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A NOTE ON TOPOLOGICAL LINEARIZATION OF LOCALLY COMPACT TRANSFORMATION GROUPS IN HILBERT SPACE

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A NOTE ON TOPOLOGICAL LINEARIZATION OF LOCALLY COMPACT

TRANSFORMATION GROUPS IN HILBERT SPACE

J. DE VRIES

§1. Introduction

The purpose of this note is to improve the results of [2] by using the results of [5] and to give some information about the structure of the universal linear transformation groups in [1] and [2].

Let G be a locally compact group. A weight function on G is a real-valued function f on G with the following properties:

(i) $f(e) = 1$, where e denotes the neutral element of G; $f(g) > 0$ for every $g \in G$.

(ii) $\sup_{g \in G} \frac{f(g)}{f(gg_0)} < \infty$ for every $g_0 \in G$.

(iii) $f \in L_2(G)$, where $L_2(G)$ is the Hilbert space of real-valued square-summable functions on G with respect to the (right) Haar measure in G.

A group G admitting a weight function is called a W-group (c.f. [1]).

A weight function f satisfying

$$(ii)_0 \qquad f(g_1g_2) \geq f(g_1)\, f(g_2) \text{ for all } g_1, g_2 \in G$$

will be called a proper weight function.

A topological transformation group (abbreviated t.t.g.) is a triple (G,X,π) with G a topological group, X a topological space and $\pi: G \times X \to X$ a continuous function satisfying $\pi(e,x) = x$ and $\pi(g_1,\pi(g_2,x)) = \pi(g_1g_2,x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Define $\pi^g: X \to X$ and $\pi_x: G \to X$ by $\pi^g(x) = \pi_x(g) = \pi(g,x)$ $(g \in G,\ x \in X)$. Then it is clear, that $\{\pi^g | g \in G\}$ is a group of autohomeomorphisms of X, and that π_x is a continuous function of G into X. The t.t.g. (G,X,π) is called effective if $g \neq h$ implies $\pi^g \neq \pi^h$ $(g,h \in G)$. Throughout this note we will use the following notation: if H is a Hilbert space, then $L(H)$ will denote the space of all bounded linear operators in H, and $GL(H)$ will denote the group of all bounded invertible linear operators in H. We always assume that $GL(H)$ is provided with the strong operator topology, that is, the point-open topology on H.

The concept of a W-group was introduced by P.C. Baayen and J. de Groot in [1]. They proved the following theorem (in a slightly different formulation):

THEOREM A. _Let_ (G,X,π) _be a topological transformation group_. _If_ X _is metrizable and if, moreover_, G _is a_ W-_group_, _then there exist a topological embedding_ τ _of_ X _into a Hilbert space_ K, _and an isomorphism_ $L\colon G \to GL(K)$, _such that for every_ $g \in G$

$$L(g) \circ \tau = \tau \circ \pi^g.$$

In a subsequent note [2], P.C. Baayen proved, that the isomorphism L is always an open map and moreover, that it is continuous (hence topological) if G has a continuous weight function f such that $\sup_{g\in G} \frac{f(g)}{f(gh)}$ is a locally bounded function of h on G (which is the case if f is continuous and has property $(ii)_0$). We shall prove, that the continuity of f may be dropped from the conditions if $1/f$ is locally bounded. The proof depends on the fact, that the space of continuous functions on G with compact support is dense in $L_2(G)$, and the proofs in [2] may easily be adapted to obtain the desired result. We shall give a slightly different proof, replacing the "Hilbert integral" of [1] by a Hilbert sum of copies of $L_2(G)$ and using a suitable representation of G in $GL(L_2(G))$.
As to the condition that $1/f$ must be locally bounded we note the following facts. In [6] A.B. Paalman-de Miranda proved, that the class of W-groups is exactly the class of σ-compact, locally compact groups, and that every W-group admits a proper weight function. In fact the author proved more. If G is a σ-compact, locally compact group, then there is a sequence $V_1 \subseteq V_2 \subseteq \dots \subseteq V_k \subseteq \dots$ of compact subsets of G, such that V_1 is a neighbourhood of e and $G = \bigcup_{k=1}^{\infty} V_k$. Then a function f, satisfying (i), $(ii)_0$ and

(iii) can be constructed in such a way that

$$m_k := \sup\{\frac{1}{f(g)} \mid g \in V_k\} < \infty \text{ for all } k = 1, 2, \dots .$$

In particular: the function $g \mapsto \frac{1}{f(g)}$ is locally bounded on G. Indeed, if $g_0 \in G$, then g_0V_1 is a neighbourhood of g_0, and for every $g \in V_1$ the inequality

$$\frac{1}{f(g_0g)} \leq \frac{1}{f(g_0)} \cdot \frac{1}{f(g)} \leq \frac{m_1}{f(g_0)}$$

holds, hence $\frac{1}{f(h)} \leq \frac{m_1}{f(g_0)}$ for every $h \in g_0V_1$.

Summarizing:

THEOREM B. Let G be a σ-compact, locally compact group. Then there is a real-valued function f on G satisfying (i), $(ii)_0$ and (iii), such that 1/f is locally bounded. In particular, 1/f is bounded on every compact subset of G.

Consequently, theorem A holds for every σ-compact, locally compact group G, and without further restrictions $L: G \to G\,L(K)$ may be assumed to be a topological isomorphism.

§2. A representation theorem

2.1. Let G be a σ-compact, locally compact topological group. Everything in this section (and hence §3) can be done if one admits complex-valued functions, but we assume all functions to be real-valued. In particular $L_2(G)$ is the real Hilbert space, consisting of all real-valued functions that are square-summable with respect to the right Haar measure in G.

Let f be a real-valued function on G that satisfies the conditions (i), $(ii)_0$ and (iii) of §1, such that 1/f is bounded on compact subsets of G. For every $g \in G$, let a mapping $\rho^g: L_2(G) \to L_2(G)$ be defined by

$$(\rho^g x)(h) = \frac{f(h)}{f(hg)}\, x(hg) \quad \text{if } h \in G \text{ and } x \in L_2(G).$$

Note that $\rho^g(x) \in L_2(G)$ for every $x \in L_2(G)$ and $g \in G$. Indeed, $\rho^g x$ is a measurable function, and

$$|(\rho^g x)(h)|^2 \leq \sup_{h' \in G} \left(\frac{f(h')}{f(h'g)}\right)^2 |x(hg)|^2 \leq \frac{1}{f(g)^2}\, |x(hg)|^2.$$

Since the function $h \mapsto f(g)^{-2}|x(hg)|^2$ is integrable with respect to the right Haar measure, the function $h \mapsto |(\rho^g x)(h)|^2$ is integrable as well, and hence $\rho^g x \in L_2(G)$.
It is easy to see, that for all $g \in G$, $\rho^g\colon L_2(G) \to L_2(G)$ is linear, and from the inequality above it follows that $||\rho^g x||_2 \leq f(g)^{-1}\; ||x||_2$. Consequently, $\rho^g \in \mathcal{L}(L_2(G))$ and $||\rho^g|| \leq f(g)^{-1}$. A simple computation shows, that for all $g, h \in G$ we have $\rho^{gh} = \rho^g \circ \rho^h$ and that ρ^e is the identity operator in $L_2(G)$. Hence $\rho^g \in \mathcal{G}\,\mathcal{L}(L_2(G))$ for all g, and the mapping $R\colon g \to \rho^g$ is a homomorphism of the group G into the group $\mathcal{G}\,\mathcal{L}(L_2(G))$.

2.2. THEOREM. The mapping $R\colon g \mapsto \rho^g$ is a faithfull representation of G as a group of bounded invertible linear operators on the Hilbert space $L_2(G)$. Moreover, R is a topological embedding of G into $\mathcal{GL}(L_2(G))$.
Proof.
Part of the theorem is proved in the preceding discussion. We only have to show that R is a topological embedding.

1°. First we show, that R is a relatively open injection. To prove openness, since R is a homomorphism of groups, it is sufficient to verify the following statement: for every neighbourhood U of e in G there are a finite set $A \subset L_2(G)$ and a real number $\varepsilon > 0$ such that

$$\{g \mid g \in G \;\&\; ||\rho^g x - x||_2 < \varepsilon \;\text{ for all } x \in A\} \subseteq U.$$

The proof is easy: let V be a compact, symmetric neighbourhood of e such that $V^2 \subseteq U$. Now there is a continuous function x_0 with support contained in V such that $\int_V |x_0(h)|^2 dh = 1$. For every $g \in G$, $g \notin U$ we have $g \notin V^2$, hence $Vg^{-1} \cap V = \emptyset$, and consequently

$$||\rho^g x_0 - x_0||_2^2 = \int_G \left|\frac{f(h)}{f(hg)} x_0(hg) - x_0(h)\right|^2 dh$$

$$= \int_{Vg^{-1}} \left|\frac{f(h)}{f(hg)} x_0(hg)\right|^2 dh + \int_V |x_0(h)|^2 dh \geq 1.$$

Taking $A = \{x_0\}$ and $\varepsilon = 1$ we have the desired result.
To prove that R is a one-to-one mapping it is sufficient to prove that $g = e$ if $||\rho^g x - x||_2 = 0$ for all $x \in L_2(G)$. Assuming $g \neq e$, there is a neighbourhood U of e such that $g \notin U$; then, using the function x_0 indicated above, we obtain:

$$||\rho^g x_0 - x_0||_2 \geq 1 \quad \text{for some } x_0 \in L_2(G),$$

contradicting the fact that $||\rho^g x - x||_2 = 0$ for all $x \in L_2(G)$.
So R is a isomorphism.

2°. To prove that $R: G \to G\,L(L_2(G))$ is strongly continuous it is sufficient to prove, that for every $x \in L_2(G)$ the mapping $g \mapsto \rho^g(x)$ from G into $L_2(G)$ is continuous in $e \in G$. The proof is in two steps:
(a) Suppose first that x is continuous and has a compact support, that is: there is a compact set C in G such that $x(g) = 0$ whenever $g \notin C$.
Let $\varepsilon > 0$; let V_1 be a compact neighbourhood of e and define $m_1 := \sup\{f(g)^{-1} \mid g \in V_1\}$. For every $g \in G$ we have

$$(1) \quad ||\rho^g x - x||_2^2 \leq \int_G \left|\frac{f(h)}{f(hg)} - 1\right|^2 |x(h)|^2 dh + \int_G \left(\frac{f(h)}{f(hg)}\right)^2 . |x(hg)-x(h)|^2 dh$$

$$\leq \int_C \left|\frac{f(h)}{f(hg)} - 1\right|^2 |x(h)|^2 dh + \frac{1}{f(g)^2} \int_G |x(hg)-x(h)|^2 dh.$$

Since x is continuous and x has a compact support, x is uniformly continuous with respect to the left uniform structure of G. Moreover, x is zero outside a set of finite Haar measure, so there is a neighbourhood W_1 of e such that

$$\int_G |x(hg) - x(h)|^2 dh < \frac{\varepsilon^2}{2m_1^2} \quad \text{for every } g \in W_1.$$

Consequently,

(2) $$\frac{1}{f(g)^2} \int_G |x(hg) - x(h)|^2 dh < \tfrac{1}{2}\varepsilon^2 \quad \text{for every } g \in V_1 \cap W_1.$$

Since CV_1 is compact, $1/f$ is bounded on CV_1, say $f(k)^{-1} \leq M$ for every $k \in CV_1$. Now

$$\left|\frac{f(h)}{f(hg)} - 1\right|^2 \leq \frac{1}{f(hg)^2} |f(h) - f(hg)|^2 \leq M^2 \, |f(h) - f(hg)|^2$$

for all $h \in C$ and $g \in V_1$, hence

(3) $$\int_C \left|\frac{f(h)}{f(hg)} - 1\right|^2 |x(h)|^2 dh \leq M^2 ||x||_0^2 \int_G |f(h) - f(hg)|^2 dh.$$

Here $||x||_0 = \sup\{|x(h)| \; | h \in C\}$. As $f \in L_2(G)$, it follows from [3], theorem (20.4), that there is a neighbourhood W_2 of e, such that

(4) $$\int_G |f(h) - f(hg)|^2 dh < \frac{\varepsilon^2}{2M^2 ||x||_0^2} \text{ for every } g \in W_2.$$

Combining (1) through (4), we get

$$||\rho^g x - x||_2 < \varepsilon \quad \text{for every } g \in V_1 \cap W_1 \cap W_2.$$

This proves that the mapping $g \mapsto \rho^g(x) = (R(g))(x)$ from G into $L_2(G)$ is continuous in e.

(b) In the general case, take $x \in L_2(G)$. Let $\varepsilon > 0$. Then there is a continuous function y with compact support, such that

$$||x - y||_2 < \frac{\varepsilon}{2(1+m_1)} .$$

For every $g \in V_1$ we have $||\rho^g|| \leq f(g)^{-1} \leq m_1$, and

$$||\rho^g x - x||_2 \leq ||\rho^g||.||x - y||_2 + ||\rho^g y - y||_2 + ||y - x||_2$$

$$< \frac{1}{2}\varepsilon + ||\rho^g y - y||_2.$$

By the result of (a) there is a neighbourhood W of e such that

$$||\rho^g y - y||_2 < \frac{1}{2}\varepsilon \quad \text{for every } g \in W,$$

hence

$$||\rho^g x - x||_2 < \varepsilon \quad \text{for every } g \in V_1 \cap W.$$

This proves that the mapping $g \mapsto \rho^g(x)$ is continuous in e.

2.3. COROLLARY. _The mapping_ ρ: $(g,x) \mapsto \rho^g(x)$ _of_ $G \times L_2(G)$ _into_ $L_2(G)$ _is continuous, so_ $(G, L_2(G), \rho)$ _is an effective t.t.g. If_ G _is regarded as a group of linear autohomeomorphisms of_ $L_2(G)$, _identifying_ g _with_ $\rho^g (g \in G)$, _then the original topology of_ G _coïncides with the point-open topology on_ $L_2(G)$.

Proof.

Everything except the continuity of ρ follows immediately from the fact that R: $g \mapsto \rho^g$ is a topological isomorphism of G into $GL(L_2(G))$. To prove the continuity of ρ, let $x \in L_2(G)$, $g \in G$ and $\varepsilon > 0$ be given. There is a (compact) neighbourhood U of g such that

$$\infty > M: = \sup_{h \in U} \frac{1}{f(h)} \geq \sup_{h \in U} ||\rho^h||.$$

The continuity of ρ in the point (g,x) of $G \times L_2(G)$ follows from the continuity of the mapping $h \mapsto \rho^h(x)$ in $g \in G$ and from the inequality

$$||\rho^h(y) - \rho^g(x)||_2 \leq ||\rho^h(y-x)||_2 + ||\rho^h(x) - \rho^g(x)||_2$$

$$\leq ||\rho^h||.||y-x||_2 + ||\rho^h(x) - \rho^g(x)||_2$$

$$\leq M.||y-x||_2 + ||\rho^h(x) - \rho^g(x)||_2$$

for all $h \in U$.

2.4. For applications in §3, we need the following generalizations of 2.2 and 2.3.
Let A be a non-empty set, $\kappa = |A|$ = the cardinality of A, and let for all $\alpha \in A$, $H_\alpha = L_2(G)$ and $\rho_\alpha^g = \rho^g$ ($g \in G$; cf. 2.1).
Take $H(\kappa,G) := \bigoplus_{\alpha\in A} H_\alpha$ and $\sigma(\kappa,G) := \bigoplus_{\alpha\in A} \rho_\alpha^g$ ($g \in G$); for short we will write $H = H(\kappa,G)$ and $\sigma^g = \sigma(\kappa,G)^g$. Note that σ^g is defined for every $g \in G$ and that it is a bounded linear operator on H with $||\sigma^g|| \leq f(g)^{-1}$, because $||\rho_\alpha^g|| \leq f(g)^{-1}$ for every $g \in G$ and every $\alpha \in A$. Moreover, it is easy to see that the mapping $S\colon g \mapsto \sigma^g$ is a homomorphism of G into $GL(H)$. Now the following generalization of 2.2 holds:

2.5. THEOREM. _The mapping_ $S\colon G \to GL(H)$ _is a topological isomorphism from_ G _into_ $GL(H)$.

Proof.
In 2.4 we remarked already that S is a homomorphism of groups. To prove that S is one-to-one and relatively open, we proceed as follows. Take any $\alpha_0 \in A$ (fixed) and a $\xi \in H_{\alpha_0} = L_2(G)$ (to be specified later), and consider $x(\xi) := (x_\alpha)_{\alpha\in A} \in H$, where $x_\alpha = 0$ for $\alpha \neq \alpha_0$ and $x_{\alpha_0} = \xi$. Then for every $g \in G$ we have by definition of the norm in H:

$$||\sigma^g x(\xi) - x(\xi)|| = \left(\sum_{\alpha\in A} ||\rho_\alpha^g x_\alpha - x_\alpha||_2^2\right)^{\frac{1}{2}} = ||\rho^g\xi - \xi||_2.$$

Now, if U is a neighbourhood of e in G, then there exist a finite subset B of $H_{\alpha_0} = L_2(G)$ and a real number $\varepsilon > 0$ such that $\{g \mid g \in G \;\&\; ||\rho^g\xi - \xi||_2 < \varepsilon \text{ for all } \xi \in B\} \subseteq U$, because $R\colon g \mapsto \rho^g$ from G into $GL(L_2(G))$ is relatively open. Hence, if $B_0 := \{x(\xi) \mid \xi \in B\}$, then B_0 is a finite subset of H, and

$$\{g \mid g \in G \;\&\; ||\sigma^g x - x|| < \varepsilon \text{ for all } x \in B_0\} \subseteq U. \tag{5}$$

This means that $S\colon g \mapsto \sigma^g$ is relatively open in e, hence relatively open on G. The fact that S is one-to-one can be proved similar to the proof of 2.2 by using (5).

To prove continuity of S, recall that there is a compact neighbourhood V_1 of e in G such that

$$\sup_{g\in V_1} ||\rho^g|| \leq \sup_{g\in V_1} \frac{1}{f(g)} = m_1 < \infty .$$

Fix any $x \in H$, $x = (x_\alpha)_{\alpha\in A}$. For every $\varepsilon > 0$ there is a finite subset $A_\varepsilon \subseteq A$ such that

$$\sum_{\alpha\in A\setminus A_\varepsilon} ||x_\alpha||^2 < \frac{\varepsilon^2}{2(m_1+1)^2} ,$$

since $\sum_{\alpha\in A} ||x_\alpha||^2 = ||x||^2 < \infty$. So for every $g \in V_1$ we have

$$\sum_{\alpha\in A\setminus A_\varepsilon} ||\rho_\alpha^g x_\alpha - x_\alpha||^2 \leq \sum_{\alpha\in A\setminus A_\varepsilon} (1+||\rho^g||)^2 \, ||x_\alpha||^2 < \tfrac{1}{2}\varepsilon^2 .$$

Using strong continuity of the mapping $g \mapsto \rho_\alpha^g$ from G into $GL(H_\alpha)$ for every α in the <u>finite</u> set A_ε, we see, that there is a neighbourhood V of e such that

$$||\rho_\alpha^g x_\alpha - x_\alpha||^2 < \frac{\varepsilon^2}{2|A_\varepsilon|} \text{ for all } \alpha \in A_\varepsilon,\ g \in V,$$

where $|A_\varepsilon|$ denotes the cardinality of A_ε. Consequently, for every $g \in V_1 \cap V$ we have

$$||\sigma^g x - x||^2 = \sum_{\alpha\in A_\varepsilon} ||\rho_\alpha^g x_\alpha - x_\alpha||^2 + \sum_{\alpha\in A\setminus A_\varepsilon} ||\rho_\alpha^g x_\alpha - x_\alpha||^2 < \varepsilon^2 ,$$

that is: $||\sigma^g x - x|| < \varepsilon$ for all $g \in V_1 \cap V$. This proves strong continuity of S in $e \in G$, hence strong continuity of S on G.

2.6. COROLLARY. <u>The mapping</u> $\sigma(\kappa,G)$: $(g,x) \mapsto \sigma^g(x)$ <u>from</u> $G \times H(\kappa,G)$ <u>into</u> $H(\kappa,G)$ <u>is continuous, and</u> $(G,H(\kappa,G),\sigma(\kappa,G))$ <u>is an effective topological transformation group. If</u> G <u>is regarded as a group of linear autohomeomorphisms of</u> $H(\kappa,G)$, <u>identifying</u> g <u>with</u> $\sigma(\kappa,G)^g$, <u>then the original topology of</u> G <u>coïncides with the point-open topology on</u> $H(\kappa,G)$.

Proof.
The proof is similar to the proof of 2.3, since $h \mapsto ||\sigma^h||$ is locally bounded.

§3. Consequences

3.1. The results of section 2 can be used to strengthen previous results on linearization in [1] and [2]; in fact, all theorems of [2], section 5, hold for σ-compact, locally compact groups G, that is, for all groups, admitting a weight function. At this point we stress the fact that the existence of a continuous weight function is not needed at all. Clearly this assertion will be a consequence of the observation, that the transformation group $(G,H(\kappa,G),\sigma(\kappa,G))$, defined in 2.6, is universal in the following sence: let (G,X,π) be a transformation group, such that X is a metrizable space of weight $\leq \kappa$; then there exists a topological embedding $\tau : X \to H(\kappa,G)$ such that for every $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} H(\kappa,G) & \xrightarrow{\sigma^g} & H(\kappa,G) \\ \uparrow{\scriptstyle\tau} & & \uparrow{\scriptstyle\tau} \\ X & \xrightarrow{\pi^g} & X \end{array}$$

Here $\sigma^g(x) = [\sigma(\kappa,G)](g,x)$, and σ^g is a bounded linear operator on $H(\kappa,G)$; this is why one may say, that the action of G on X is linearized in H (c.f. [1]). Moreover, because of the fact that the mapping $S: g \mapsto \sigma^g$ of G into $GL(H(\kappa,G))$ is a topological embedding, one may speak of a topological linearization (c.f. [2]).
Before outlining a proof of our observation we have to make a remark about the condition in [1] and [2] that G is a continuous homomorphic image of a W-group F (that is: a locally compact, σ-compact group F; c.f. [5]). That condition is in fact equivalent to the condition that

the action on X comes from a σ-compact, locally compact group. For if (G,X,π) is a t.t.g., $h: F \to G$ a continuous homomorphism of a topological group F onto G, then h and π induce an action $\overline{\pi}$ of F in X such that the homeomorphism groups $\{\overline{\pi}^{\phi} \mid \phi \in F\}$ and $\{\pi^{g} \mid g \in G\}$ of $(F,X,\overline{\pi})$ and (G,X,π) are exactly the same; one has only to take $\overline{\pi}(\phi,x) = \pi(h(\phi),x)$ for all $\phi \in F$ and $x \in X$. So without loss of generality we consider a t.t.g. (G,X,π) where G itself is a σ-compact, locally compact group.

3.2. Let G be a σ-compact, locally compact group, and let $f: G \to \mathbb{R}$ be a weight function as considered in 2.1. Finally, let $(G,L_2(G),\rho)$ be the transformation group, defined in 2.3.
If (G,X,π) is any t.t.g. such that X is a Hausdorff topological space, and if $\phi: X \to \mathbb{R}$ is a bounded continuous function, say $|\phi(x)| \leq 1$ for all $x \in X$, then we define a mapping $\overline{\phi}: X \to L_2(G)$ by

$$(\overline{\phi}(x))(g) := f(g) \,.\, (\phi \circ \pi_x)(g) = f(g) \,.\, \phi(\pi(g,x)).$$

That $\overline{\phi}(x) \in L_2(G)$ whenever $x \in X$ follows from the facts that $\phi \circ \pi_x: G \to \mathbb{R}$ is bounded and continuous, that $f \in L_2(G)$, and that $\overline{\phi}(x)$ is the pointwise product of both functions.

3.3. THEOREM. <u>The mapping</u> $\overline{\phi}: X \to L_2(G)$ <u>is continuous, and</u>

$$\rho^{g} \circ \overline{\phi} = \overline{\phi} \circ \pi^{g} \quad \text{for every } g \in G.$$

<u>Moreover,</u> $\overline{\phi}$ <u>is one-to-one if and only if</u> ϕ <u>has the property that</u> $\{\phi \circ \pi^{g} \mid g \in G\}$ <u>separates the points of</u> X.
Proof.
It is a simple computation to show that $\rho^{g} \circ \overline{\phi} = \overline{\phi} \circ \pi^{g}$ for all $g \in G$, so we leave it to the reader. To show continuity, note that for any $x, y \in X$ we have

$$||\overline{\phi}(x) - \overline{\phi}(y)||_2^2 = \int_G f(g)^2 \; |\phi(\pi(g,x)) - \phi(\pi(g,y))|^2 dg.$$

Now the proof can be finished by the proof of theorem A of §1, as it is given in [1], page 370, but for completeness sake we reproduce here that proof. Let $\varepsilon > 0$ be given. Since G is σ-compact, there is a compact subset C of G such that

$$\int_{G\setminus C} f(g)^2 \, dg < \frac{\varepsilon^2}{8} .$$

From the continuity of $\phi \circ \pi : G \times X \to \mathbb{R}$ and the compactness of C it follows by standard arguments, that for x fixed in X, there is a neighbourhood U of x such that

$$|(\phi\circ\pi)(g,x) - (\phi\circ\pi)(g,y)| < \frac{\varepsilon}{2||f||_2} \quad \text{for all } g \in C,\ y \in U.$$

Consequently, for all $y \in U$ we have

$$||\overline{\phi}(x) - \overline{\phi}(y)||_2^2 \leq \int_C f(g)^2 . |(\phi\circ\pi)(g,x)-(\phi\circ\pi)(g,y)|^2 dg + 4 \int_{G\setminus C} f(g)^2 dg$$

$$< \frac{1}{4}\varepsilon^2 + \frac{1}{2}\varepsilon^2 < \varepsilon^2,$$

hence $\overline{\phi}$ is continuous.

Now assume $\{\phi \circ \pi^g \mid g \in G\}$ separates the points of X, that is: if $x, y \in X$, $x \neq y$, then there is a $g_0 \in G$ such that

$$(\phi\circ\pi_x)(g_0) \neq (\phi\circ\pi_y)(g_0).$$

Since $\phi \circ \pi_x$ and $\phi \circ \pi_y$ are continuous, there is a neighbourhood U of g_0 in G such that $(\phi\circ\pi_x)(g) \neq (\phi\circ\pi_y)(g)$ for all $g \in U$. Consequently, $(\overline{\phi}(x))(g) \neq (\overline{\phi}(y))(g)$ for all $g \in U$, where U has positive Haar measure. This means, that $\overline{\phi}(x)$ and $\overline{\phi}(y)$ are different as elements of $L_2(G)$. Conversely, it is easy to see that $\{\phi \circ \pi^g \mid g \in G\}$ separates the points of X if $\overline{\phi}$ is one-to-one.

3.4. Remark. The property of the t.t.g. $(G,L_2(G),\rho)$ described in 3.3 may be expressed by saying that $(G,L_2(G),\rho)$ is quasi-universal for all t.t.g.

(G,X,π): if (G,X,π) is any t.t.g., then there exists a continuous mapping $\overline{\phi}$: $X \to L_2(G)$ such that for every $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} L_2(G) & \xrightarrow{\ \rho^g\ } & L_2(G) \\ \uparrow \overline{\phi} & & \uparrow \overline{\phi} \\ X & \xrightarrow{\ \pi^g\ } & X \end{array}$$

One might ask for conditions that $\overline{\phi}$ be a topological embedding. The following two conditions are obviously necessary:

(a) X is metrizable and the weight of X is less than or equal to the weight of $L_2(G)$ which equals, as is well known, the Hilbert dimension of $L_2(G)$, that is, the cardinality of an orthogonal base of $L_2(G)$.

(b) The set of invariant points in X, that is, the set $\{x \in X \mid \forall g \in G: \pi^g x = x\}$, is homeomorphic to a subset of $\mathbb{R}$.

As to condition (b), this follows trivially from the fact that the only invariant points of the t.t.g. $(G,L_2(G),\rho)$, where G is a σ-compact, locally compact group, are the points λf in $L_2(G)$. Here $\lambda \in \mathbb{R}$ and f is the weight function, used in the definition of ρ. Thus the set of invariant points $(G,L_2(G),\rho)$ is homeomorphic to $\mathbb{R}$. We only know about one special case in which the conditions (a) and (b) are sufficient: the case that $G = \mathbb{R}$ and X is compact (we disregard the trivial, though not unimportant, case that X is a subset of $\mathbb{R}$ and G an arbitrary σ-compact, locally compact group).

3.5. THEOREM. *Let* $(\mathbb{R},X,\pi)$ *be a t.t.g. If* X *is a compact, metrizable space, and if the action of* $\mathbb{R}$ *on* X *by* π *is such that the set of invariant points in* X *is homeomorphic to a subset of* $\mathbb{R}$, *then there is a topological embedding* $\overline{\phi}$: $X \to L_2(\mathbb{R})$ *such that*

$$\rho^t \circ \overline{\phi} = \overline{\phi} \circ \pi^t \quad \textit{for all } t \in \mathbb{R}.$$

Remark. A weight function f on $\mathbb{R}$ that satisfies all conditions of 2.1 is given by $f(t) = \exp(-|t|)$. C.f. [1], p. 367. Hence $\rho^t: L_2(\mathbb{R}) \to L_2(\mathbb{R})$ may be defined by

$$(\rho^t x)(s) = \frac{\exp(-|s|)}{\exp(-|s+t|)}\, x(s+t) \quad \text{if } x \in L_2(\mathbb{R}) \text{ and } s, t \in \mathbb{R}.$$

Proof of the theorem:

In [4], S. Kakutani has proved that the assumptions of our theorem imply the existance of a continuous function $\phi: X \to \mathbb{R}$ such that $\{\phi \circ \pi^t \mid t \in \mathbb{R}\}$ separates the points of X. Consequently, the corresponding mapping $\overline{\phi}: X \to L_2(\mathbb{R})$ is continuous and one-to-one, by theorem 3.3, hence a topological embedding, since X is compact.

If X is metrizable and weight $(X) \geq$ weight $(L_2(G))$ then X cannot be embedded into $L_2(G)$. Instead, we have theorem A of §1, which is, in fact, the following variant of theorem 3.3:

3.6. THEOREM. *Let* (G,X,π) *be a topological transformation group, with* G *a* σ-*compact, locally compact group and* X *a metrizable space of weight* κ. *Then there is a topological embedding* τ *of* X *into the Hilbert space* $H(\kappa,G)$ *such that*

$$\tau \circ \pi^g = \sigma^g \circ \tau \quad \text{for every } g \in G.$$

Here $\sigma = \sigma(\kappa,G)$. (*c.f.* 2.6.).

Proof.

Let A be a set with cardinality κ. It is well known that X may be regarded as a subset of the unit ball of a Hilbert space H_0 with Hilbert dimension κ (see [1] for references). Let $(..|..)$ denote the inner product in H_0, let $||..||_0$ be the norm in H_0 and let $\{e_\alpha \mid \alpha \in A\}$ be an orthogonal normed base of H_0. Note, that for all $x \in X$ and $\alpha \in A$ we have $|(x|e_\alpha)| \leq ||x||_0 \, ||e_\alpha||_0 \leq 1$ and that

(6) $$||x||_0 = \sum_{\alpha\in A} |(x|e_\alpha)|^2.$$

Now a function $\tau: X \to H(\kappa,G) = \bigoplus_{\alpha\in A} H_\alpha$ where $H_\alpha = L_2(G)$ for all $\alpha \in A$, can be defined by

$$\tau(x) = (\xi_\alpha)_{\alpha\in A} \text{ , with } \xi_\alpha \in L_2(G) \text{ such that}$$

$$\xi_\alpha(g) = f(g) \; . \; (\pi(g,x)|e_\alpha) \quad \text{for all } g \in G \text{ and } \alpha \in A.$$

Indeed, by the Lebesgue theorem and formula (6) we have

$$\begin{aligned}\sum_{\alpha\in A} ||\xi_\alpha||_2^2 &= \sum_{\alpha\in A} \int_G f(g)^2 \; |(\pi(g,x)|e_\alpha)|^2 dg \\ &= \int_G f(g)^2 \sum_{\alpha\in A} |(\pi(g,x)|e_\alpha)|^2 dg \\ &= \int_G f(g)^2 \; ||\pi(g,x)||_0^2 \, dg \\ &\leq \int_G f(g)^2 dg < \infty,\end{aligned}$$

hence $(\xi_\alpha)_{\alpha\in A} \in \bigoplus_{\alpha\in A} H_\alpha$ (note that $\xi_\alpha \in H_\alpha = L_2(G)$ by a similar argument as in the proof of 3.3). We have proved, that $\tau(x) \in H(\kappa,G)$ and that

$$||\tau(x)||^2 = \int_G f(g)^2 \; ||\pi(g,x)||_0^2 \, dg.$$

Similarly, one shows that for any x, $y \in X$

$$||\tau(x) - \tau(y)||^2 = \int_G f(g)^2 \; ||\pi(g,x) - \pi(g,y)||_0^2 \, dg.$$

Now the proof can be completed by the arguments given in [1], page 370.

3.7. COROLLARY. *Let* κ *be a cardinal and* G *a* σ-*compact, locally compact group. Then the topological transformation group* $(G,H(\kappa,G),\sigma(\kappa,G))$, *defined in* 2.6, *is universal for all t.t.g.* (G,X,π) *with* X *a metrizable space of weight* $\leq \kappa$. *That is: if* (G,X,π) *is any t.t.g. with* X *metrizable* of weight $\leq \kappa$, then there is a topological embedding $\tau: X \to H(\kappa,G)$ such that for all $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} H(\kappa,G) & \xrightarrow{\ \sigma(\kappa,G)^g\ } & H(\kappa,G) \\ \uparrow \tau & & \uparrow \tau \\ X & \xrightarrow{\ \pi^g\ } & X \end{array}$$

3.8. Remark. If A is a non-void set and λ a cardinal, $\lambda \geq \max(|A|,\aleph_0)$, and if for each $\alpha \in A$, H_α is a Hilbert space of Hilbert dimension λ, then the Hilbert space $H = \bigoplus_{\alpha\in A} H_\alpha$ has dimension λ, and H is topological isomorphic as a Hilbert space with each of the H_α. With this in mind one might expect that in 3.7 $(G,H(\kappa,G),\sigma(\kappa,G))$ may be replaced by $(G,L_2(G),\rho)$ if $\kappa \leq \dim(L_2(G))$. However, this is not possible in general, for several reasons. In the first place, there cannot be a topological isomorphism ϕ from $H(\kappa,G)$ onto $L_2(G)$ such that $\rho^g\circ\phi = \phi\circ\sigma(\kappa,G)^g$ for all $g \in G$ if $\kappa = |A| > 1$. Suppose there is such a ϕ; then ϕ maps the set of invariant points of $(G,H(\kappa,G),\sigma(\kappa,G))$ onto the set of invariant points of $(G,L_2(G),\rho)$. Since the first set may be identified with $\bigoplus_{\alpha\in A} R_\alpha$ with $R_\alpha = \mathbb{R}$ for all $\alpha \in A$ and the second set with $\mathbb{R}$, this is impossible unless $|A| = 1$.

Secondly, if the set of invariant points of (G,X,π) is not homeomorphic with a subset of $\mathbb{R}$, X cannot be imbedded into $L_2(G)$ in such a way that the action of G on X (by π) becomes a restriction of the action of G on $L_2(G)$ by ρ.

However, if $\kappa \leq \dim(L_2(G))$, then there is actually a linear isometrical mapping ϕ of $H(\kappa,G)$ onto $L_2(G)$ if $\dim(L_2(G)) \geq \aleph_0$. Since the transformation $t \mapsto \phi\circ t\circ\phi^{-1}$ is a topological isomorphism of the group $GL(H(\kappa,G))$ onto the group $GL(L_2(G))$, it is easy to see that the following theorem holds:

3.9. THEOREM. Let G be a σ-compact, locally compact group such that the Hilbert dimension κ of $L_2(G)$ is not finite. Then there is a mapping

$\bar{\rho}: G \times L_2(G) \to L_2(G)$ _with the following properties_:

1°. $(G, L_2(G), \bar{\rho})$ _is an effective t.t.g._

2°. _The mapping_ $\bar{R}: g \mapsto \bar{\rho}^g$ _is a topological isomorphism of the group_ G _into the group_ $G\,L(L_2(G))$.

3°. _If_ (G, X, π) _is any t.t.g. with_ X _a metrizable space of weight_ $\leq \kappa$, _then there is a topological embedding_ $\bar{\tau}$ _of_ X _into_ $L_2(G)$ _such that_

$$\bar{\rho}^g \circ \bar{\tau} = \bar{\tau} \circ \pi^g \quad \text{for every } g \in G.$$

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