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A NOTE ON CERTAIN OSCILLATING SUMS

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A note on certain oscillating sums

by

A.E. Brouwer and J. van de Lune

ABSTRACT: Let $S(N,\alpha) = \sum_{n=1}^{N} (-1)^{[n\alpha]}$.

A characterization is given of all real α for which $S(N,\alpha) \geq 0$ for all N. In addition it is shown that the set consisting of all these α has Lebesgue measure zero.

KEY WORDS & PHRASES: *exponential sums, continued fractions*

0. INTRODUCTION

In this note we investigate sums of the form

$$(0.1)\qquad S_N(\alpha) = \sum_{n=1}^{N} (-1)^{[n\alpha]}, \quad (\alpha\in\mathbb{R}).$$

In particular we shall characterize the set P and the irrational elements of N where

$$(0.2)\qquad P = \{\alpha\in\mathbb{R} \mid S_N(\alpha) \geqq 0 \quad \text{for all } N \in \mathbb{N}\}$$

and

$$(0.3)\qquad N = \{\alpha\in\mathbb{R} \mid S_N(\alpha) \leqq 0 \quad \text{for all } N \in \mathbb{N}\}.$$

These characterizations (see theorem 2.1 and 4.1) will be given in terms of the regular continued fraction expansions of the corresponding α.
In addition it will be shown that P and N have (Lebesgue) measure 0.

1. PREPARATIONS

We start dealing with P.
It is clear that

$$(1.1)\qquad 0 \in P$$

and

$$(1.2)\qquad \alpha \in P \iff \alpha + 2 \in P.$$

Hence, without loss of generality, we may assume that $\alpha > 0$. For the time being we also assume α to be *irrational*.

A simple counting process reveals that if α is positive then

$$(1.3)\qquad S_N(\alpha) = \sum_{k=1}^{M} (-1)^{k-1} \{[k\beta] - [(k-1)\beta]\} + (-1)^M \{N-[M\beta]\}$$

where $M = [N\alpha]$ and $\beta = \frac{1}{\alpha}$.
Observe that for any $M \in \mathbb{N}$

$$S_{[M\beta]}(\alpha) = \sum_{k=1}^{M} (-1)^{k-1} \{[k\beta] - [(k-1)\beta]\}. \tag{1.4}$$

It is easily seen that (for positive α) $\alpha \in P$ if and only if

$$\sum_{k=1}^{2K} (-1)^{k-1} \{[k\beta] - [(k-1)\beta]\} \geqq 0 \quad \text{for all } K \in \mathbb{N}. \tag{1.5}$$

Since 2K is even (sic!) it follows that $\alpha \in P$ if and only if for some $z \in \mathbb{Z}$

$$\sum_{k=1}^{2K} (-1)^{k-1} \{[k(\beta+z)] - [(k-1)(\beta+z)]\} \geqq 0 \quad \text{for all } K \in \mathbb{N}. \tag{1.6}$$

If we choose $\beta + z > 0$ it follows that

$$\alpha \in P \iff \frac{1}{\beta+z} \in P. \tag{1.7}$$

In particular, taking $z = -[\beta]$ we obtain

$$\alpha \in P \iff \frac{1}{\frac{1}{\alpha} - [\frac{1}{\alpha}]} \in P. \tag{1.8}$$

For any irrational α with regular continued fraction expansion

$$\alpha = \langle a_0; a_1, a_2, a_3, \dots \rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

we define

$$g(\alpha) = \langle a_2; a_3, a_4, a_5, \dots \rangle \tag{1.9}$$

and

$$(1.10) \qquad \rho_k(\alpha) = < 0; a_k, a_{k+1}, a_{k+2}, \ldots > , \qquad (k \in \mathbb{N}).$$

It is clear that

$$(1.11) \qquad 0 < \rho_k(\alpha) < 1 \quad \text{for all } k \in \mathbb{N}$$

and

$$(1.12) \qquad \frac{1}{\rho_k(\alpha)} = < a_k; a_{k+1}, a_{k+2}, \ldots >$$

so that

$$(1.13) \qquad \frac{1}{\rho_k(\alpha)} = a_k + \rho_{k+1}(\alpha)$$

and

$$(1.14) \qquad [\frac{1}{\rho_k(\alpha)}] = a_k.$$

Hence

$$(1.15) \qquad \frac{1}{\frac{1}{\rho_k(\alpha)} - [\frac{1}{\rho_k(\alpha)}]} = \frac{1}{\rho_{k+1}(\alpha)} = a_{k+1} + \rho_{k+2}(\alpha).$$

LEMMA 1.1. *If* α *is positive and irrational then*

$$(1.16) \qquad \alpha \in P \iff (g(\alpha) \in P \text{ and } a_0 \equiv 0 \pmod 2).$$

PROOF. ($\Rightarrow$) Let $\alpha \in P$. Then $a_0 \equiv 0 \pmod 2$. Indeed, if $a_0 \not\equiv 0 \pmod 2$ we would have $S_1(\alpha) = -1$ so that $\alpha \notin P$.
Hence, by (1.2), it follows that $\alpha - a_0 \in P$, so that by (1.8) we have

$$(1.17)\qquad \frac{1}{\dfrac{1}{\alpha-a_0} - \left[\dfrac{1}{\alpha-a_0}\right]} \in P.$$

Since the left hand side of (1.17) equals $g(\alpha)$ this part of the proof is complete.

($\Leftarrow$). If $g(\alpha) \in P$ then by (1.17) and (1.8) we have that $\alpha-a_0 \in P$. Since $a_0 \equiv 0 \pmod 2$ it follows from (1.2) that $\alpha \in P$. □

Define

$$(1.18)\qquad P_N = \{0 \leqq \alpha < 1 \mid S_n(\alpha) \geqq 0 \quad \text{for all} \quad n \leqq N\}.$$

From this definition it is clear that

$$(1.19)\qquad P_1 \supset P_2 \supset P_3 \supset \dots$$

and

$$(1.20)\qquad P \cap [0,1) = \bigcap_{N=1}^{\infty} P_N.$$

Let F_N be the Farey series of order N, restricted to the interval [0,1).

LEMMA 1.2. *P_N is a (non-empty) union of a finite number of intervals of the form [a,b) with a < b where a and b are (rational) points of F_N.*

PROOF. It is easily seen that

$$(1.21)\qquad [0,\tfrac{1}{N}) \subset P_N \text{ and } [\tfrac{N-1}{N}, 1) \subset P_N$$

proving the "non-empty" part of the lemma.

Now let a and b be consecutive points of F_N. Then the proof is complete if we can show that

$$(1.22)\qquad a \in P_N \Rightarrow [a,b) \subset P_N.$$

By definition, $a \in P_N$ implies that

$$(1.23) \qquad S_n(\alpha) = \sum_{k=1}^{n} (-1)^{[k\alpha]} \geqq 0 \text{ for all } n \leqq N.$$

Since for every fixed $k \leqq N$ the function $[kx]$ is constant on each of the intervals $[0,\frac{1}{k})$, $[\frac{1}{k}, \frac{2}{k})$, ... , $[\frac{k-1}{k}, 1)$ and since $[a,b)$ is always contained in one of these intervals, the lemma follows from (1.23) by a right-continuity argument. □

COROLLARY 1.1. *P is left-closed. In other words:*

If $\{\alpha_n\}_{n-1}^{\infty}$ is a non-increasing sequence in P with limit α then also $\alpha \in P$.

COROLLARY 1.2. *P is (Lebesgue) measurable.*

LEMMA 1.3. *Let α be irrational and positive.*
If

$$\beta = \frac{1}{\alpha},\ M \in \mathbb{N},\ \ N = [2M\beta],\ z \in \mathbb{Z},\ \ \beta + z > 0,\ K = [2M(\beta+z)]$$

then

$$(1.24) \qquad S_N(\alpha) = S_K(\frac{1}{\beta+z}).$$

PROOF. This is a simple consequence of (1.4). □

If we choose $z = -[\beta]$ in lemma 1.3 then

$$(1.25) \qquad K = [2M(\beta-[\beta])] \leqq [2M\beta] = N.$$

2. CHARACTERIZATION OF P

THEOREM 2.1. *Let α be irrational and positive with regular continued fraction expansion*

$$\alpha = < a_0; a_1, a_2, a_3, \dots >.$$

Then

$$(2.1) \qquad \alpha \in P \iff (a_{2i} \equiv 0 \pmod 2 \text{ for all } i \geqq 0).$$

PROOF. ($\Rightarrow$). Let $\alpha \in P$. Then, by lemma 1.1 we have $a_0 \equiv \pmod 2$ and $g(\alpha) \in P$. Observing that $g(\alpha) = < a_2; a_3, a_4, \dots >$ we must have $a_2 \equiv 0 \pmod 2$ etc.
($\Leftarrow$). Now assume that $a_{2i} \equiv 0 \pmod 2$ for all $i \geqq 0$. Suppose that $\alpha \notin P$. Then also $\rho_1 \overset{\text{def}}{=} \alpha - a_0 \notin P$.
Hence

$$(2.2) \qquad S_N(\rho_1) < 0 \text{ for some } N \in \mathbb{N}.$$

Choose N such that the inequality in (2.2) holds true and such that N is *minimal*. Since $0 < \rho_1 < 1$ we may consider the position of ρ_1 with respect to the Farey series of order N.
For every $n \in \mathbb{N}$ such that $n \leqq N$, the function $[nx]$ is constant on the canonical (= smallest) intervals of the form $[a,b)$ corresponding to F_N. Hence, since ρ_1 is irrational, there exists an open interval I containing ρ_1 such that

$$(2.3) \qquad S_N(\gamma) < 0 \text{ for all } \gamma \in I.$$

Because of the minimality of N there exists an $M \in \mathbb{N}$ such that $N = \left[\frac{2M}{\rho_1}\right]$.
From continuity arguments concerning regular continued fractions it follows that there exists an $\ell \in \mathbb{N}$ such that all irrational numbers $x > 0$ defined by

$$(2.4) \qquad x = < 0; a_1, a_2, \dots, a_{2\ell-1}, a_{2\ell}, m_{2\ell+1}, m_{2\ell+2}, \dots >$$
$$(\text{with } m_j \in \mathbb{N},\ j \geqq 2\ell+1)$$

are such that

(2.5) $\qquad x \in I$ and $\left[\frac{2M}{x}\right] = \left[\frac{2M}{\rho_1}\right] = N.$

Observe that (since $a_{2i} \equiv 0 \pmod 2$)

(2.9) $$S_{N_1}\left(\frac{1}{\frac{1}{x_0} - \left[\frac{1}{x_0}\right]}\right) = S_{N_1}(< a_2; a_3, a_4, \ldots, a_{2\ell}, N, 1, 1, 1, \ldots >) =$$

$$= S_{N_1}(< 0; a_3, a_4, \ldots, a_{2\ell}, N, 1, 1, 1, \ldots >) = S_{N_1}(\rho_3(\alpha)).$$

Without loss of generality we may assume that N_1 is the smallest natural number for which

$$S_{N_1}(\rho_3(\alpha)) < 0.$$

Continuing this reduction we will ultimately find a natural number N_ℓ such that

(2.10) $\qquad S_{N_\ell}(<0;N,1,1,1,\ldots>) < 0$ with $N_\ell \leqq N.$

On the other hand, since

(2.11) $\qquad < 0;N,1,1,1,\ldots > = \frac{1}{N+\delta} \;(< \frac{1}{N})$

for some $\delta > 0$ and since $N_\ell \leqq N$ we have

(2.12) $\qquad S_{N_\ell}(<0;N,1,1,1,\ldots>) > 0.$

Since this contradicts (2.10) the proof is complete.

THEOREM 2.2. *If* α *is rational then* $\alpha \in P$ *if and only if the canonical continued fraction expansion of* α *is of the form*

(2.13a) $$\alpha = < a_0; a_1, a_2, \ldots, a_{2\ell-1}, a_{2\ell} >$$

with

(2.13b) $$\alpha_{2i} \equiv 0 \pmod 2 \text{ for all } 0 \leqq i \leqq \ell.$$

PROOF. Suppose α satisfies (2.13).
Then

(2.14) $$\alpha < \alpha_N \text{ for all } N \in \mathbb{N}$$

where

(2.15) $$\alpha_N = < a_0; a_1, a_2, \ldots, a_{2\ell-1}, a_{2\ell}, 2N, 2N, 2N, \ldots >.$$

Observing that P is left closed and that

(2.16) $$\lim_{N\to\infty} \alpha_N = \alpha \text{ and } \alpha_N \in P.$$

it follows from (2.14) that $\alpha \in P$.
Now assume that (2.13) is not satisfied. Observe that if α is positive and rational then (compare (1.3))

(2.16) $$S_N(\alpha) = \lim_{\varepsilon\downarrow 0} \left\{ \sum_{k=1}^{M} (-1)^k \{[k(\beta-\varepsilon)] - [(k-1)(\beta-\varepsilon)]\} + \right.$$
$$\left. + (-1)^M \{N - [M(\beta-\varepsilon)]\} \right\}$$

where

$$M = \lim_{\varepsilon\downarrow 0} [N(\alpha+\varepsilon)] = [N\alpha].$$

From this we obtain that (for positive α) $\alpha \in P$ if and only if for all $K \in \mathbb{N}$

$$\lim_{\varepsilon \downarrow 0} \sum_{k=1}^{2K} (-1)^{k-1} \{[k(\beta-\varepsilon)] - [(k-1)(\beta-\varepsilon)]\} \geqq 0 \tag{2.17}$$

so that, similarly as in section 1, for $\alpha > 0$ and $\alpha \in Q$ we have

$$\alpha \in P \iff \left(\frac{1}{\beta+z} \in P \text{ for some } z \in \mathbb{Z}\right). \tag{2.18}$$

In particular we use (2.18) with $z = -[\beta]$.

CASE 1. $\alpha = \langle a_0; a_1, a_2, \ldots, a_{2\ell-1} \rangle$

Assuming that $\alpha \in P$ we would ultimately obtain that $\langle a_{2\ell-2}; a_{2\ell-1} \rangle \in P$ so that we must have $a_{2\ell-2} \equiv 0 \pmod 2$ and hence

$$\langle 0; a_{2\ell-1} \rangle = \frac{1}{a_{2\ell-1}} \in P. \tag{2.19}$$

However, it is easily verified that P does *not* contain any of the numbers $\frac{1}{n}$, $n \in \mathbb{N}$.

CASE 2. $\alpha = \langle a_0; a_1, a_2, \ldots, a_{2\ell-1}, a_{2\ell} \rangle$
with $a_{2i} \not\equiv 0 \pmod 2$ for some i.
Repeated use of (2.18) reveals that $\alpha \notin P$. □

3. THE MEASURE OF THE SET P.

THEOREM 3.1. *The set P has measure* 0.

PROOF. Define $P^* = \{P \backslash Q\} \cap [0,1)$.

Let (a_i, b_i) be some countable system of open intervals such that $0 \leqq a_i < b_i$ for all i and

$$(3.1) \qquad E \overset{\text{def}}{=} \bigcup_{i=1}^{\infty} (a_i, b_i) \supset P^*.$$

From the characterization of the irrational points belonging to P it is clear that

$$(3.2) \qquad P^* = \bigcup_{\substack{k,a=1 \\ x \in P^*}}^{\infty} \left\{ \frac{1}{k + \frac{1}{2a + x}} \right\}$$

so that

$$(3.3) \qquad P^* \subset \bigcup_{\substack{k,a=1 \\ x \in E}}^{\infty} \left\{ \frac{2a+x}{k(2a+x)+1} \right\}.$$

Since for all fixed k, a $\in \mathbb{N}$ the function

$$(3.4) \qquad \frac{2a+x}{k(2a+x)+1}, \qquad (x>0)$$

is increasing we obtain that (λ denoting Lebesgue measure)

$$(3.5) \qquad \lambda(P^*) \leqq \sum_{i,k,a=1}^{\infty} \left\{ \frac{2a+b_i}{k(2a+b_i)+1} - \frac{2a+a_i}{k(2a+a_i)+1} \right\} =$$

$$= \sum_{i,k,a=1}^{\infty} \frac{b_i - a_i}{\{k(2a+b_i)+1\}\{k(2a+a_i)+1\}} \leqq$$

$$\leqq \sum_{i,k,a=1}^{\infty} \frac{b_i - a_i}{4k^2 a^2} = \frac{1}{4} \left(\frac{\pi^2}{6}\right)^2 \cdot \lambda(E).$$

It follows that

$$(3.6) \qquad \lambda(P^*) \leqq \frac{\pi^4}{144} \lambda(E) < \frac{7}{10} \lambda(E).$$

Since P^* is measurable and E may be chosen such that

$$\lambda(E) < \lambda(P^*) + \varepsilon, \tag{3.7}$$

it follows easily that we must have

$$\lambda(P^*) = 0 \tag{3.8}$$

and hence

$$\lambda(P) = 0. \tag{3.9}$$

4. THE SET N

THEOREM 4.1. *If α is irrational then*

$$\alpha \in N \iff -\alpha \in P. \tag{4.1}$$

PROOF. Observe that

$$[x] + [-x] = -1 \quad \text{for all} \quad x \in \mathbb{R}\setminus\mathbb{Z}. \tag{4.2}$$

Hence, if α is irrational then

$$S_N(\alpha) = \sum_{n=1}^{N} (-1)^{[n\alpha]} = \sum_{n=1}^{N} (-1)^{-1-[-n\alpha]} = \tag{4.3}$$

$$= -\sum_{n=1}^{N} (-1)^{-[-n\alpha]} = -\sum_{n=1}^{N} (-1)^{[n(-\alpha)]}$$

so that

$$S_N(\alpha) \leqq 0 \iff S_N(-\alpha) \geqq 0, \tag{4.4}$$

proving the theorem.

REMARK. In general, formula (4.1) does not hold true for $\alpha \in Q$ as may be seen from the following example: $1 \in N$, $-1 \notin P$.

COROLLARY. *The set* N *has measure zero.*

5. ONE MORE PROPERTY OF P (resp. N)

THEOREM 5.1. *For every irrational* $\alpha \in P$ *we have that*

(5.1) $\qquad S_N(\alpha) = 0$ for infinitely many $N \in \mathbb{N}$.

In order to prove this we use the following

LEMMA 5.1. *If the positive integers* p *and* q *are such that* p *is even and* $(p,q) = 1$ *then*

(5.2) $\qquad S_{q-1}\left(\frac{p}{q}\right) = 0.$

PROOF. Consider the $q-1$ numbers

$$\frac{p}{q}, \frac{2p}{q}, \ldots, \frac{(q-1)p}{q}.$$

Since $(p,q) = 1$ none of these numbers is an integer and since p is even q is odd so that $q-1$ is even.
Since p is even we have for $1 \leqq r \leqq \frac{q-1}{2}$ that the integers

$$[r\,\tfrac{p}{q}] \text{ and } [(q-r)\cdot\tfrac{p}{q}]$$

have different parity from which it is clear that $S_{q-1}\left(\frac{p}{q}\right) = 0$.

PROOF OF THEOREM 5.1.

Without loss of generality, we may assume that $0 < \alpha < 1$.

Let $\alpha = < 0; a_1, a_2, \ldots >$ and let

$$\frac{A_0}{B_0} = \frac{0}{1}, \frac{A_1}{B_1} = \frac{1}{a_1}, \frac{A_2}{B_2} = \frac{a_2}{a_1 a_2 + 1}, \ldots, \frac{A_n}{B_n}, \ldots$$

be the corresponding convergents.
Since $\alpha \in P$ we have that $a_{2i} \equiv 0 \pmod 2$ for all $i \geqq 1$ from which it is easily seen that A_{2n} is even for all n.

In order to prove the theorem it suffices to show that for all $n \in \mathbb{N}$

$$\sum_{k=1}^{B_{2n}-1} (-1)^{[k\alpha]} = 0. \tag{5.3}$$

Since A_{2n} is always even it follows from lemma 5.1 that

$$\sum_{k=1}^{B_{2n}-1} (-1)^{\left[k\frac{A_{2n}}{B_{2n}}\right]} = 0, \tag{5.4}$$

so that our proof will be complete if we can show that

$$[k\alpha] = \left[k\,\frac{A_{2n}}{B_{2n}}\right] \text{ for } 1 \leqq k \leqq B_{2n} - 1. \tag{5.5}$$

We proceed by contradiction.
If (5.5) is not true then (note that $\frac{A_{2n}}{B_{2n}} < \alpha$)

$$k\frac{A_{2n}}{B_{2n}} < m < k\,\alpha \quad \text{for some} \quad m \in \mathbb{N}. \tag{5.6}$$

Hence

$$\frac{1}{B_{2n}} \leqq m - k\frac{A_{2n}}{B_{2n}} < k\alpha - k\frac{A_{2n}}{B_{2n}} = k\left(\alpha - \frac{A_{2n}}{B_{2n}}\right) < (B_{2n}-1) \cdot \frac{1}{B_{2n}^2} < \frac{1}{B_{2n}}. \tag{5.7}$$

This contradiction completes the proof.

CORROLLARY. For every irrational $\alpha \in N$ we have that

(5.8) $\qquad S_N(\alpha) = 0$ for infintely many $N \in \mathbb{N}$.

ADDENDUM.

During the preparation of this note J. VAN DE LUNE and H.J.J. TE RIELE proved the following (more general)

THEOREM. *If* α *is irrational then* $S_n(a) = 0$ *for infinitely many* $n \in \mathbb{N}$.

REMARK: From now on all fractions $\frac{p}{q}$ are assumed to be irreducible.

In order to prove the theorem we use the following

LEMMA. *If* p *is odd then* $S_{2q}(\frac{p}{q}) = 0$.

PROOF: *Observe that the numbers*

$$\left[r\frac{p}{q}\right] \text{ and } \left[(q+r)\frac{p}{q}\right], \quad 1 \le r \le q$$

have different parity. $\square$

In addition we will use the following well-known

THEOREM (of HURWITZ). *If* $\alpha \in \mathbb{R}$ *is irrational then there exist infinitely many rationals* $\frac{p}{q}$ *such that*

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2\sqrt{5}}.$$

PROOF OF THE THEOREM. *Let* H *be the set of all fractions* $\frac{p}{q}$ *such that*

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2\sqrt{5}}.$$

It is clear that the proof will be complete if we can show that for every $\frac{p}{q} \in H$ we have either $S_{q-1}(\alpha) = 0$ or $S_{2q}(\alpha) = 0$.
We consider a number of cases.

CASE 1. $\frac{p}{q} \in H$, p even.

Then we have $S_{q-1}(\alpha) = 0$.
In order to see this it is clearly sufficient to prove that $S_{q-1}(\alpha) = S_{q-1}(\frac{p}{q})$.
Hence it is sufficient to show that

$$[k\alpha] = [k\frac{p}{q}] \text{ for } 1 \leqq k \leqq q-1.$$

Assuming this does not hold true we have for some k, $1 \leqq k \leqq q-1$, that there exists an $m \in \mathbb{Z}$ such that either

$$k\frac{p}{q} \leqq m < k\alpha \text{ (in case } \frac{p}{q} < \alpha)$$

or

$$k\alpha < m \leqq k\frac{p}{q} \text{ (in case } \frac{p}{q} > \alpha).$$

Since $1 \leqq k \leqq q-1$, equality in the above cases is impossible and thus

$$\frac{1}{q} \leqq \left| m - k\cdot\frac{p}{q} \right| < k \left| \alpha - \frac{p}{q} \right| < \frac{q-1}{q^2\sqrt{5}} < \frac{1}{q},$$

which is a contradiction.

CASE 2. $\frac{p}{q} \in H$, p odd.

In this case we have $S_{2q}(\alpha) = 0$. In order to see this we need only show that $S_{2q}(\alpha) = S_{2q}(\frac{p}{q})$.

CASE 2.1. $\frac{p}{q} < \alpha$, p odd.

It suffices to show that

$$[k\alpha] = [k\frac{p}{q}] \text{ for } 1 \leqq k \leqq 2q.$$

Since this may be established similarly as in case 1 we consider

CASE 2.2. $\frac{p}{q} > \alpha$, p odd.

We observe that

$$[q\cdot\frac{p}{q}] = p, \ [2q\frac{p}{q}] = 2p$$

and

$$[q\alpha] = p-1, \ [2q\alpha] = 2p-1$$

so that (since p is odd) it suffices to show that

$$[k\alpha] = [k\frac{p}{q}] \text{ for } 1 \leqq k \leqq 2q, \ k \neq q, \ k \neq 2q.$$

Since this may be shown similarly as before the proof is complete. □

REMARK. From the above considerations it is easily seen that
(i) if p is *even* then $S_{nq}(\frac{p}{q}) = n$ for all $n \in \mathbb{N}$.

(ii) if p is *odd* then $S_{2nq}(\frac{p}{q}) = 0$ for all $n \in \mathbb{N}$.