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On the range of functions of bounded variation

by

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On the range of functions of bounded variation.

This paper contains two theorems which give necessary and sufficient conditions on a set W of real numbers which guarantee that there exists a function of bounded variation defined on the closed unit interval which has precisely W as its range.

The first theorem deals with the family of all functions of bounded variation; in the second theorem we admit only one to one functions of bounded variation. We restrict ourselves to functions with real domain and range. Without loss of generality we can assume that 0 is the greatest lower bound and 1 is the least upperbound of the range (except for the trivial case in which the range consists of one point).

Throughout this paper we denote the total variation of a function f by V(f) and the complement of a set A with respect to the set of all real numbers by A^c.

The problems which lead to this paper were posed by Dr. M.A. Maurice.

<u>Theorem 1</u>. A bounded set W of real numbers is the range of some function f of bounded variation defined on $[0,1]$ if and only if W^c has at most a countable number of components.

Proof. a). Sufficiency. Suppose that W is a bounded set of real numbers such that W^c has at most a countable number of components. We assume that $\sup_{y\in W} y = 1$ and $\inf_{y\in W} y = 0$ (if this is not the case then either W consists of one point or there exists a linear transformation of $\mathbb{R}$ onto $\mathbb{R}$ which maps W onto a set with the required properties).
In order to prove that W is the range of some function f of bounded variation, we construct a convergent sequence of functions such that the limit function satisfies the required properties.

Let $\{C_i | i \in \mathbb{N}\}$ be an enumeration of the components of W^c, such that C_1 is not bounded below and C_2 is not bounded above. Choose an $\varepsilon > 0$ and define $p_1 = 0$. For every natural number $i \geq 2$ we define $p_i = \inf_{y \in C_i} y$, and we choose a point q_i for every $i \in \mathbb{N}$ such that $q_i \in W$ and $|p_i - q_i|^i < \frac{\varepsilon}{2^i}$. Let the function f_0 be the identity mapping from the unit interval of the domain onto the unit interval in the range space.

Define $f_i(x)$ for every $i \in \mathbb{N}$ by $\begin{cases} f_i(x) = f_0(x) \text{ iff } x \notin \bigcup_{k \leq i} f_0^{-1}(C_k) \\ f_i(x) = q_k \text{ iff } x \in f_0^{-1}(C_k); k \leq i \end{cases}$

and a function $f(x)$ by

$$\begin{cases} f(x) = f_0(x) \text{ iff } x \in f_0^{-1}(W) \\ f(x) = q_k \text{ iff } x \in f_0^{-1}(C_k). \end{cases}$$

Cleary $f(x)$ is the limit function of $\{f_i(x)\}_{i=1}^{\infty}$, and W is precisely the range of $f(x)$ since $q_i \in W$ for every i.

Furthermore it is obvious that $V(f_{i+1}) \leq V(f_i) + 2|p_{i+1} - q_{i+1}|$ for every non-negative integer i. This means that $V(f_{i+1}) \leq V(f_i) + \frac{\varepsilon}{2^i}$ for every i. Since $V(f_0) = 1$ we obtain that $V(f_i) \leq 1 + 2.\varepsilon(1 - \frac{1}{2^i}) < 1 + 2\varepsilon$ for every i. We now prove that $V(f) \leq 1 + 2\varepsilon$. Let $\{a_v\}_{v=1}^{n}$ be an arbitrary subdivision of the interval. Then there exists some i such that $a_v \notin \bigcup_{k>i} C_k$ for all $v = 1, \ldots, n$. Hence $f(a_v) = f_i(a_v)$ and

$$\sum_{v=1}^{n} |f(a_v) - f(a_{v-1})| = \sum_{v=1}^{n} |f_i(a_v) - f_i(a_{v-1})| \leq V(f_i) \leq 1 + 2\varepsilon.$$

We conclude that $V(f) \leq 1 + 2\varepsilon$. This proves the sufficiency.

b). Necessity. We suppose that f is a function of bounded variation defined on $[0,1]$ with range W. We restrict ourselves again to cases in which $\sup_{y \in W} y = 1$ and $\inf_{y \in W} y = 0$. Suppose that W^c has an uncountable number of components, then W^c has an uncountable number of components which consist of one single point. Let ζ be a point-component of W^c, then there exists a sequence $\{\xi_i\}$ in W which converges to ζ. We choose a sequence $\{x_i\}$ in $[0,1]$ with the property that $f(x_i) = \xi_i$. Since $[0,1]$ is compact, there exists a subsequence of x_i, say z_j, with limit $z \in [0,1]$.

We have $\lim_{j\to\infty} z_j = z$ and $\lim_{j\to\infty} f(z_j) = \lim_{i\to\infty} \xi_i = \zeta$. Since $\zeta \notin W$ z must be a point of discontinuity of f. We can assign to every point-component ζ_α of W^c a point of discontinuity z_α of f. Since f is a function of bounded variation, f has at most a countable number of points of discontinuity. Therefore there exists a point of discontinuity of the domain z_0, such that every member of an uncountable collection $\{\zeta_\alpha\}_{\alpha\in A}$ of point components of the complement of the range can be assigned to it.

Choose three points ζ_1, ζ_2, and ζ_3 of the collection $\{\zeta_\alpha\}_{\alpha\in A}$. Following the same technique as before we can find three sequences $\{z_{1j}\}$, $\{z_{2j}\}$ and $\{z_{3j}\}$ which converge to z_0 and which have the property that $\lim_{j\to\infty} f(z_{kj}) = \zeta_k$ for k = 1,2 and 3. Without loss of generality we may assume that $\{z_{kj}\}_{j=1}^{\infty}$ is either monotonously increasing or monotonously decreasing. In this case there are two monotonous sequences of the same sort, say for example that $\{z_{1j}\}$ and $\{z_{2j}\}$ are monotonously increasing. Then there exist two subsequences $\{x_{1l}\}$ of $\{z_{1j}\}$ and $\{x_{2l}\}$ of $\{z_{2j}\}$ such that $x_{1l} \leq x_{2l} \leq x_{1l+1}$ for every l. Since $\lim_{l\to\infty} f(x_{1l}) = \zeta_1 \neq \zeta_2 = \lim_{l\to\infty} f(x_{2l})$ it is clear that for every natural number N there exists some n such that $\sum_{l=1}^{n} |f(x_{1l})-f(x_{2l})| > N$. Therefore $V(f) > N$ and f cannot be of bounded variation. This contradiction proves the necessity of the condition.

In order to prove the second theorem we will prove first a number of lemma's, which will give an outline of the proof of theorem 2.

Lemma 1. A set W of real numbers is the range of a strictly monotonous function defined on $[0,1]$ iff W with the order-topology is homeomorphic with the unit interval in the usual topology.

Proof. A strictly monotonous function is an order-isomorphism and hence a homeomorphism with respect to the order topology. On the other hand, if W is homeomorphic with I in the order-topology then every homeomorphism from W onto I is an order-isomorphism and hence a strictly monotonous function, which implies that the inverse function is also strictly monotonous.

Definition. A point p of a set W is called a condensation point of W iff every neighbourhood of p contains at least an uncountable number of elements of W.

Convention. The distance of a real number to the empty set is considered to be infinite.

Lemma 2. If f is a one to one function of bounded variation defined on $[0,1]$ with range W, then every accumulation point of W is a condensation point of W.

The proof of this lemma is left to the reader.

Lemma 3. Let f be a function of bounded variation and let $\{y_i\}_{i=1}^{\infty}$ be a countable subset of the range of f. Let $\{x_i\}_{i=1}^{\infty}$ be a subset of the domain of f such that $f(x_i) = y_i$ for every natural number i. Let $\{\xi_{ij}\}_{j=1}^{\infty}$ be a strictly monotonous sequence in $[0,1]$ converging to x_i. Then $\lim_{j\to\infty} f(\xi_{ij})$ exists for every i, and moreover, the function f_0 defined by

$$f_0(x) = f(x) \text{ iff } x \notin \{x_i\}_{i=1}^{\infty}$$

$$f_0(x_i) = \lim_{j\to\infty} f(\xi_{ij}); \quad (i = 1,2,\dots)$$

is of bounded variation $V(f_0) \leq V(f)$.

The proof is left to the reader.

Lemma 4. Let f be a function of bounded variation and let g be a function which is defined on the same domain D, such that $\sum_{x\in D} |f(x)-g(x)| < \infty$. Then also the function g is of bounded variation and

$$V(g) \leq V(f) + 2 \{ \sum_{x\in D} |f(x)-g(x)| \}.$$

The proof is left to the reader.

The next two lemma's show, that in many cases a set W is the range of a one to one function of bounded variation if a set V is the range of a one to one function of bounded variation, and the symetric difference between V and W is countable.

Lemma 5. If a set W is the range of a real valued one to one function of bounded variation f which is defined on the unit interval, and if B is a subset of W such that $W\setminus B$ consists of at most countably many points, then also B is the range of a one to one function of bounded variation which is defined on the unit interval.

Proof. Choose an $\varepsilon>0$. Let $y_1, y_2, \ldots$ be an enumeration of the points of $W\setminus B$, and suppose that $x_i = f^{-1}(y_i)$ for every natural number i. Choose some strictly monotonous sequence $\{\xi_{ij}\}_{j=1}^{\infty}$ in $f^{-1}(B)$ for every i, which converges to x_i. Then also $\{f(\xi_{ij})\}_{j=1}^{\infty}$ converges to some limit, say z_i. We choose for every i a subsequence $\{\eta_{ij}\}$ of $\{\xi_{ij}\}$ such that $\sum_{j=1}^{\infty} |f(\eta_{ij}) - z_i| \leq \varepsilon . 2^{-i-2}$, and such that $\eta_{ij} = \eta_{kl}$ if and only if i=k and j=l. Now we define three functions, f_0, g_0 and g, and we prove that g satisfies the required properties.

1). $f_0(x) = f(x)$ iff $x \notin \{x_i\}_{i=1}^{\infty}$

$f_0(x_i) = z_i$ for $i \in \mathbb{N}$.

2). $g_0(x) = f(x)$ for all $x \notin \bigcup_{i=1}^{\infty} ((\bigcup_{j=1}^{\infty} \{\eta_{ij}\}) \cup \{x_i\})$

$g_0(x) = z_i$ for all $x \in (\bigcup_{j=1}^{\infty} \{\eta_{ij}\}) \cup \{x_i\}$; $i \in \mathbb{N}$.

3). $g(x) = f(x)$ for all $x \notin \bigcup_{i=1}^{\infty} ((\bigcup_{j=1}^{\infty} \{\eta_{ij}\}) \cup \{x_i\})$,

$g(x_i) = f(\eta_{i1})$ for every $i \in \mathbb{N}$;

$g(\eta_{ij}) = f(\eta_{ij+1})$ for every i and $j \in \mathbb{N}$.

According to lemma 3 the function f_0 is of bounded variation: $V(f_0) \leq V(f\ )$.

Since $\sum_{x\in I} |f_0(x)-g_0(x)| = \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} |f(\eta_{ij})-z_i| \le \sum_{i=1}^{\infty} \varepsilon.2^{-i-2} = \varepsilon/_4$.
we can apply lemma 4 and we obtain: g_0 is a function of bounded variation and

$$V(g_0) \le V(f_0) + 2(\sum_{x\in I} |f_0(x)-g_0(x)|) \le V(f) + \varepsilon/_2.$$

In precisely the same way we obtain for g:

$$V(g) \le V(g_0) + 2(\sum_{x\in I} |g_0(x)-g(x)|) \le V(f) + \varepsilon/_2 + \varepsilon/_2 = V(f) + \varepsilon.$$

We have shown in fact, that g is a function of bounded variation. Since g is defined entirely by means of f, the range of g is contained in W; since f is one to one and no x_i is an argument of f in the definition of g no y_i is contained in the range of g and hence the range of g is contained in B. Since every member of $I\setminus\{x_i\}_{i=1}^{\infty}$ occurs precisely one time as an argument of f in the definition of g, it follows easily that g is one to one and B is its range. We conclude that g is a function which satisfies the required properties.

<u>Lemma 6</u>. Let f be a one to one function of bounded variation defined on $[0,1]$ with range W, and let K be the collection of accumulation points of W. Let $\{y_i\}_{i=1}^{\infty}$ be a countable set of real numbers such that $\{y_i\}_{i=1}^{\infty}\cap W = \phi$ and $\sum_{i=1}^{\infty} (\inf_{q\in K}|y_i-q|) < \infty$, then there exists a one to one function g of bounded variation with range $W\cup\{y_i\}_{i=1}^{\infty}$.

Proof. Choose an $\varepsilon>0$, and choose for every $i\in \mathbb{N}$ some element $z_i\in K$ such that

$$|y_i-z_i| \le \inf_{q\in K} |y_i-q| + \varepsilon.2^{-i-3}.$$

Furthermore, we choose for every $i\in\mathbb{N}$ a sequence $\{\eta_{ij}\}_{j=1}^{\infty}\subset W$ such that $\sum_{j=1}^{\infty} |z_i-\eta_{ij}| < \varepsilon.2^{-i-3}$ and $\eta_{ij} = \eta_{kl}$ if and only if i=k and j=l.
Let ξ_{ij} be $f^{-1}(\eta_{ij})$ for every i and j. Since f is one to one the points ξ_{ij} are uniquely determined.

We now define two functions g_0 and g.

1). $g_0(x) = f(x)$ iff $x \notin \bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} \{\xi_{ij}\} \right)$,

$g_0(\xi_{ij}) = z_i$ for every i and j from $\mathbb{N}$.

2). $g(x) = f(x)$ iff $x \notin \bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} \{\xi_{ij}\} \right)$,

$g(\xi_{i1}) = y_i$ for every $i \in \mathbb{N}$,

$g(\xi_{ij+1}) = \eta_{ij} = f(\xi_{ij})$ for every i and j from $\mathbb{N}$.

Since
$$\sum_{x \in I} |f(x)-g_0(x)| = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f(\xi_{ij})-g_0(\xi_{ij})| = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\eta_{ij}-z_i| \leq \sum_{i=1}^{\infty} \varepsilon.2^{-i-3} = \varepsilon/8$$
it follows from lemma 4 that g_0 is a function of bounded variation and $V(g_0) \leq V(f) + 2.\varepsilon/8 \leq V(f) + \varepsilon/2$. Moreover, $\sum_{x \in I} |g_0(x)-g(x)| = \sum_{i=1}^{\infty} \left(|z_i-y_i| + \sum_{j=1}^{\infty} |\eta_{ij}-z_i| \right) \leq \sum_{i=1}^{\infty} \left(\inf_{q \in K} |y_i-q| + \varepsilon.2^{-i-3} + \varepsilon.2^{-i-3} \right) \leq \sum_{i=1}^{\infty} \inf_{q \in K} |y_i-q| + \varepsilon/4$, and it follows from lemma 4 that g is a function of bounded variation. $V(g) \leq V(g_0) + 2\left(\sum_{i=1}^{\infty} \inf_{q \in K} |y_i-q| + \varepsilon/4 \right) \leq V(f) + 2\left(\sum_{i=1}^{\infty} \inf_{q \in K} |y_i-q| \right) + \varepsilon$.

Since f is one to one and since W and $\{y_i\}$ are disjoint it follows from the definition of g that g is one to one, and it is also easily verified that g has the required range.

Theorem 2. Let W be a bounded subset of the set of real numbers, let K denote the collection of all condensation points of W and let, for every real number p, $D(p)$ be the distance between p and K (i.e. $D(p) = \inf_{q \in K} |p-q|$). Then there exists a function of bounded variation f, defined on the closed unit interval with range W if and only if:

(i) The number of components of W^c is at most countable and

(ii) $\sum_{p \in W} D(p)$ is convergent.

Proof. a) Necessity. The necessity of condition (i) is proved in theorem 1. In order to prove the necessity of condition (ii) we suppose that $\sum_{p\in W} D(p)$ does not converge. Assume that f is a one to one function of bounded variation with range W. Choose an arbitrary number N and choose a finite subset F of W such that $\sum_{p\in F} D(p) > N$. Suppose that F contains m elements, $p_1, p_2, \ldots, p_m$. Order $\{f^{-1}(p) | p\in F\}$ and call these points $a_1, a_2, \ldots, a_m$. We assume that $a_1 < a_2 < \ldots < a_m$. For every natural number $k \le m-1$ the open interval (a_k, a_{k+1}) consists of uncountably many points, and hence the image f of such an interval contains a condensation point y_k.

Let b_k be $f^{-1}(y_k)$.

Then $V(f) \geq \sum_{k=1}^{m-1} |f(a_k) - f(b_k)| + \sum_{k=1}^{m-1} |f(b_k) - f(a_{k+1})| \geq$

$$\geq \sum_{k=1}^{m-1} |f(a_k) - f(b_k)| + |f(b_{m-1}) - f(a_m)| \geq \sum_{k=1}^{m} D(f(a_k)) = \sum_{p\in F} D(p) > N.$$

We conclude that the variation of f cannot be bounded and this contradiction shows the necessity of the second condition.

b) Sufficiency. In order to prove the sufficiency of the conditions (i) and (ii), we assume that W is a bounded set of real numbers, which satisfies (i) and (ii). Let B be the collection, consisting of sup W and of all real numbers p such that every right neighbourhood of p contains uncountably many points of W. Then B satisfies the following conditions:

1) Every component of B^c has non-zero length.
2) The complement of B has at most a countable number of components.
3) Every component of B^c is a right open and left closed interval, except the two unbounded ones.
4) Every subset of B has a uniquely defined supremum and a uniquely defined infimum relative to B as an ordered set
5) For every two points ξ and η of B, such that $\xi<\eta$, there exists a point ζ of B with $\xi<\zeta<\eta$.
6) B contains its supremum and its infimum.
7) $W\setminus B$ and $B\setminus W$ are both at most countable.

Proof of 1). Suppose that $\xi \notin B$, then there exists an $\varepsilon>0$ such that the open interval $(\xi,\xi+\varepsilon)$ contains only countably many points of W. If η is an arbitrary point of $(\xi,\xi+\varepsilon)$, then also the interval $(\eta,\xi+\varepsilon)$ contains at most countably many points of W and hence $\eta \notin B$. This implies that the component of B^c which contains ξ has at least the length ε.

Proof of 2). Follows immediately from 1.

Proof of 3). Let G_i be a bounded component of B^c. Then $\forall \xi \in G_i \, \exists \varepsilon_\xi > 0$ such that $[\xi,\xi+\varepsilon_\xi) \subset G_i$. It follows that G_i is right open. Let ξ_0 be the infimum of G_i. Then there exists an $\varepsilon>0$ such that $\xi_0+\varepsilon$ is contained in G_i. Since G_i is connected it follows that the interval $(\xi_0,\xi_0+\varepsilon)$ is a subset of G_i. For every member η of this interval there is an ε_η such that $[\eta,\eta+\varepsilon_\eta)$ contains at most countably many points of W. The collection of sets $\{[\eta,\eta+\varepsilon_\eta)\}$ covers $(\xi_0,\xi_0+\varepsilon)$. Since the half open interval space is hereditarily Lindelöff, it follows that there exists a countable subcover. Every member of this subcover contains at most a countable number of members of W and hence $[\xi_0,\xi_0+\varepsilon)$ contains at most a countable number of members of W. this implies that $\xi_0 \in B^c$ and hence $\xi_0 \in G_i$. This proves that every G_i is left closed.

Proof of 4). Let A be a subset of B. Let ξ_0 be the infimum of A and suppose that $\xi_0 \in B^c$. Then there exist a right neighbourhood of ξ_0 which is contained in B^c (cf. 2) and 3)) which implies that ξ_0 cannot be the infimum of A. This proves that $\xi_0 \in B$. Therefore ξ_0 is the g.l.b. of A relative to B. Let ξ_1 be the supremum of A. If $\xi_1 \in B$ then the l.u.b. of A relative to B exists; if $\xi_1 \notin B$, then ξ_1 is a member of some bounded component G_i of the complement of B, and in that case sup G_i is the least upper bound of A relative to B, since $\sup G_i \in B$ (cf. 3)). We conclude that B is order-complete.

Proof of 5). Suppose that ξ and η are two points of B, such that $\xi<\eta$, and such that there exists no point ζ of B with $\xi<\zeta<\eta$. Then (ξ,η) is a component of B^c, whereas all bounded components of B^c are left closed.

Proof of 6) follows immediately from 4).

Proof of 7). Let $\zeta \in B \setminus W$, and let C_i be the component of W^c which contains ζ. Then $\zeta = \sup C_i$, since every right neighbourhood of ζ contains points of W. Since the number of components of W^c is at most countable, also $B \setminus W$ is at most countable.

Let us suppose that $\eta \in W \setminus B$. Then $\eta \in W \cap G_i$ for some component G_i of B^c. Since $W \cap G_i$ is always countable (cf. 3) it follows that $W \setminus B$ is countable.

From 4, 5 and 6 it follows, that B is order-isomorphic with the closed unit interval and hence there exists an order-isomorphism (cf. lemma 1) between B and $[0,1]$. Let f_0 be this function, then f_0 is strictly monotonous and thus one to one and f_0 has a bounded variation $V(f_0) = |\sup B - \inf B|$.

According to lemma 5 there exists a one to one function of bounded variation f_1 from $[0,1]$ to $B \cap W$, since $B \setminus W$ is countable. Moreover, for every $\varepsilon > 0$ the function f_1 can be constructed in such a way that

$$V(f_1) \leq V(f_0) + \varepsilon = |\sup B - \inf B| + \varepsilon.$$

According to lemma 2 every accumalation point of $B \cap W$ is a condensation point, and moreover for every $p \in W$ $\inf\limits_{q \in K} |p-q| = \inf\limits_{q \in B \cap K} |p-q|$ and hence $\sum\limits_{p \in W} \inf\limits_{q \in K} |p-q| = \sum\limits_{p \in W} \inf\limits_{q \in B \cap K} |p-q| < \infty$. Clearly $\sum\limits_{p \in W \setminus B} \inf\limits_{q \in B \cap K} |p-q| < \infty$ and we can apply lemma 6 in order to construct a function f from f_1 which has a bounded variation $V(f)$, such that W is the range of f.

In this case we can construct f in such a way that

$$V(f) \leq V(f_1) + \varepsilon + 2\Big(\sum_{p \in W \setminus B} \inf_{q \in B \cap K} |p-q| \Big) \leq V(f_0) + 2\varepsilon + 2 \sum_{p \in W} D(p)$$

and hence $V(f) \leq \sup W - \inf W + 2\varepsilon + 2 \sum\limits_{p \in W} D(p)$.

This proves the sufficiency of the conditions.

<u>Remark</u>. In this paper we have restricted ourselves to functions, defined on a closed interval. This restriction is convenient for the proofs, but not necessary. In case of open or half open intervals the theorems 1 and 2 remain both true with the same conditions. This is a corollary of the fact, that the lemma's 5 and 6 describe methods to omit points from the range and methods to add points to the range.