stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 14/73 APRIL

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS (MOS) subject classification scheme (1970): 02E10, 02F27, 06A10, 06A40

Abstract: We show an explicit algorithm that a set-algebra defined by A. Mostowski in 1938 is not only universal for countable Boolean Algebra's and partial orderings but that these embeddings are effective whenever the embedded structures are recursive.

Introduction: A. Mostowski defines in 1938 [Mo] a partial order $(M,\subseteq)$ which is universal for countable partial orderings; i.e. for every countable partial ordering (X,R) there exists an order-preserving embedding of X in M. This ordered set M is in fact an algebra of sets, ordered by inclusion.

The proof of this universality is not given in [Mo]. Mostowski refers to earlier results concerning general Boolean algebra. In modern words the argument is as follows: Under the Stone representation theorem [St] the Mostowski algebra corresponds to the Cantor space. The topological theorem that each metrizable zero dimensional compact space can be embedded into (is a continuous image of) the Cantor space yields that each countable Boolean algebra, having such a space as Stone space is a homomorphic image (subalgebra) of the Mostowski algebra.

The algebra M is easily seen to be a recursive algebra in a suitable indexing. The universality property shows that in particular all recursive partial orderings can be embedded. The classical proofs however do not yield these embeddings to be effective. In order to get effective embeddings we present a complete proof and an explicit algorithm to embed orders. The construction works correctly for initial complete set-configurations, a concept defined in this report which embraces both Boolean algebra's and partial orders as particular examples.

§1. Notations, definitions and the free structures

Notations 1 : $\mathbb{N}$ denotes the set of non negative integers ($0 \in \mathbb{N}$). For h and $k \in \mathbb{N}$ we denote the uniquely determined 0 - 1 sequence $\langle e_0, \ldots, e_k \rangle$ satisfying $\sum_{i=0}^{k-1} e_i * 2^{**}i \equiv n \mod 2^{**}k$ by $\underline{bn}(n,k)$.

The element e_i is denoted by $\underline{bn}(n,k)[i]$.

E denotes the set of 0 - 1 functions with finite domain. We have some fixed canonical indexing $\eta : \mathbb{N} \to E$. The domain of a function f is denoted by $\mathcal{D}f$.

$E_k \subset E$ is the subset of functions with domain $[0,k)$ Let $E_\infty = \bigcup_k E_k$; E_∞ has the structure of a binary tree. The binary representations $\underline{bn}(n,k)$ are considered as members of E_k also;

Two functions f and $g \in E$ are called compatible iff $f \mid \mathcal{D}f \cap \mathcal{D}g = g \mid \mathcal{D}f \cap \mathcal{D}g$, and they are called incompatible if these two restrictions are distinct. The union of two compatible functions f and g is denoted by $f \cup g$.

If $X = (X_i)_i$ is a collection of sets in some domain E then we denote $X^0 = X$ and $X^1 = E \setminus X$. For $f \in E$ we have $X(f) = \bigcap_{j \in \mathcal{D}f} X_j^{f(j)}$. $X(n,k)$ denotes the set $X(\underline{bn}(n,k))$. By convention $X(n,0) = E$.

The same conventions are used for a sequence of elements in a Boolean algebra A with operations un, ints, cpm, eqv, and incl, representing union, intersection, complementation, equality, and inclusion. The relation eqv is not assumed to be identity; it is however a congruence relation with respect to all operations. $A/\underline{eqv}$ is a (distributive) Boolean algebra in the usual sense.

For elements $x \in A$ we denote $x^0 = x$; $x^1 = \underline{cpm}\ x$. For $f \in E$ and for a sequence $x = (x_i)_i$ the element

$\underline{\text{Ints}}\{x_j^{f(j)} \mid j \in \mathcal{D}f\}$ is denoted by $\chi(f)$. Again $\chi(n,k) = \chi(\underline{\text{bn}}(n,k))$.

Representants of the zero and one class are denoted by $\emptyset$ and u.

__Proposition__ 2: For compatible f and g one has $X(f) \cap X(g) = X(f \cup g)$ $(\chi(f)\ \underline{\text{ints}}\ \chi(g) = \chi(f \cup g))$. If f and g are incompatible $X(f) \cap X(g) = \emptyset \quad (\chi(f)\ \underline{\text{ints}}\ \chi(g) = \emptyset)$

__Proof__ : trivial.

__Definition__ 3 : A Boolean algebra $A = \langle A, \underline{\text{un}}, \underline{\text{ints}}, \underline{\text{cpm}}, \underline{\text{eqv}}, \underline{\text{incl}}\rangle$ is called recursive provided A is $= \mathbb{N}$ (or A is indexed by a fixed canonical indexing) and all operations and relations __un__, __ints__, __cpm__, __eqv__ and __incl__ are total recursive.

For $A = \mathbb{N}$ (or A is canonically indexed) we denote by $A(f)$ the element $\underline{\text{Ints}}\ \{j^{f(j)} \mid j \in \mathcal{D}f\}$ in A.

__Definition__ 4 : Let A be a Boolean algebra and let $K \subset P(E)$ be a set algebra. A homomorphism $\psi : A \to K$ is called a (faithfull) representation iff for each pair x, y $\in A$ $\psi(x) = \psi(y)$ iff x __eqv__ y. (By a homomorphism we mean a mapping satisfying $\psi(x\ \underline{\text{un}}\ y) = \psi(x) \cup \psi(y)$; $\psi(x\ \underline{\text{ints}}\ y) = \psi(x) \cap \psi(y)$ and $\psi(\underline{\text{cpm}}\ x) = E \setminus \psi(x)$).

__Lemma__ 5 : A homomorphism $\psi : A \to K$ is a representation iff for each $x \in A$ $\psi(x) = \emptyset$ iff x __eqv__ $\emptyset$.

__proof__ : $\Longleftarrow$: one has x __eqv__ y iff

$(x\ \underline{\text{ints}}\ (\underline{\text{cpm}}\ y))\ \underline{\text{vn}}\ ((\underline{\text{cpm}}\ x)\ \underline{\text{ints}}\ y)\ \underline{\text{eqv}}\ \emptyset$ __iff__

$(\psi(x)\cap\psi(y)^1) \cup (\psi(x)^1\cap\psi(y)) = \emptyset$ __iff__

$\psi(x) = \psi(y)$

The other implication is trivial.

<u>Definition</u> 6 : [The free Boolean algebra on a countable sequence of independent elements]:

Define a language L (terms) by

(i) for $k \in \mathbb{N}$ $v[k] \in L$

(ii) if $t_1, t_2 \in L$ then $(t_1 \underline{\text{un}}\ t_2)$, $(t_1 \underline{\text{ints}}\ t_2)$ and $(\underline{\text{cpm}}\ t_1) \in L$

The above syntactical constructions define the operations <u>un</u>, <u>ints</u>, and <u>cpm</u> on L.

(iii) A relation <u>eqv</u> is defined on L to be the smallest congruence relation containing all pairs resulting from substitution of terms for the variables in the left and right hand side of axioms for a Boolean algebra. For example we consider the following (not minimal) axiomatization:

$$x\ \underline{\text{un}}\ y = y\ \underline{\text{un}}\ x \qquad x\ \underline{\text{ints}}\ y = y\ \underline{\text{ints}}\ x$$

$$(x\ \underline{\text{un}}\ y)\ \underline{\text{un}}\ z = x\ \underline{\text{un}}\ (y\ \underline{\text{un}}\ z) \qquad (x\ \underline{\text{ints}}\ y)\ \underline{\text{ints}}\ z = x\ \underline{\text{ints}}\ (y\ \underline{\text{ints}}\ z)$$

$$x\ \underline{\text{un}}\ x = x \qquad x\ \underline{\text{ints}}\ x = x$$

$$x\ \underline{\text{ints}}(y\ \underline{\text{un}}\ z)=(x\ \underline{\text{ints}}\ y)\underline{\text{un}}(x\ \underline{\text{ints}}\ z) \qquad x\ \underline{\text{un}}(y\ \underline{\text{ints}}\ z)=(x\ \underline{\text{un}}\ y)\underline{\text{ints}}(x\ \underline{\text{un}}\ z)$$

$$\underline{\text{cpm}}\ (\underline{\text{cpm}}\ x) = x$$

$$\underline{\text{cpm}}\ (x\ \underline{\text{un}}\ y) = (\underline{\text{cpm}}\ x)\ \underline{\text{ints}}\ (\underline{\text{cpm}}\ y) \qquad \underline{\text{cpm}}\ (x\ \underline{\text{ints}}\ y) = (\underline{\text{cpm}}\ x)\ \underline{\text{un}}\ (\underline{\text{cpm}}\ y)$$

$$x\ \underline{\text{un}}\ (y\ \underline{\text{ints}}(\underline{\text{cpm}}\ y)) = x \qquad x\ \underline{\text{ints}}\ (y\ \underline{\text{un}}(\underline{\text{cpm}}\ y)) = x$$

(iv) The relation <u>incl</u> is defined by

$t_1\ \underline{\text{incl}}\ t_2$ iff $(t_1\ \underline{\text{ints}}\ t_2)\ \underline{\text{eqv}}\ t_1$

Note that $t\ \underline{\text{un}}\ (\underline{\text{cpm}}\ t)$ $(t\ \underline{\text{ints}}(\underline{\text{cpm}}\ t))$ represent the one (zero) element in L for each $t \in L$.

The sequence $(v[i])_i$ is denoted by ν

<u>fact</u> 7 : For a suitable canonical indexing of L the free Boolean algebra defined above becomes a recursive Boolean algebra.

The recursiveness of the relation <u>eqv</u> as defined above is nothing but the decidabillity of propositional calculus.

<u>Definition</u> 8 : [The Mostowski Algebra]. Let E be some (infinite) set. Let there be defined a decomposition of E into non empty subsets $B(n,k)$ for $k \in \mathbb{N}$, $0 \le n < 2{**}k$ satisfying:

(i) $B(0,0) = E$

(ii) $B(n,k) = B(n,k+1) \cup B(n+2{**}k,k+1)$;

$$B(n,k+1) \cap B(n+2{**}k,k+1) = \emptyset$$

Let $\mathcal{B} = \{B(n,k) \mid k \in \mathbb{N},\ 0 \le n < 2{**}k\}$ and let M be the set-algebra generated by the collection $\mathcal{B}$. In fact M consists of all finite unions of members from $\mathcal{B}$.

M is called the Mostowski Algebra.

<u>proposition 9</u>: Let $C_k = \cup\, \{B(n,k) \mid \underline{bn}\ (n,k)\ [k-1] = 0\}$ and let $C = (C_k)_k$. Then for $k \in \mathbb{N}$, $0 \le n < 2{**}k$ we have $B(n,k) = C(n,k)$

<u>proof</u> : By induction on k ⊠

<u>fact</u> 10 : Define $\psi : L \to M$ by

(i) $\psi(v[k]) = C_k$

(ii) $\psi(t_1\ \underline{un}\ t_2) = \psi(t_1) \cup \psi(t_2)$

$\psi(t_1\ \underline{ints}\ t_2) = \psi(t_1) \cap \psi(t_2)$

$\psi(\underline{cpm}\ t) = E \setminus \psi(t)$

Then ψ is a representation from L in M which is faithfull. This assertion is in fact nothing but the completeness of propositional calculus.

<u>remark</u> 11 : The homomorphism ψ satisfies the relation $\psi(\upsilon(f)) = C(f)$. If there is no danger for confusion we write $\psi(f)$ in stead of $\psi(\upsilon(f))$.

Examples 12 : Two physical representations of the Mostowki Algebra are:

$$(\mathrm{I})\quad E = \mathbb{N}\ ;\ B(n,k) = \{m \mid m \equiv n \mod 2**k\}$$

$$(\mathrm{II})\quad E = \prod_{i=0}^{\infty} \{0,1\} \quad \text{(the Cantor space)} \quad \text{and} \quad C_k = \pi_k^{-1}(0)$$

Since the Cantor space is in fact the Stone space [St] of the free Boolean algebra L it is not amazing that the clopen subsets in the Cantor space represent L. Considering the natural numbers in $\mathbb{N}$ as a subset of the ring of 2-adic integers with the 2-adic topology we conclude that the first representation of M consists of the intersections of the clopen sets in a Cantor space with a countable dense subset.

§2. Set configurations and their representations

The free structures defined in the preceding section describe the configuration of a countable sequence of independent sets; sets for which no non-trivial set-theoretical relation holds. In this section we consider the situation of a non free configuration; there is given a number of relations which must be made valid.

For example we want to satisfy $X_0 \subseteq X_1$; $X_1 \cap X_2 \subseteq X_3$; $X_4 \cup X_5 \subset X_7 \cup X_8$ etc. .... It is not difficult to see that these relations can be translated into a series of relations in a normal form of the type $\bigcap_{j \in F} X_j^{f(j)} = \emptyset$ where F is the finite domain of a function $f \in \mathcal{E}$. The three relations above become after translation $X_0^0 \cap X_1^1 = \emptyset$; $X_1^0 \cap X_2^0 \cap X_3^1 = \emptyset$; $X_4^0 \cap X_7^1 \cap X_8^1 = \emptyset$; $X_5^0 \cap X_7^1 \cap X_8^1 = \emptyset$ etc. .... .

The complete configuration is described this way by a collection of 0 - 1 functions with finite domain; i.e. a subset of $\mathcal{E}$. The same relations can also be interpreted within a Boolean algebra. This explains our next definition.

<u>Definition</u> 13 : A <u>(set)-configuration</u> $\mathcal{F}$ is a subset of $\mathcal{E}$. A (set) configuration is called recursive iff $\mathcal{F}$ is a recursive subset of $\mathcal{E}$ (in some fixed canonical indexing of $\mathcal{E}$).

In general the relations in $\mathcal{F}$ will force other relations to become valid which are not necessarily contained in $\mathcal{F}$. If $X_i \cap X_j = \emptyset$ is a relation in $\mathcal{F}$ then $X_i \cap X_j \cap X_k^\varepsilon = \emptyset$ for each k and each $\varepsilon \in \{0,1\}$. Moreover if both $X_i \cap X_j \cap X_k = \emptyset$ and $X_i \cap X_j \cap X_k^1 = \emptyset$ are relations in $\mathcal{F}$ we can derive that $X_i \quad X_j = \emptyset$ also; it is inconsistent to assume that a non empty set has a empty intersection with both X_k and X_k^1.

<u>Definition</u> 14 : A (set) configuration $\mathcal{F}$ is called <u>extensive</u> iff $\mathcal{F}$ contains each extension $\mathcal{E}$ of each member of $\mathcal{F}$. $\mathcal{F}$ is called <u>consistent</u> iff each $f \in \mathcal{E}$ with the property that for some $k \notin \mathcal{D}f$ both $f \cup \{<k,0>\}$ and $f \cup \{<k,1>\}$ are members of $\mathcal{F}$ is a member of $\mathcal{F}$ itself. A configuration $\mathcal{F}$ wich is both extensive and consistent is called <u>complete</u>.

<u>Proposition</u> 15: The intersection of extensive (consistent, complete) configurations is again extensive (consistent, complete). Each configuration $\mathcal{F}$ is contained in a minimal extensive (consistent, complete) configuration which is denoted by <u>ext</u> $\mathcal{F}$ (<u>cs</u> $\mathcal{F}$, <u>cp</u> $\mathcal{F}$).

<u>proof</u> : The first assertion is trivial and implies the second by taking the intersection of all extensive (consistent, complete) extensions of $\mathcal{F}$.

The configuration <u>cs</u> $\mathcal{F}$ can be defined inductively as follows:

Let <u>cs1</u> $\mathcal{F} = \mathcal{F} \cup \{g \in \mathcal{E} \mid \exists l\ [l \notin \mathcal{D}g$ <u>and</u> $g \cup \{<l,0>\} \in \mathcal{F}$ <u>and</u>
<u>and</u> $g \cup \{<l,1>\} \in \mathcal{F}\ ]\}$

Let $\mathcal{F}_0 = \mathcal{F}$ and let $\mathcal{F}_{k+1} =$ <u>cs1</u> $\mathcal{F}_k$. Then one easily verifies that <u>cs</u> $\mathcal{F} = \bigcup_k \mathcal{F}_k$.
If $g \in$ <u>cs</u> $\mathcal{F}$ then the least k so that $g \in \mathcal{F}_k$ is called the length of a proof for $g \in$ <u>cs</u> $\mathcal{F}$.

<u>Lemma</u> 16 : The following three configurations are equal:

$\mathcal{F}_1 =$ <u>cp</u> $\mathcal{F}$
$\mathcal{F}_2 =$ <u>cs</u> <u>ext</u> $\mathcal{F}$
$\mathcal{F}_3 = \{g \mid \exists k \in \mathbb{N}\ [\mathcal{D}g \subset [0,k)$ <u>and</u> <u>ext</u> $\{g\} \cap \mathcal{E}_k \subset$ <u>ext</u> $\mathcal{F}]\}$

<u>proof</u> : $\mathcal{F}_2 \subseteq \mathcal{F}_1$ is trivial since $\mathcal{F} \subset \mathcal{F}_1$ and $\mathcal{F}_1$ is complete.
$\mathcal{F}_3 \subseteq \mathcal{F}_2$: Let $g \in \mathcal{F}_3$ and let $g \subset [0,k)$ and let <u>ext</u> $\{g\} \cap \mathcal{E}_k \subset$ <u>ext</u> $\mathcal{F}$. By induction on $\#\ ([0,k)\setminus\mathcal{D}g)$ one proves $g \in$ <u>cs</u> <u>ext</u> $\mathcal{F}$.

$\mathcal{F}_1 \subseteq \mathcal{F}_3$: Since $\mathcal{F} \subset \mathcal{F}_3$ is trivial it is sufficient to show that $\mathcal{F}_3$ is both extensive and consistent.

Let $g \in \mathcal{F}_3$ and suppose $h \in \underline{\text{ext}}\ \{g\}$.
Suppose $\mathcal{D}g \in [0,k)$ and $\underline{\text{ext}}\ \{g\} \cap \mathcal{E}_k \subset \underline{\text{ext}}\ \mathcal{F}$. Then for each $l \geq k$ we have $\underline{\text{ext}}\ \{g\} \cap \mathcal{E}_l \subset \underline{\text{ext}}\ \mathcal{F}$. If $\mathcal{D}h \subset [0,l)$ we then have $\underline{\text{ext}}\ \{h\} \cap \mathcal{E}_l \subset \underline{\text{ext}}\ \{g\} \cap \mathcal{E}_l \subset \underline{\text{ext}}\ \mathcal{F}$.
Hence $h \in \mathcal{F}_3$, and consequently $\mathcal{F}_3$ is entensive.

Next suppose that $f \in \mathcal{E}$, $n \notin \mathcal{D}f$ and both $f_0 = f \cup \{<n,0>\}$ and $f_1 = f \cup \{<n,0>\} \in \mathcal{F}_3$. For sufficiently large l one has $\mathcal{D}_f \subset \mathcal{D}_{f_0} = \mathcal{D}_{f_1} \subset [0,l)$ and
$\underline{\text{ext}}\ \{f\} \cap \mathcal{E}_l = (\underline{\text{ext}}\ \{f_0\} \cap \mathcal{E}_l) \cup (\underline{\text{ext}}\ \{f_1\} \cap \mathcal{E}_l) \subset \underline{\text{ext}}\ \mathcal{F}$
Hence $f \in \mathcal{F}_3$ which proves $\mathcal{F}_3$ to be consistent.

This completes the proof of the Lemma.

Remark 17 : One has not generally $\underline{\text{cp}}\ \mathcal{F} = \underline{\text{ext}}\ \underline{\text{cs}}\ \mathcal{F}$.

Remark 18 : We do not exclude the case that $\underline{\text{cp}}\ \mathcal{F} = \mathcal{E}$; i.e. the empty function ε is a member of $\underline{\text{cp}}\ \mathcal{F}$. A configuration $\mathcal{F}$ such that $\underline{\text{cp}}\ \mathcal{F} = \mathcal{E}$ is called paradoxical.

Proposition 19 : Let A be a (recursive) Boolean algebra. Then the collection $\mathcal{F} = \{f \in \mathcal{E} \mid A(f)\ \underline{\text{eqv}}\ \emptyset\}$ is a (recursive) complete configuration. $\mathcal{F}$ is non paradoxical unless $\underline{\text{eqv}}$ is the universal relation $\mathbb{N} \times \mathbb{N}$.

proof : trivial.

Prop.19 shows that a Boolean algebra determines a configuration. We now wants to reverse this relation:
given a configuration $\mathcal{F}$ we want to construct a Boolean algebra or a set-algebra satisfying the relations of $\mathcal{F}$ but no relations which are not derivable from $\mathcal{F}$.

In this report we consider three constructions.

The first construction which can be called a syntactical construction yields a quotient algebra of the free algebra L. We extend eqv to the smallest congruence relation containing both eqv and all pairs $\langle \nu(f), \emptyset \rangle$ for $f \in F$.
The resulting algebra is denoted

$L_F = \langle L, \underline{un}, \underline{ints}, \underline{cpm}, \underline{eqv}_F, \underline{incl}_F \rangle$.

The second construction starts with a Mostowski algebra on some domain E. We delete from D each set $C(f)$ for $f \in F$. The resulting set $F \subset E$ $F = E \setminus \cup \{C(f) \mid f \in F\}$ determines a set algebra M_F consisting of the sets $A \cap F$ for $A \in M$. This construction can be called an excision construction. The mapping $A \to A \cap F$ clearly is a homomorphism from M onto M_F.

Let ψ be the mapping $\psi : L \to M$ defined in fact 10.
We define $\psi_F : L \to M_F$ by $\psi_F(\nu(f)) = C(f) \cap F$. It is easy to see that ψ_F is in fact a homomorphism from L_F onto M_F; see our next proposition.
If there is no danger for confusion we write $\psi_F(f)$ in stead of $\psi_F(\nu(f))$.

The third construction which is described in section 4 yields a representation of L_F in the Mostowski algebra M by a suitable exchange between the sets C_k and their complements. In this way we make the set $C(f)$ for $f \in F$ vanish without deleting material from E.

This construction is correct if the configuration F satisfies certain conditions which are introduced in the next section.

Proposition 20 : Let the Mostowski algebra M consist of clopen subsets of a compact space (the Cantor space for example).

Then the following assertions are equivalent:

(i) $f \in \underline{cp}\ F$

(ii) $\nu(f)\ \underline{eqv}_F\ \emptyset$

(iii) $\psi_F(f) = \emptyset$

<u>proof</u> :

(i) $\rightarrow$ (ii): If $f \in \mathcal{F}$ then $\nu(f)$ $\underline{eqv}_{\mathcal{F}}$ $\emptyset$ by definition. If $g \in \underline{ext}\{g\}$ then $\nu(g)$ $\underline{inc}$ $\nu(f)$ $\underline{eqv}_{\mathcal{F}}$ $\emptyset$ and hence $\nu(g)$ $\underline{eqv}_{\mathcal{F}}$ $\emptyset$. Finally suppose $g \in \underline{cp}\ \mathcal{F}$. Then there exists a $k \in \mathbb{N}$ so that $\mathcal{D}g \subset [0,k)$ and $\underline{ext}\ \{g\} \cap \mathcal{E}_k \subset \underline{ext}\ \mathcal{F}$. Then $\nu(g)$ $\underline{eqv}$ $\underline{Un}$ $\{\nu(h) \mid h \in \underline{ext}\ \{g\} \cap \mathcal{E}_k\}$.
Consequently $\nu(g)$ $\underline{eqv}_{\mathcal{F}}$ $\underline{Un}$ $\{\emptyset \mid h \in \underline{ext}\ \{g\} \cap \mathcal{E}_k\}$ $\underline{eqv}$ $\emptyset$.

(ii) $\rightarrow$ (iii): Let $\underline{eqv}_F$ be the "kernel" of $\psi_{\mathcal{F}}$:

t_1 $\underline{eqv}_F$ t_2 $\underline{iff}$ $\psi_{\mathcal{F}}(t_1) = \psi_{\mathcal{F}}(t_2)$. Then $\underline{eqv}_F$ is a congruence relation on $\mathcal{L}$ which contains both $\underline{eqv}$ and all pairs $<\nu(f), \emptyset>$ for $f \in \mathcal{F}$ since $\psi_{\mathcal{F}}(f) = \mathcal{C}(f) \cap F \subseteq \mathcal{C}(f) \setminus \mathcal{C}(f) =$ $= \emptyset = \psi\ (\emptyset)$.
Consequently $\underline{eqv}_{\mathcal{F}} \subseteq \underline{eqv}_F$ which proves the implication.

(iii) $\rightarrow$ (i): Suppose $\psi_{\mathcal{F}}(f) = \emptyset$. Using <u>Lemma</u> 16 we show $f \in \underline{cp}\ \mathcal{F}$.

$$\psi_{\mathcal{F}}(f) = \psi(f) \cap F = \psi(f) \setminus \bigcup_{g \in \mathcal{F}} \mathcal{C}(g) = \emptyset.$$

Each $\mathcal{C}(g)$ is an open set and $\psi(f)$ is compact; consequently there exists a finite union $\mathcal{C}(g_1) \cup \ldots \cup \mathcal{C}(g_k)$ wich contains $\psi(f) = \mathcal{C}(f)$; $g_i \in \mathcal{F}$.

Select an integer l so that $\mathcal{D}f \subset [0,l)$ and so that $\mathcal{D}g_i \subset [0,l)$ for $i = 1, \ldots, k$. Using $\mathcal{C}(f)$: $\cup\ \{\mathcal{C}(h) \mid h \in \underline{ext}\ \{f\} \cap \mathcal{E}_l\}$ and similar relations for the $\mathcal{C}(g_i)$ we conclude

$$\cup\ \{\mathcal{C}(h) \mid h \in \underline{ext}\ \{f\} \cap \mathcal{E}_l\} \subseteq \cup\ \{\mathcal{C}(h) \mid h \in \underline{ext}\{g_i,\ldots,g_k\} \cap \mathcal{E}_l\}$$

Now the sets $\mathcal{C}(h)$ for $h \in \mathcal{E}_l$ are either equal or disjoint. The above inclusion implies therefore

$$\underline{ext}\ \{f\} \cap \mathcal{E}_l \subseteq \underline{ext}\ \{g_1,\ldots,g_k\} \cap \mathcal{E}_l \subseteq \underline{ext}\ \mathcal{F} \cap \mathcal{E}_l.$$

Hence $f \in \underline{cp}\ \mathcal{F}$ by <u>lemma</u> 16.

<u>Remark</u> 21 : The compactness of the elements of the Mostowski algebra is necessary in the above proposition. Taking repre-

sentation (I) of example 12 we can construct an example where (iii) $\rightarrow$(i) does not hold.

Let f_k be defined by $f_k(x)$ = if $x < k$ then 1 else if $x = k$ then 0 else ∞ and let $\mathcal{F} = (f_k)_k$. The intended meaning of this configuration reads:

$$X_0 = \emptyset \ ; \quad X_1 \subseteq X_0 \ ; \quad X_2 \subseteq X_1 \cup X_0 \ ; \ \ldots \text{ etc.}$$

Hence for each k $X_k = \emptyset$.

If we apply the excision construction for this configuration we find

$$F = E \setminus \bigcup_{k\in\mathbb{N}} C(f_k) = E \setminus \bigcup_{k\in\mathbb{N}} B(2{**}k-1,k+1).$$

In representation (I) where $E = \mathbb{N}$ this is the set

$F = \{z \mid \forall_{k>1} [\underline{bn}(z,k)[1] = 1]\}$. This set however is empty. Consequently $M_{\mathcal{F}}$ is trivial whereas $X_0^1 = \emptyset$ is not derivable from the relations in $\mathcal{F}$.

To complete this section we consider the case that $\mathcal{F}$ is recursive. In this situation ext $\mathcal{F}$ is also recursive but cp $\mathcal{F}$ may fail to be recursive. This can be seen from our next example:

Example 22 : [A recursive configuration with a non recursive completion]. Define the functions $f_{n,k}$ and $g_{n,k}$ by:

$$\mathcal{D}f_{n,k} = \mathcal{D}g_{n,k} = \{\langle j,k\rangle \mid 0 \le j \le n\}$$

$$f_{n,k}(\langle j,k\rangle) = \underline{if}\ j < n\ \underline{then}\ 0\ \underline{else}\ \underline{if}\ j = n\ \underline{then}\ 1\ \underline{else}\ \infty.$$

$$g_{n,k}(\langle j,k\rangle) = \underline{if}\ j \le n\ \underline{then}\ 0\ \underline{else}\ \infty$$

The intended meanings of $f_{n,k}$ and $g_{n,k}$ are

$$X_{\langle 0,k\rangle} \cap \ldots \cap X_{\langle n-1,k\rangle} \subseteq X_{\langle n,k\rangle} \text{ c.q.}$$

$$X_{\langle 0,k\rangle} \cap \ldots \cap X_{\langle n-1,k\rangle} \cap X_{\langle n,k\rangle} = \emptyset .$$

Consequently $g_{n,k-1} \in \underline{cs}\ \{f_{n,k}, g_{n,k}\}$.

We now define a configuration $\mathcal{F}$ $\mathcal{F} = \cup \{\mathcal{F}_{n,k} | k\in\mathbb{N},\ n\geq 1\}$ where $\mathcal{F}_{n,k}$ is defined by:

$$\mathcal{F}_{n,k} = \begin{cases} \{f_{n,k}\} & \underline{\text{iff}} \quad \Phi_k(k) > n-1 \qquad (*) \\ \{f_{n,k}, g_{n,k}\} & \underline{\text{iff}} \quad \Phi_k(k) = n-1 \\ \emptyset & \text{otherwise} \end{cases}$$

It is clear that $\mathcal{F}$ is recursive. However one easily proves that $g_{0,k} \in \underline{cp}\ \mathcal{F}$ (i.e. $X_{<0,k>} = \emptyset$) iff $\Phi_k(k) < \infty$. This reduces the halting problem to the set $\underline{cp}\ \mathcal{F}$ and consequently $\underline{cp}\ \mathcal{F}$ is not recursive.

This example illustrates a fact which is known in propositional calculus: it is possible to construct recursive systems of axioms which generate an undecidable theory.

(*) Φ_k denotes the runtime of the program ϕ_k in an effective Gödelnumbering of all computable functions: $\mathcal{D}\Phi_k = \mathcal{D}\phi_k$ and $\Phi_k(x) = y$ is decidable. Readers unfamiliar with the concept of a Complexity measure [Bl] can replace the condition $\Phi_k(k) = n - 1$ by $T(k,k,n-1)$ where T is the Kleene predicate.

§3 Initial configurations

Definition 23 : A configuration $\mathcal{F}$ is called *initial* if $\mathcal{F} \subset \mathcal{E}_\infty$. For $f \in \mathcal{E}$ we denote the integer $\mu z[\mathcal{D}f \subset [o,z)]$ by $\underline{\text{len}}\ f$ and we let $\underline{\text{init}}\ \{f\} = \underline{\text{ext}}\ \{f\} \cap \mathcal{E}_{\underline{\text{len}}\ f}$. $\underline{\text{len}}\ f$ is called the *length* of f.

The initial configuration $\underline{\text{init}}\ \mathcal{F} = \cup\ \{\underline{\text{init}}\{f\} \mid f \in \mathcal{F}\}$ is called the initiation of $\mathcal{F}$.

Remark 24 : The initiation of $\mathcal{F}$ contains in fact all information of $\mathcal{F}$; in particular one "reconstructs" $\mathcal{F}$ from $\underline{\text{init}}\ \mathcal{F}$ since $\mathcal{F} \subset \underline{\text{cs}}\ \underline{\text{init}}\ \mathcal{F}$.

Notations 25 : Let $f \in \mathcal{E}_k$. Then $\uparrow f$ denotes the function $f \cup \{\langle k,0\rangle\}$ and $\downarrow f$ denotes the function $f \cup \{\langle k,1\rangle\}$. $\uparrow f$ and $\downarrow f$ are the two extensions of f in $\mathcal{E}_{k+1}$. Note that $C(f) = C(\uparrow f) \cup C(\downarrow f)$.

Definition 26 : A configuration $\mathcal{F}$ is called *initial extensive* if $\underline{\text{ext}}\ \mathcal{F} \cap \mathcal{E}_\infty \subset \mathcal{F}$. An initial configuration $\mathcal{F}$ is called *initial consistent* if each $f \in \mathcal{E}_\infty$ with the property that both $\uparrow f$ and $\downarrow f$ are members of $\mathcal{F}$ is a member of $\mathcal{F}$ itself. An arbitrary configuration $\mathcal{F}$ is called initial consistent if its initiation is initial consistent. A configuration which is both initial consistent and initial extensive is called *initial complete*.

Notations 27 : For (initial) $\mathcal{F}$ $\underline{\text{iext}}\ \mathcal{F}$ ($\underline{\text{ics}}\ \mathcal{F}$, $\underline{\text{icp}}\ \mathcal{F}$) denotes the smallest initial extensive (initial consistent, initial complete) extension of $\mathcal{F}$. ($\underline{\text{ics}}\ \mathcal{F}$ is only defined for initial $\mathcal{F}$)

Lemma 28 : Let $\mathcal{F}$ be initial. Then we have:

(i) $\underline{\text{iext}}\ \mathcal{F} = \underline{\text{ext}}\ \mathcal{F} \cap \mathcal{E}_\infty$

(ii) $\underline{\text{ics}}\ \mathcal{F} = \underline{\text{cs}}\ \mathcal{F} \cap \mathcal{E}_\infty$

(iii) $\underline{\text{icp}}\ \mathcal{F} = \underline{\text{cp}}\ \mathcal{F} \cap \mathcal{E}_\infty$

(iv) $\underline{\text{icp}}\ \mathcal{F} = \underline{\text{ics}}\ \underline{\text{iext}}\ \mathcal{F} = \underline{\text{iext}}\ \underline{\text{icp}}\ \mathcal{F}$

<u>proof</u> :

(i) : is trivial

(ii) : The inclusion <u>ics</u> $F \subset$ <u>cs</u> $F \cap E_\infty$ is trivial, since <u>cs</u> $F \cap E_\infty$ is initial consistent. The converse inclusion is proved by induction on the length of a proof for $f \in$ <u>cs</u> F.

<u>Induction assumption</u>: If $f \in$ <u>cs</u> F by a proof of length $\leq k$ then <u>init</u> $\{f\} \in$ <u>ics</u> F.

<u>Base for induction argument</u>: If $f \in F$ then $f \in$ <u>ics</u> F since F is initial. This covers length 0.

<u>Induction step</u>: Let $f \in$ <u>cs</u> F by a proof of length $k + 1$. Then there exists a $l \notin f$ so that both $f_0 = f \cup \{<l,0>\}$ and $f_1 = f \cup \{<l,1>\}$ are members of <u>cs</u> F by a proof of length $\leq k$. Consequently <u>init</u> $\{f_0\} \cup$ <u>init</u> $\{f_1\} \subset$ <u>ics</u> F.

Let $n =$ <u>len</u> f. If $l < n$ then <u>init</u> $\{f\} =$ <u>init</u> $\{f_0\} \cup$ <u>init</u> $\{f_1\}$ and consequently <u>init</u> $f \subset$ <u>ics</u> F. If however $n \leq l$ one concludes only that <u>ext</u> $\{f\} \cap E_{l+1} \subset$ <u>ics</u> F.
From this one derives by induction on $l - n$ that <u>init</u> $\{f\} \subset$ <u>ics</u> F.

The induction assumption yields the inclusion <u>cs</u> $F \cap E_\infty \subset$ <u>ics</u> F as a straight forward corollary.

(iii) : Since <u>cp</u> $F \cap E_\infty$ is initial complete the inclusion <u>icp</u> $F \subseteq$ <u>cp</u> $F \cap E_\infty$ is trivial. The converse inclusion is proved by:

<u>cp</u> $F \cap E_\infty =$ <u>cs</u> <u>ext</u> $F \cap E_\infty =$ <u>ics</u> (<u>ext</u> $F \cap E_\infty$) = <u>ics</u> <u>iext</u> $F \subset$ <u>icp</u> F.

(iv) : The relation <u>ics</u> <u>iext</u> $F =$ <u>icp</u> F is proved in (iii). Now the initial extension of an initial consistent configuration F is initial consistent: Suppose $f \in E_\infty$; $\uparrow f$ and $\downarrow f \in$ <u>iext</u> F then either $\uparrow f$ and $\downarrow f \in F$ or $f \in F$ and in both cases we are done. But from this one derives
<u>icp</u> $F =$ <u>ics</u> <u>iext</u> $F \subseteq$ <u>ics</u> <u>iext</u> <u>ics</u> $F =$ <u>iext</u> <u>ics</u> $F \subseteq$ <u>icp</u> F.
This completes the proof.

§4 Effective embedding of configurations in the Mostowski algebra.

In section 2 we constructed a representation $\psi_{\mathcal{F}}$ of a configuration $\mathcal{F}$ by excision of a set F in the domain E of the Mostowski algebra $\mathcal{M}$. In order to yield a faithful representation we had to assume that E was compact.

In the present section we define a representation which is based on a suitable "trade off" between the sets C_k and their complements. The trade off is defined inductively. The assumption that E is compact becomes superfluous but we must assume that F is non paradoxical and initial complete.

Theorem 29 : Let $\mathcal{F}$ be an (initial) complete non paradoxical configuration. Then there exists a sequence of sets $\mathcal{D} = (D_k)_k$ with the following properties:

(i) D_k is a finite union of sets $B(n,k+1)$

(ii) $\mathcal{D}(n,k) = \emptyset$ iff $\underline{bn}(n,k) \in \mathcal{F}$.

proof : For k = 0 we take

$$D_0 = \begin{cases} \emptyset & \text{iff } \underline{bn}(0,1) \in \mathcal{F} \\ E & \text{iff } \underline{bn}(1,1) \in \mathcal{F} \\ B(0,1) & \text{otherwise.} \end{cases}$$

The case that both $\underline{bn}(0,1)$ and $\underline{bn}(1,1)$ are contained in $\mathcal{F}$ is excluded since $\mathcal{F}$ is non paradoxical.

Assume that D_m is constructed for $0 \le m < k$, satisfying (i) and (ii). Consider a set $B(n,k)$; $n < 2{**}k$. Since each "piece" $B(n,k)$ is entirely contained in $B(n,j)$ for $j < k$ and since D_j consists of entire pieces D_j we find a unique sequence $\langle e_0, e_1, \ldots, e_{k-1}\rangle$ $e_i = 0,1$ so that $B(n,k) \subset \mathcal{D}_j^{e_j}$.

Let $n' = \sum_{j=0}^{k-1} e_j * 2{**}j$ then $\langle e_0, \ldots, e_{k-1}\rangle = \underline{bn}(n',k) \notin \mathcal{F}$.

Since F is initial complete this implies that not both $\underline{bn}(n',k+1) = \langle e_0,\dots,e_{k+1},0\rangle$ and $\underline{bn}(n'+2^{**}k,k+1) = \langle e_0,\dots,e_{k-1},1\rangle$ are members of F.

Hence we can safely issue the following instructions:

if $\underline{bn}(n',k+1) \in F$ then $B(n,k) \subset D_k^1$

if $\underline{bn}(n'+2^{**}k,k+1) \in F$ then $B(n,k) \subset D_k$

otherwise $B(n,k+1) \subset D_k$ and $B(n+2^{**}k,k+1) \subset D_k^1$.

The above procedure tells us how to distribute the pieces $B(n,k+1)$ over D_k and D_k^1. This yields a definition of D_k satisfying (i).

To verify condition (ii) we consider the following

Induction hypothesis:

if $\underline{bn}(n,k) \notin F$ then $B(n,k) \subset \mathcal{D}(n,k)$ and

if $\underline{bn}(n,k) \in F$ then $\mathcal{D}(n,k) = \emptyset$.

This induction hypothesis is easily seen to be valid for $k = 1$.

Induction step: Suppose first that $\underline{bn}(n,k+1) \notin F$. Then also $\underline{bn}(n,k) \notin F$ and consequently $B(n,k) \subset \mathcal{D}(n,k)$. One easily checks from the procedure described above that $B(n,k+1) \subset \mathcal{D}(n,k+1)$, and consequently $\mathcal{D}(n,k+1) \neq \emptyset$. If $\underline{bn}(n,k+1) \in F$ and $\underline{bn}(n,k) \in F$, then $\mathcal{D}(n,k) = \emptyset$; however $D(n,k+1) \subseteq \mathcal{D}(n,k)$ and consequently $\mathcal{D}(n,k+1) = \emptyset$ also.

Finally suppose $\underline{bn}(n',k+1) \in F$ and $\underline{bn}(n',k) \notin F$.

Let $n < 2^{**}k$ be an integer so that $B(n,k) \subset \mathcal{D}(n',k)$.

Let $e_k = \underline{bn}(n',k+1)\ [k]$. From the procedure defined above we conclude that $B(n,k)$ is inserted entirely in $D_k^{1-e_k}$ and consequently $B(n,k) \cap \mathcal{D}(n',k+1) = \emptyset$. Since this holds for each piece $B(n,k)$ contained in $\mathcal{D}(n',k)$

we conclude that $(n',k+1) = \emptyset$. This completes the proof. The reader should note that for functions $f \notin E_\infty$ one has $\mathcal{D}(f) = \emptyset$ iff $\cup \{\mathcal{D}(h) \mid h \in \underline{\text{init}}\,\{f\}\} = \emptyset$ iff $\underline{\text{init}}\,\{f\} \in F$ iff $f \in \underline{\text{cp}}\, F$.

The test whether a piece $B(n,k+1)$ belongs to D_k or not consits in the description of the above procedure of two parts: first we determine the piece $\mathcal{D}(n',k)$ to which $B(n,k)$ belongs; next one determines how the pieces $B(n,k)$ are distributed over D_k and its complement.

These two tests are combined into a single algorithm which is described by the following program.

Algorithm 30:

```
comment   The procedure binrep (n,k,e) stores the binary representation
          bn(n,k) in the array e. The procedure condition (k,e) yields an
          answer to the question "is the sequence e[0 : k - 1] a member of F".
          belongs to (n,k,condition) delivers an answer whether B(n,k+1) is
          contained in Dk or in Dk1;

procedure binrep(n,k,e); value n,k; integer array e; integer n,k;

begin     integer i, m;
          for m := n while m < 0 do n := n + 2**k;
          for i := 0 step 1 until k - 1 do
          begin m := n ÷ 2; e[i] := n - 2*m; n := m end

end       binrep;

Boolean procedure belongs to (n,k,condition); value n, k; k

          integer n,k; Boolean procedure condition;

begin     integer array e[0 : k]; integer i;
          binrep (n,k+1,e);
          for i := 0 step 1 until k do
          if condition(i,e) then e[i] := 1 - e[i];
          belongs to := e[k] = 0

end       belongs to;
```

To convince himself of the correctness of the above program the reader should note that the element e_j = bn(n,k+1)[j] represents the "a priori" destination of B(n,k+1) with respect to D_j. In the preceding stages of the construction some of these a priori destinations e_j are disregarded because of conditions in $\mathcal{F}$; this fact is accounted for by inverting the corresponding element e_j. The assumption that $\mathcal{F}$ is initial complete quarantees that not both destinations $e_j = 0$ and $e_j = 1$ are forbidden by conditions in $\mathcal{F}$; otherwise e[0 : j - 1] was already a condition in $\mathcal{F}$ and e_{j-1} should have been inverted before. Since $\mathcal{F}$ is assumed to be non paradoxical this argument is founded.

It is clear now that belongs to delivers the "a posteriori" destination of B(n,k+1) with respect to Dk.

For non paradoxical initial complete $\mathcal{F}$ the above construction yields a faithfull representation of $L_{\mathcal{F}}$ in the Mostowski algebra. Define a homomorphism $\chi_{\mathcal{F}} : L_{\mathcal{F}} \to M$ by $\chi_{\mathcal{F}}(v(f)) = \mathcal{D}(f)$. Then for $f \in E_\infty$ one has $\chi_{\mathcal{F}}(\nu(f)) = = \mathcal{D}(f) = \emptyset$ <u>iff</u> $f \in \mathcal{F}$; from which one concludes for general $f \in E$: $\chi_{\mathcal{F}}(\nu(f)) = \emptyset$ <u>iff</u> $f \in$ <u>cp</u> $\mathcal{F}$.

Moreover algorithm 30 is recursive with respect to a program for the procedure "condition", i.e. with respect to $\mathcal{F}$. This yields the following corollary:

<u>Corollary</u> 31 : Each recursive Boolean algebra can effectively be represented in the Mostowski algebra.

§5 Effective embedding of orders in M.

Let R be a partial order on $\mathbb{N}$ (or some other countable set with an indexing). Replacing the elements $i \in \mathcal{D}R$ by the sets $L_i := \{j \mid j\ R\ i\}$ the order is translated into a sequence of sets ordered by inclusion. The pairs $\langle i,j\rangle \in R$ become conditions of the type $L_i \subseteq L_j$.

There exists many configurations satisfying these inclusions. For R-incomparable elements i and j we do not know whether $L_i \cap L_j = \emptyset$ unless an element k can be found which precedes both i and j in the ordering. It is not difficult to construct an example of a recursive partial order R where it is undecidable whether a pair $\langle i,j\rangle$ of uncomparable elements has a common R-lower bound. See example 32.

This implies that we should not try to represent the configuration of the sets L_i in M since by representing the order R as a subcollection of M ordered by inclusion the emptiness of intersections becomes decidable. To represent the order R this way we complete the configuration generated by the inclusions: intersections are not empty unless we are forced to take them empty. In the situation of two incomparable elements i and j we will take two representing sets X_i and X_j with a not empty intersection ($X_i \cap X_j = \emptyset$ says nothing about the presence of an other set $X_k \subset X_i \cap X_j$).

The Key lemma which allows us to apply the completion process described in §2 and §3 is the fact that the initial extension of the configuration consisting of the inclusion pairs corresponding to R is initial consistent.

Since the initial extension of a recursive configuration is again recursive this yields an effective representation by theorem 29.

Example 32 : [A recursive partial order with an undecidable common lower-bound problem].

The example is defined by disconnecting a recursive subset of elements in a product-ordering making them uncomparable with every element except themselves.

Let R_1 on $\mathbb{N}$ be defined by $x\ R_1\ y$ iff $x \geq y$ (infinite descending chain).

Let R_2 on $\mathbb{N}$ be defined by $x\ R_1\ y$ iff $y = 0$. (0 is the common upperbound of a countable sequence of mutually incomparable elements).

Consider the product ordering $R_1 \times R_2$ on $\mathbb{N} \times \mathbb{N}$.

Let $A \subset \mathbb{N} \times \mathbb{N}$ be the set $A = \{<n,k> \mid n > 0 \text{ and } \Phi_k(k) \geq n\}$

Define $R = (R_1 \times R_2 \mid (\mathbb{N} \times \mathbb{N}) \setminus A) \cup \{<a,a> \mid a \in A\}$.

Hence writing $\geq$ for R we have $\qquad (k,l \neq 0)$

(i) $<0,0> \geq <k,0>$; $<0,0> \geq <0,l>$; $<k,0> \geq <k+1,0>$

(ii) $<k_1,i>$ and $<k_2,i>$ are $\leq$ incomparable for $k_1,k_2 \neq 0$ $i \in \mathbb{N}$ $k_1 \neq k_2$.

(iii) $<k,l>$ is $\leq$ - incomparable with every other element if $<k,l> \in A$

(iv) if $<k,l> \notin A$ then $<k,l> \geq <k+1,l>$; $<k,l> \leq <0,l>$;

$<k,l> \leq <k,0>$.

One now has the following equivalence:

"$<1,0>$ and $<0,l>$ have a common lowerbound" <u>iff</u> $\Phi_l(l) < \infty$.

Hence the common lower bound problem is undecidable.

<u>Notations</u> 33: For i,j $\mathbb{N}$, $i \neq j$ the function $f_{i,j}$ is defined by $f_{i,j}(x) =$ if $x = i$ <u>then</u> 0 <u>else</u> <u>if</u> $x = j$ <u>then</u> 1 <u>else</u> ∞.

For a partial order R on $\mathbb{N}$ we let $\mathcal{R} = \{f_{i,j} \mid i \; R \; j\}$.

$\mathcal{R}$ is called the inclusion configuration corresponding to R.

<u>Lemma</u> 34 : If R is a partial order on $\mathbb{N}$ and if $\mathcal{R}$ is the inclusion configuration corresponding to R, then <u>iext</u> $\mathcal{R}$ is initial consistent.

<u>proof</u> : Let $f \in E_k$ and suppose that both $\uparrow f$ and $\downarrow f$ are members of <u>iext</u> $\mathcal{R}$. Then there exist integers $i \neq j$ and $n \neq m$ so that $f_{i,j}$ and $f_{n,m} \in \mathcal{R}$ and so that $\uparrow f$ extends $f_{i,j}$ and $\downarrow f$ extends $f_{n,m}$.

There are two possibilities:

(i) : $k \notin \{i,j\} \cap \{n,m\}$.

In this case f is already an extension of $f_{i,j}$ or $f_{n,m}$ and consequently $f \in$ iext $\mathcal{R}$.

(ii) : $k \in \{i,j\} \cap \{n,m\}$.

In this case we have $k = i = m$. Moreover

$f(j) = \uparrow f(j) = f_{i,j}(j) = 1$ and

$f(n) = \downarrow f(n) = f_{n,m}(j) = 0$.

Since $\langle i,j\rangle \in R$ and $\langle n,m\rangle \in R$ we conclude $n\ R\ k\ R\ j$ and hence $n\ R\ j$. Therefore $f_{n,j} \in \mathcal{R}$.

Now f extends $f_{n,j}$ and therefore $f \in$ iext $\mathcal{R}$.

Remark 35 : The configuration cp $\mathcal{R}$ contains no functions $f_{i,j}$ which are not already contained in $\mathcal{R}$. This can be seen as follows. Consider the trivial representation of the order R by the sets L_i. For i and j which are R incomparable we have $L_i \not\subseteq L_j$ and $L_j \not\subseteq L_i$. Since the configuration of the sets $(L_i)_i$ satisfies all conditions in $\mathcal{R}$ but neither $f_{i,j}$ nor $f_{j,i}$ we conclude that $f_{i,j}, f_{j,i} \notin$ cp $\mathcal{R}$.

Corollary 36 : Each recursive partial order can be embedded effectively in $\mathcal{M}$; this embedding is uniform in R.

proof : Take $\mathcal{F}$ = iext $\mathcal{R}$ in theorem 29. Then we have

$$
\begin{aligned}
D_i \subseteq D_j \quad & \text{iff } \mathcal{D}(f_{i,j}) = \emptyset \\
& \text{iff } f_{i,j} \in \text{cp } \mathcal{R} \\
& \text{iff } f_{i,j} \in \mathcal{R} \quad (\text{by remark } 35) \\
& \text{iff } \langle i,j\rangle \in R.
\end{aligned}
$$

Hence the $(D_i)_i$ represent the order R in $\mathcal{M}$.

Since the embedding procedure is uniform in $\mathcal{F}$ = iext $\mathcal{R}$ which is itself uniform in R the embedding procedure is uniform in R.

We complete this section by an algorithm specially designed to represent partial orders.

Algorithm 37:

```
comment     belongs to 1(n,k,order) decides whether B(n,k+1) belongs to
            Dk or not. The configuration represented is the inclusion con-
            figuration corresponding to the order which is computed by the
            Boolean procedure order (i,j);

Boolean procedure belongs to 1 (n,k,order); value n, k;
            integer n, k; Boolean procedure order;

begin       integer array e[0 : k];
            integer i, j, ei ;
            Boolean no condition;
            binrep(n,k+1,e);
            for i := 1 step 1 until k do
            begin no condition := true; ei := e[i];
                 for j := 0, j + 1 while j < i and no condition do
                 if     if e[j] = ei then false
                   else if ei = 0 then order (i,j) else order (j,i)
                 then begin e[i]:= 1 - ei; no condition := false end.
            end for i;
            belongs to 1 := e[k] = 0;

end         belongs to 1;
```

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