

$\underline{\text{Final}}^{++}(d,v,\ulcorner A\urcorner) \;:\equiv\; \underline{\text{Final}}_2^{++}(d,v,\ulcorner A\urcorner) \;\vee\; \underline{\text{Final}}_3^{++}(d,v,\ulcorner A\urcorner)$

where

$\underline{\text{Final}}_2^{++}(d,v,\ulcorner A\urcorner) \;:\equiv\; \underline{\text{Final}}_2^{+}(d,v,\ulcorner A\urcorner) \;\&\; \neg\underline{\text{Tr}}_{\Sigma^0_1}(\ulcorner F^{d,v*\langle 0\rangle}\urcorner)$

$\underline{\text{Final}}_3^{++}(d,v,\ulcorner A\urcorner) \;:\equiv\; \underline{\text{Final}}_3(d,v,\ulcorner A\urcorner) \;\&\; \underline{\text{Tr}}_{QF}(\ulcorner F^{d,v*\langle 0\rangle}\urcorner)$

$\underline{\text{Final}}^{++}$ corresponds to a real termination of the search described in 5.2.3. Contrary to $\underline{\text{Final}}^{+}$ however $\underline{\text{Final}}^{++}$ is a Π^0_1 predicate, and not a Δ^0_1 one.

5.4 - 5.6. PROOF OF 4.4(1): the existence of a critical node (first part of the proof theoretic reductions)

5.4. LEMMA.

$$\vdash_{Y_0+BI} [\; E^* \;\&\; \underline{\text{E-Der}}(d) \;\&\; \underline{\text{Start}}(d,u) \;] \;\rightarrow\; \neg\neg\exists w \succ u\; \underline{\text{Crit}}_1(d,w).$$

PROOF. Denote the formula to be proven by R(u). First, we prove below by BI, and using the well-foundedness of the proof-tree d, the (open) formula

$$S(u) :\equiv [\; E^* \;\&\; \underline{\text{E-Der}}(d) \;\&\; \underline{\text{Start}}(d,u) \;\&\; \neg R_4(d,u) \;] \;\rightarrow\; \neg\neg\exists w \succ u\; \underline{\text{Start}}(d,w).$$

Assuming $\forall u S(u)$, we can now prove R(u) by a second use of BI, where S(u) is to be used for the induction step.

Towards proving S(u) by BI, assume the premise of S(u), assume $\forall n S(u*\langle n\rangle)$, and consider cases for $\rho^{d,u}$, which by the normality of d are only the following:

(i) $\rho^{d,u}$ is [T]. This contradicts $R_1(d,u)$. $\rho^{d,u}$ is also not [TE] by $R_3(d,u)$.

(ii) $\rho^{d,u}$ is [FE]. As in 3.1.2, the normality of d implies then that $\rho^{d,u*\langle 0\rangle}$ is [T], and so $F^{d,u} \in \underline{a}^{d,u}$, contradicting $R_2(d,u)$.

(iii) $\rho^{d,u}$ is a propositional rule, [∃I] or [∀E]. If $\neg\neg\underline{Pr}_{L_1A}(\ulcorner s^{d,u*\langle n\rangle}\urcorner)$ for all n<3, then of course $\neg\neg\underline{Pr}_{L_1A}\ulcorner s^{d,u}\urcorner$, since all the rules considered in this case are (isomorphic to) rules of L_1. This contradicts $R_1(d,u)$. So $\neg\neg\exists n{<}3\ \neg\underline{Pr}_{L_1A}(\ulcorner s^{d,u*\langle n\rangle}\urcorner)$. For the cases considered the subformula property of d implies trivially $R_j(d,u) \rightarrow R_j(d,u*\langle n\rangle)$ for j = 2,3, and so we conclude that $\neg\neg\exists n{<}3\ \underline{Start}(d,u*\langle n\rangle)$.

(iv) $\rho^{d,u}$ is $[\exists E^*]$. Let $\bar{p}$ be the first numeral which does not occur in $s^{d,u}$, $s^{d,u*\langle 0\rangle}$, and prove

(*) $\neg\neg[\ \underline{Start}(d,u*\langle 0\rangle)\ \ v\ \ \underline{Start}(d,u*\langle p+1\rangle)\ ]$

like in (iii), using 5.1(i). That is, for the u considered

$$\neg\underline{Start}(d,u*\langle j\rangle) \rightarrow \neg R_1(d,u*\langle j\rangle) \rightarrow \neg\neg\underline{Pr}_{L_1A}\ulcorner s^{d,u*\langle j\rangle}\urcorner$$

while by the choice of p and 5.1(i)

$$\underline{Pr}_{L_1A}\ulcorner s^{d,u*\langle 0\rangle}\urcorner\ \&\ \underline{Pr}_{L_1A}\ulcorner s^{d,u*\langle p+1\rangle}\urcorner \rightarrow \underline{Pr}_{L_1A}\ulcorner s^{d,u}\urcorner \rightarrow \neg\underline{Start}(d,u).$$

Since this contradicts the assumed premise of S(u), one gets (*) by intuitionistic prop. logic (cf. KLEENE [52], p.119,*60i,g).

(v) $\rho^{d,u}$ is [∀I]. Let $\bar{p}$ be the first numeral which does not occur in $s^{d,u}$, and proceed to prove $\neg\neg\underline{Start}(d,u*\langle p+1\rangle)$ like in (iii), using 5.1(ii).

(vi) $\rho^{d,u}$ is $[\exists E^1]$, $F^{d,u*\langle 0\rangle} \equiv: \exists zCz$, where Cz is q.f.. Since $R_1(d,u)$, i.e. $-\ \neg\underline{Pr}_{L_1A}\ulcorner s^{d,u}\urcorner$, we get from 4.2.1 $\forall mR_1(d,u*\langle m+1\rangle)$. $R_3(d,u)$ implies $\forall mR_3(d,u*\langle m+1\rangle)$ trivially. Finally, for each m $R_2(d,u)$ and $C\bar{m}$ imply outright $R_2(d,u*\langle m+1\rangle)$. Summing up we hence get

(*) $\underline{Start}(d,u)\ \&\ \exists zCz \rightarrow \exists z\ \underline{Start}(d,u*\langle z\rangle)$.

But by the subformula property of d $\exists zCz$ is a subformula of the Π^0_2 sentence E^*, and so $E^* \rightarrow \exists zCz$, while by the assumed $\forall nS(u*\langle n\rangle)$,

$$\underline{Start}(d,u*\langle z\rangle) \rightarrow \neg\neg\exists w \succ u*\langle z\rangle\ \underline{Crit}_1(d,w)$$

So we get from (*)

$$\underline{\text{Start}}(d,u) \ \& \ E^{*} \ \rightarrow \ \neg\neg\exists w \succ u \ \underline{\text{Crit}}_1(d,w)$$

as wished. □

5.5.1. LEMMA.

$$\vdash_{\mathcal{V}_0+\text{BI}} \underline{\text{E-Der}}(d) \ \& \ \underline{\text{Crit}}_1(d,u) \ \& \ \neg\exists v \succ u \ \underline{\text{Start}}(d,v) \ \rightarrow \ \underline{\text{Crit}}_2(d,u).$$

We prove this lemma as a corollary of

5.5.2. LEMMA. *Let* A *be an* E-*sentence. Then*

$$\begin{aligned}\vdash_{\mathcal{V}_0+\text{BI}} \underline{\text{E-Der}}(d) \ \& \ \underline{\text{Crit}}_1(d,u) \ \& \ w = u*\langle m\rangle*z \ \& \ \underline{\text{Selected}}(d^{u*\langle m\rangle},z) \\ \& \ \forall v \succ u \ \neg\underline{\text{Start}}(d,v) \ \& \ \underline{\text{Bar}}(d^{w},x) \\ \& \ \forall y \in x \ \neg\underline{\text{Final}}^{+}(d^{w},y,\ulcorner F^{d,u}\urcorner) \\ \rightarrow \ \neg\neg\underline{\text{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u}\urcorner)\end{aligned}$$

5.5.3. *Proof that 5.5.2 implies 5.5.1*

Assume the premise of 5.5.1. For each $m \in \omega$ this implies the first five conjuncts of 5.5.2 for $w = u*\langle m\rangle$, $z = \langle\rangle$, and also

$$\neg\underline{\text{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,u*\langle m\rangle} \Rightarrow F^{d,u}\urcorner)$$

since $\underline{a}^{d,u*\langle m\rangle} = \underline{a}^{d,u}$ here. So, by the contrapositive of 5.5.2, and quantifying over m,

$$\forall m,x \ [\ \underline{\text{Bar}}(d^{u*\langle m\rangle},x) \ \rightarrow \ \exists y \in x \ \underline{\text{Final}}^{+}(d^{u*\langle m\rangle},y,\ulcorner F^{d,u}\urcorner)\ ]$$

(note that $\underline{\text{Final}}^{+}$ is decidable); i.e. - $\underline{\text{Crit}}_2(d,u)$ as required. □

5.5.4. *Proof of 5.5.2*

Write S(w) for the formula to be proven. By BI the problem reduces to showing

$$\forall n\ S(w*\langle n\rangle) \rightarrow S(w).$$

So assume

(1) $\forall n\ S(w*\langle n\rangle)$ and

(2) the premise $S^{-}(w)$ of $S(w)$.

Note first that the definition of <u>Selected</u> implies, by a trivial induction on <u>lth</u>(w)

(3) $F^{d,w} \equiv F^{d,u*\langle m\rangle} \equiv \exists y\ E\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle$

(4) $R_2(d,w) :\equiv$ "all equations in $\underline{a}^{d,w}$ are true".

Consider now cases for $\rho^{d,w}$.

(i) [T]. Then $F^{d,w} \in \underline{a}^{d,w}$. But by the subformula property of d no Σ^0_1 sentence may be discharged in d, because an E-sentence has no subformula of the form G∨H, G→H or ∃zG where G is Σ^0_1. So this case is ruled out. A similar argument excludes the cases [&E] and [→E].

(ii) [⊥]. Then $s^{d,w*\langle 0\rangle} = \underline{a}^{d,w} \Rightarrow \bot$, while ¬<u>Start</u>(d,w*⟨0⟩) implies (by (4))

$$\neg\neg\underline{\mathrm{Pr}}_{L_1A}(\ulcorner \underline{a}^{d,w} \Rightarrow \bot \urcorner),$$

so

$$\neg\neg\underline{\mathrm{Pr}}_{L_1A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner).$$

(iii) [∀E]. Then (3) implies

(5) $F^{d,w*\langle 0\rangle} \equiv F^{d,u}$.

On the other hand ¬Start(d,w*⟨0⟩) implies

(6) $\neg\neg\underline{\mathrm{Pr}}_{L_1A}(\ulcorner s^{d,w*\langle 0\rangle} \urcorner)$.

Here $\underline{a}^{d,w*\langle 0\rangle} = \underline{a}^{d,w}$ so (5) and (6) yield $\neg\neg\underline{\mathrm{Pr}}_{L_1A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner)$.

(iv) $[\exists E^1]$, $F^{d,w*\langle 0\rangle} \equiv: \exists z Cz$. Let <u>Bar</u>($d^w$,x).

<u>Subcase [a]</u>. ⟨⟩ ∈ x. Then ¬<u>Final</u>$^+(d^w,\langle\rangle,\ulcorner F^{d,u}\urcorner)$ by $S^{-}(w)$, and so by the definition of <u>Final</u>$^+$ for this case $F^{d,w*\langle 0,0\rangle} \equiv F^{d,u}$, and we get as in (iii)

$$\neg\neg\underline{\mathrm{Pr}}_{L_1A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner).$$

Subcase [b]. $\langle\rangle \notin x$. Then, since $x \neq \emptyset$ by the definition of Bar, we must have $\exists zCz$ and so for some p $\underline{\text{Step}}(d^{u*\langle w\rangle},z,p)$. We thus get by the BI hyp. (1) applied to $w*\langle p\rangle$

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w}, C(\bar{p}) \Rightarrow F^{d,u}\urcorner).$$

But $C(\bar{p})$ is here an equation, so by 4.2.1

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u}\urcorner).$$

(v) $[\exists E^*]$, $F^{d,w*\langle 0\rangle} :\equiv: \exists zCz$. Let $p \dot{-} 1$ be the first numeral which does not occur in $s^{d,w}, s^{d,w*\langle 0\rangle}$. We have then as in (iv)[b]

(7) $$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w}, C(\bar{p}) \Rightarrow F^{d,u}\urcorner)$$

and as in (iii) we get

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner s^{d,w*\langle 0\rangle}\urcorner) \equiv \neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow \exists zCz\urcorner)$$

which together with (7) yields

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u}\urcorner).$$

(vi) $[\vee E]$, $F^{d,w*\langle 0\rangle} :\equiv: G_1 \vee G_2$. Let

$$x^{(j)} :\equiv \{\, y \mid \langle j\rangle * y \in x \,\} \qquad (j=1,2).$$

Then, by the definition of Bar, $S^-(w)$ implies

$$\underline{\text{Bar}}(d^{w*\langle j\rangle}, x^{(j)}) \quad \& \quad \forall y \in x^{(j)}\ \neg\underline{\text{Final}}^+(d^{w*\langle j\rangle}, y, \ulcorner F^{d,u}\urcorner)$$

while trivially

$$\underline{\text{Selected}}(d^{u*\langle w\rangle}, z*\langle j\rangle) \qquad (j=1,2).$$

Apply now, as in (iv) and (v), the BI hyp. (1) to $w*\langle j\rangle$ $(j=1,2)$, to yield

(8) $$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w}, G_j \Rightarrow F^{d,u}\urcorner) \qquad (j=1,2).$$

On the other hand we get as in (iii)

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner s^{d,w*\langle 0\rangle}\urcorner) \equiv \neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow G_1 \vee G_2\urcorner)$$

which together with (8) yields

$$\neg\neg \underline{Pr}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u}\urcorner).$$

(vii) [∃I]. Then the definition of Bar implies

(9) $$\underline{Bar}(d^w,x) \longrightarrow x = \{\langle\rangle\}.$$

For this case, $\underline{Final}^+(d^w,\langle\rangle,\ulcorner F^{d,u}\urcorner)$ automatically, while by $S^-(w)$ (9) implies $\neg\underline{Final}(d^w,\langle\rangle,\ulcorner F^{d,u}\urcorner)$, so this case is ruled out. □

5.6. PROPOSITION.

$$\vdash_{Y_0+BI} E^* \ \ \& \ \ \underline{E\text{-}Der}(d) \ \ \& \ \ \underline{Start}(d,u) \ \longrightarrow \ \neg\neg\exists v \succ u \ \underline{Crit}(d,v).$$

PROOF. Straightforward from 5.4 and 5.5.1 using BI and the well-foundedness of the proof-tree d. □

Applying proposition 5.6 to $u = \langle\rangle$ we get assertion 4.4.(1).

5.7 - 5.11. PROOF OF 4.4(4). (Second part of the proof theoretic reduction)

5.7. LEMMA.

$$\vdash_{Y_0+BI} \underline{E\text{-}Der}(d) \ \ \& \ \ R_5(d,w) \ \longrightarrow \ \exists x \ \underline{Bar}(d^w,x).$$

PROOF. Straightforward by BI and the well-foundedness of d. □

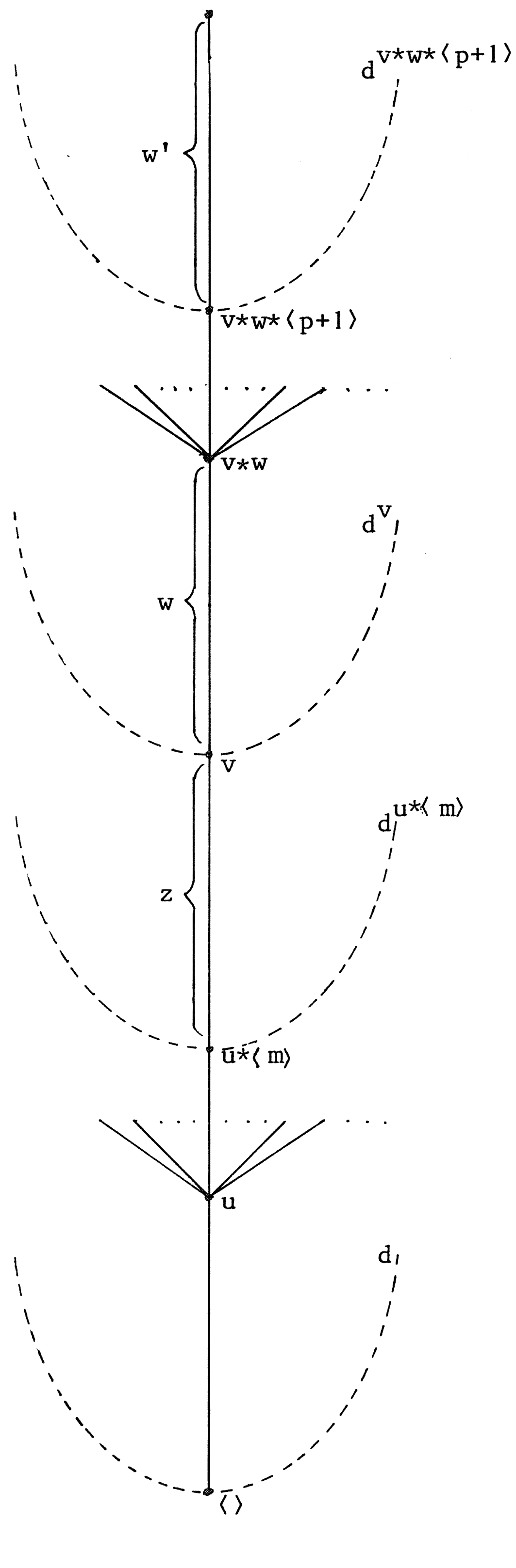
5.8.1. LEMMA.

$$\vdash_{Y_0+BI} \underline{E\text{-}Der}(d) \ \ \& \ \ \underline{Crit}(d,u) \ \ \& \ \ v = u*\langle m\rangle*z \ \ \& \ \ \underline{Selected}(d^{u*\langle m\rangle},z)$$
$$\longrightarrow \ \neg\neg\exists w \ \underline{Final}^{++}(d^v,w,\ulcorner F^{d,u}\urcorner)$$

PROOF. Fix v, assume the formula to hold for v', $v' \succ v$, and assume the premise for v. By 5.7 then

$$\underline{Bar}(d^v,x) \qquad \text{for some } x,$$

and so by $\underline{Crit}_2(d,u)$


d^v*w*⟨p+1⟩
w'
v*w*⟨p+1⟩
v*w
d^v
w
v
d^u*⟨m⟩
z
u*⟨m⟩
u
d
⟨⟩

$$\exists w \in x\ \underline{\mathrm{Final}}^{+}(d^{v}, w, \ulcorner F^{d,u} \urcorner).$$

Fix w, and observe the two possible cases for $\rho^{d,v*w}$.

(i) If $\rho^{d,v*w} = [\exists E^{1}]$, $F^{d,v*w*\langle 0\rangle} \equiv: \exists z Cz$, assume

$$(1) \qquad \neg\underline{\mathrm{Tr}}_{\Sigma_1^0}(\ulcorner \exists z Cz \urcorner) \ \vee\ \underline{\mathrm{Tr}}_{\Sigma_1^0}(\ulcorner \exists z Cz \urcorner).$$

If $\neg\underline{\mathrm{Tr}}_{\Sigma_1^0}(\ulcorner \exists z Cz \urcorner)$ then $\underline{\mathrm{Final}}_{2}^{++}(d^{v}, w, \ulcorner F^{d,u} \urcorner)$ by definition. If $\underline{\mathrm{Tr}}_{\Sigma_1^0}(\ulcorner \exists z Cz \urcorner)$ let $p := \mu z.Cz$; then $\underline{\mathrm{Selected}}(d^{v}, w*\langle p+1\rangle)$, hence $\underline{\mathrm{Selected}}(d^{u*\langle m\rangle}, z*w*\langle p+1\rangle)$ and so by BI hypothesis

$$\exists w'\ \underline{\mathrm{Final}}^{++}(d^{v*w*\langle p+1\rangle}, w', \ulcorner F^{d,u} \urcorner).$$

Since $\underline{\mathrm{Selected}}(d^{v}, w*\langle p+1\rangle)$ this implies $\exists w'\ \underline{\mathrm{Final}}^{++}(d^{v}, w', \ulcorner F^{d,u} \urcorner)$. Hence we have

$$\neg\neg\exists w'\ \underline{\mathrm{Final}}^{++}(d^{v}, w', \ulcorner F^{d,u} \urcorner)$$

without assumption (1) (cf. KLEENE [52], p.119 *58b-c,*51a).

(ii) If $\rho^{d,v*w} = [\exists I]$, assume $\neg\underline{\mathrm{Tr}}_{QF}(\ulcorner F^{d,v*\langle 0\rangle} \urcorner)$. By our definition of normality (cf. 1.3) $\rho^{d,v*w*\langle 0\rangle}$ cannot be $[\bot]$, and by the subformula property it cannot be other than $[T]$ (see 3.1.2). But this contradicts $R_2(d,v)$, which is seen outright to hold because $R_2(d,u*\langle w\rangle)$ and $\underline{\mathrm{Selected}}(d^{u*\langle w\rangle}, z)$. Since $\underline{\mathrm{Tr}}_{QF}$ is a decidable predicate we thus get $\underline{\mathrm{Tr}}_{QF}(\ulcorner F^{d,w*\langle 0\rangle} \urcorner)$ and so $\underline{\mathrm{Final}}_{3}^{++}(d^{v}, w, \ulcorner F^{d,u} \urcorner)$. □

5.8.2. COROLLARY

$$\vdash_{\mathcal{V}_0 + BI} \underline{\text{E-Der}}(d)\ \&\ \underline{\mathrm{Crit}}(d,u) \ \rightarrow\ \forall m \neg\neg\exists w\ \underline{\mathrm{Final}}^{++}(d^{u*\langle m\rangle}, w, \ulcorner F^{d,u} \urcorner).$$

PROOF. Apply 5.8.1 to $v = u*\langle m\rangle$. □

5.9. LEMMA. *There are prim.rec. functions* f_j (j=2,3) s.t.

$$\vdash_{\mathcal{V}_0} \underline{\text{E-Der}}(d)\ \&\ \underline{\mathrm{Final}}_{j}^{++}(d, v, \ulcorner E_{i}^{n}(\vec{t}) \urcorner)$$

$$\rightarrow\ \underline{\mathrm{Pr}}_{rec}^{\infty}(f_j(d, v, \langle i, n, \langle\vec{t}\rangle\rangle), \ulcorner B[\langle i, n, \langle\vec{t}\rangle\rangle] \Rightarrow F^{d,v} \urcorner).$$

(i) Let $\{f_2(d,v,\langle i,n,\langle\vec{t}\rangle\rangle)\}$ describe the tree

$$\cfrac{\cfrac{\cfrac{[T]\ B[\,] \Rightarrow B[\langle i,n,\langle\vec{t}\rangle\rangle] \qquad [\forall E]\ B[\,] \Rightarrow \underline{Ineq}(\langle j,k,\langle\vec{t}\rangle\rangle,\langle i,n,\langle\vec{t}\rangle\rangle) \to F^{d,v*\langle 0,0\rangle} \qquad [TE]\ B[\,] \Rightarrow \underline{Ineq}(\ )}{[\to E]\ \ B[\,] \Rightarrow F^{d,v*\langle 0,0\rangle}}}{[\forall E]\ \ B[\,] \Rightarrow F^{d,v*\langle 0\rangle} \qquad \{\Gamma_m\}_{m<\omega}}}{[\exists E^1]\ \ B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,v}}$$

where $F^{d,v*\langle 0,0\rangle} \equiv E^k_j(\vec{s})$ and $F^{d,v*\langle 0\rangle} \equiv \exists zCz$, and where

$$\Gamma_m :\equiv \begin{array}{ll} [T] & B[\langle i,n,\langle\vec{t}\rangle\rangle],C\bar{m} \Rightarrow C\bar{m} \\ [FE] & B[\langle i,n,\langle\vec{t}\rangle\rangle],C\bar{m} \Rightarrow \bot \\ [\bot] & B[\langle i,n,\langle\vec{t}\rangle\rangle],C\bar{m} \Rightarrow F^{d,v} \end{array}$$

(ii) Let $\{f_3(d,v,\langle i,n,\langle\vec{t}\rangle\rangle)\}$ describe the tree

$$\cfrac{[TE]\ \ B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,v*\langle 0\rangle}}{[\exists I]\ \ B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,v}}$$

$f_j(\ldots)$ are indeces of functions recursive in $\{d\}$, and by the s.m.n.-theorem f_j are indeed prim.rec. functions. The proof of the lemma for these functions is now straightforward. The only less trivial detail is the correctness of the [TE] inferences in the definition of f_2. From $\underline{Final}^{++}_2(d,v,\ulcorner E^n_i(\vec{t})\urcorner)$ we only know that $E^k_j(\vec{s})$ and $E^n_i(\vec{t})$ are not syntactically identical, but this does not exclude, prima facie, that $\vec{s}$ and $\vec{t}$ are numerically equal. Recall, however, that by our definition of <u>E-Der</u> in 4.1 $\vec{t}$ and $\vec{s}$ are tuples of numerals, and therefore their numerical equality implies their syntactical identity. □

5.10. COROLLARY.

$$\vdash_{\mathcal{V}_0+BI} \underline{E\text{-}Der}(d)\ \ \&\ \ \underline{Crit}(d,u)\ \ \&\ \ \text{"}F^{d,u} \equiv E^n_i(\vec{t})\text{"}$$

$$\to\ \forall m\neg\neg\exists x\ \underline{Prf}^{\infty}_{rec}(x,\ulcorner B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,u*\langle n\rangle}\urcorner)$$

PROOF. Immediate from 5.8 and 5.9. □

5.11. PROPOSITION.

$$\vdash_{\mathcal{Y}_0^C+AC_{00}} \text{E-Der}(d) \ \& \ \underline{\text{Crit}}(d,u) \ \& \ \text{"}F^{d,u} \equiv E_i^n(\vec{t})\text{"}$$
$$\rightarrow \exists\phi \ \underline{\text{Prf}}^{\infty}(\phi,\ulcorner s[\langle i,n,\langle\vec{t}\rangle\rangle]\urcorner).$$

PROOF. Note, first, that $\mathcal{Y}_0^C+AC_{00} \supset \mathcal{Y}_0+BI$. Assume the premise. Then by 5.10 and AC_{00}, for some function ψ

$$\forall m \ \underline{\text{Prf}}^{\infty}_{\text{rec}}(\psi m,\ulcorner B[\langle i,n,\langle\vec{t}\rangle\rangle]\Rightarrow F^{d,u*\langle m\rangle}\urcorner).$$

Define now ϕ by

$$\phi\langle\rangle := \langle\ulcorner\forall I\urcorner,\ulcorner s[\langle i,n,\langle\vec{t}\rangle\rangle]\urcorner\rangle$$

$$\phi(\langle m\rangle * u) := \{\psi m\}u$$

and the antecedent follows from 5.10. □

Applying prop. 5.11 to u = ⟨⟩, we get 4.4.(4).

6. SOLUTION OF THE REDUCED PROBLEM FOR L_1 (proof of 4.5(10))

<u>6.1. PROPOSITION</u> (= 4.5(10)). *Let S be a* Σ_2^0 *enumerated theory (with proof-predicate* $\exists x \forall y\ \underline{Prf}_S(x,y,\ulcorner F \urcorner)$ *say). Then there is a q.f. formula* E(x) *s.t., in the notation of 4.1,*

$$\vdash_{A+\underline{Con}(S)+\underline{Comp}_{\Sigma_1^0}(S)} \forall x \neg \underline{Pr}_S \ulcorner s^E(x) \urcorner \quad \& \quad \neg\neg E^*.$$

The proof given below is based on KRIPKE [63].

<u>6.2. LEMMA</u>. *For S as above, there exists a* Σ_2^0 *predicate* J(x) *s.t.*

(i) $\vdash_A \forall x,y\ [\ J(x)\ \ \&\ \ J(y)\ \ \rightarrow\ \ x=y\ ]$

(ii) $\nvdash_S \neg J(\bar{m})$ *for every numeral* $\bar{m}$.

<u>PROOF</u>. Let <u>neg</u> and $\underline{sub}_2$ be prim.rec. functions s.t. for every formula F

$$\underline{neg}(\ulcorner F \urcorner) = \ulcorner \neg F \urcorner$$

$$\underline{sub}_2(\ulcorner F \urcorner, x, y) = \ulcorner F[\bar{x}/a][\bar{y}/b] \urcorner$$

where $\bar{x}$ is the numeral with numeric value equals to x, and where F[t/a] is the formula which comes from F by replacing every occurrence of the parameter-letter a by (the closed term) t. Define

$$K(x,n,m) \quad :\equiv \quad \forall y\ \underline{Prf}_S(x,y,\underline{neg}(\underline{sub}_2(n,n,m)))$$

$$L \ \equiv\ L(a,b) \quad :\equiv \quad \exists x\ [\ K(x,a,b)\ \ \&\ \ \forall z{<}x \forall w{<}z\ \neg K(z,a,w)\ ]$$

$J(m) \;:\equiv\; L(\ulcorner L \urcorner, m)$ (here the g.n. $\ulcorner L \urcorner$ is the code of the fixed formula $L(a,b)$, while the defining symbol L is understood as a <u>predicate</u>)

Assuming that the g.n. of a syntactic object is larger than the g.n.'s of its partial syntactic objects, we have

(1) $$L(m,n) \;\leftrightarrow\; L^*(m,n)$$

where L^* is defined like L, except that the bounded quantifier $\forall w{<}z$ is replaced by an unbounded $\forall w$. For suitable Gödel numbering (e.g. - the standard ones) the property mentioned above is provable in A, hence

(2) $$\vdash_A \forall x,y\, [\, J(x) \;\&\; J(y) \;\rightarrow\; x{=}y \,].$$

Now suppose

(3) $$\vdash_S \neg J(\bar{m}) \qquad \text{for some } \bar{m},$$

i.e. -

$$\exists m \exists x \forall y\, \underline{Prf}_S(x,y,\ulcorner \neg L(\ulcorner L \urcorner, \bar{m}) \urcorner).$$

Then

(4) $$\neg\neg \exists m \exists x\, [\, \forall y\, \underline{Prf}_S(x,y,\ulcorner \neg L(\ulcorner L \urcorner,\bar{m})\urcorner) \;\&\; \forall z{<}x \forall w \exists y\, \neg \underline{Prf}_S(z,y,\ulcorner \neg L(\ulcorner L \urcorner,\bar{m})\urcorner) \,]$$

which is just $\neg\neg\exists m\, L(\ulcorner L \urcorner, m)$ by (1) and the definition of L.
But by $\underline{Comp}_{\Sigma^0_2}(S)$

(5) $$\forall m\, [\, L(\ulcorner L \urcorner, m) \;\rightarrow\; \exists x \forall y\, \underline{Prf}_S(x,y,\ulcorner L(\ulcorner L \urcorner, m)\urcorner) \,],$$

while the definition of L implies

(6) $$\forall m\, [\, L(\ulcorner L \urcorner, m) \;\rightarrow\; \exists x \forall y\, \underline{Prf}_S(x,y,\neg \ulcorner L(\ulcorner L \urcorner, m)\urcorner) \,],$$

so (4), (5), (6) together imply $\neg\neg\exists x \forall y\, \underline{Prf}_S(x,y,\ulcorner \bot \urcorner)$, contradicting $\underline{Con}(S)$. □

<u>6.3. LEMMA</u>. *For S as above there is a* Σ^0_2 *predicate* M(x), *s.t. for every q.f. predicate* P(x)

$$\not\vdash_S \neg\forall x\, [\; M(x) \;\leftrightarrow\; P(x)\;].$$

<u>PROOF</u>. Let U(n,x) be a binary q.f. predicate which enumerate all unary q.f. predicates (by Kleene's enumeration theorem, cf. e.g. KLEENE [52], §58), and let J be as in 6.2. Define

$$M(x) \;:\equiv\; \exists y\, [\; J(y) \;\&\; U(y,x)\;].$$

By 6.2(i) then

$$J(\bar{m}) \vdash_A \forall x\, [\; M(x) \;\leftrightarrow\; U(\bar{m},x)\;] \qquad \text{for every numeral } \bar{m}.$$

But by 5.2(ii)

$$\not\vdash_S \neg J(\bar{m}),$$

so

$$\not\vdash_S \neg\forall x\, [\; M(x) \leftrightarrow U(\bar{m},x)\;] \qquad \text{for every m, as wished.} \quad \Box$$

<u>6.3.2. LEMMA</u>. *Lemma 6.3.1 holds also when* M *is required to be* Π^0_2.

<u>PROOF</u>. Replace M by ¬M. □

<u>6.4. PROOF OF 6.1 (concluded)</u>. Let M(z) be given by 6.3.2, and write M(z) as ∀x∃y E⟨x,y,z⟩.

(i) Assume now $\underline{Pr}_S \ulcorner s^E(n) \urcorner$ for some n (i.e. $\vdash \exists x \forall y\, \underline{Prf}_S(x,y,\ulcorner s^E(n)\urcorner)$). By the form of the sequent $s^E(n)$ we have then

$$\vdash_S \forall z \neq \bar{n} M(z) \;\rightarrow\; M(\bar{n})$$

and therefore

$$\vdash_S \neg\forall z\, [\; z \neq \bar{n} \;\leftrightarrow\; M(z)\;]$$

contradicting 6.3.2.

(ii) Assume $\neg E^*$, i.e. $\vdash \neg\forall z M(z)$. Then, by $\underline{Comp}_{\Sigma^0_2}(S)$, $\neg\neg\underline{Pr}_S(\ulcorner \neg\forall z M(z)\urcorner)$. But taking P(z) :≡ z=z in 6.3.2 we get $\not\vdash_S \neg\forall z M(z)$, a contradiction. So $\neg\neg E^*$. □

7. CONCLUDING REMARKS

7.1. A COUNTEREXAMPLE TO A CONJECTURE OF H. FRIEDMAN

H. FRIEDMAN [73] has conjectured that every sequence of classically independent Σ^0_1 sentences may serve as a (meta-) substitution for the absoluteness of L_0. This is however false already for schemata over two propositional letters.

Construct a counter-example as follows. Let A,B be Σ^0_1 sentences independent over A, and s.t.

(1) $\vdash_A A \to \neg B$

(such sentences exist, by KRIPKE [63]).

Let C_1 be A-independent over B, and define

$$C :\equiv B \mathbin{\&} C_1 \quad ; \quad D :\equiv A \vee C$$

[1] {B,D} is classically independent, because

(i) $B \vdash D \Rightarrow B \vdash A \vee C \underset{\text{(by (1))}}{\Rightarrow} B \vdash C \Rightarrow B \vdash C_1$

contradicting the choice of C_1.

(ii) $D \vdash B \Rightarrow A \vdash B \underset{\text{(by (1))}}{\Rightarrow} \vdash \neg A$

contradicting the choice of A.

(iii) $D \vdash \neg B \Rightarrow C_1 \vdash \neg B \Rightarrow B \vdash \neg C_1$

again a contradiction.

[2] $\vdash D \rightarrow B \vee \neg B$,

because $A \vdash \neg B$ and $C \vdash B$; $\{B,D\}$ is therefore not a (meta-) substitution for the absoluteness of L_0. A similar counter-example was discovered independently by D.H.J. de Jongh.

7.2. OPEN PROBLEMS

7.2.1. Is L_0 absolute (for A, say) with a universal Σ_1^0 metasubstitution (independent of the number of propositional letters in a schema)?

7.2.2. In LEIVANT [75] it is shown that L_1 is not absolute (for A, say) with Σ_1^0 metasubstitutions. This is a pleasant bound on possible improvements of theorem II. There remains however the question whether the theorem holds with Δ_2^0 metasubstitutions. More generally, the problem may be referred to a whole intuitionistic hierarchy of arithmetical predicates between Σ_1^0 and Π_2^0, e.g. - $\Pi_1^0 \rightarrow \Sigma_1^0$, $\Pi_1^0 \vee \Pi_1^0$, $\Sigma_1^0 \rightarrow \Pi_2^0$, etc.

7.2.3. A more philosophically inclined (and hence - technically vague) problem is the following.

Let us propose as a thesis that an arithmetical sentence is true only if it is provable is some (constructively acceptable) number-theory, belonging to some (fixed) constructively generated class of theories. This thesis is a claim for a proof-theoretic criterion for constructive truth, and thus establishes a connection between absoluteness of L_1 for the class of theories considered, and completeness of L_1 for a more abstract notion of truth.

To make our thesis precise, we have, however, to specifiy a class of theories, and to justify the claim of exhaustiveness of this class for constructive truth. The relevance of the results given above to the abstract completeness of L_1 depends then on the relation between the proposed class, and the classes of regular and strongly regular number-theories.

Some technical results related to the general problem above will be given elsewhere.

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