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Relativistic Toda Systems*

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Abstract. We present and study Poincaré-invariant generalizations of the Galilei-invariant Toda systems. The classical nonperiodic systems are solved by means of an explicit action-angle transformation.

Contents

1. Introduction

In recent years it has been shown that the well-known Galilei–invariant Calogero–Moser N-particle systems admit Poincaré–invariant generalizations. These relativistic particle systems are not only completely integrable at the classical level, but can also be quantized in such a fashion that integrability survives [1, 2]. In this paper we show that relativistic integrable generalizations of the nonrelativistic Toda systems [3–5] exist, too. Moreover, we solve the nonperiodic classical systems by constructing an explicit action-angle transformation.

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In Sect. 2 we describe how the new Toda type systems naturally arise by taking the relativistic Calogero–Moser systems as a paradigm. Just as for the latter systems, integrability at the classical and quantum levels amounts to certain functional equations for the "potential." The technical details are relegated to Appendix A.

In the remainder of the paper we only study the nonperiodic classical systems. In Sect. 3 we find the Lax matrix for these systems by exploiting the fact that they may be viewed as a strong coupling limit of the (hyperbolic) relativistic Calogero–Moser systems. The flow generated by the Hamiltonian that equals the trace of the Lax matrix is then used to prove that the Lax matrix has positive and simple spectrum. (An argument due to Moser plays a crucial role in this proof [6, 7, 1].)

The latter flow is further studied in Sect. 4. It is shown that the position part is given by the same formula as in the nonrelativistic case [8], by isolating a general result that can be applied to both contexts.

Section 5 is concerned with the construction of action-angle transformations. We handle both the relativistic and the nonrelativistic case in a similar fashion and detail how the relativistic quantities reduce to their nonrelativistic counterparts when the speed of light is taken to infinity. Certain matrices introduced and studied in Appendix B turn out to be crucial in both contexts. Some analytic aspects of the construction are dealt with in Appendix C.

The final Sect. 6 contains some further results. We discuss the scattering occurring for a large class of Hamiltonians (i), study integrable systems living on the action-angle phase space (ii), find Lax pair formulations for a class of Hamiltonian flows (iii), and introduce integrable generalizations of the Toda systems associated to the root systems C_l and BC_l (iv).

This paper owes much to previous work on Toda type systems, especially by Moser [6, 7], Olshanetsky and Perelomov [8, 4] and Kostant [9]. In particular, the explicit formula (5.46) for the nonrelativistic case can be gleaned from Kostant's monumental work [9] by specializing to the root system A_{N-1}, cf. also [7, 10, 11]. We nevertheless present complete proofs for the nonrelativistic case, too. This is because our approach for obtaining an action-angle transformation in the relativistic case also applies to the nonrelativistic case, where it is perhaps more easily understood. Moreover, our arguments do not involve more than elementary linear algebra and analysis (with the possible exception of Appendix C), in contrast to the very extensive use of Lie algebra and Lie group theory in [9].

Most of the results reported here were already obtained in 1985 [12], but for various reasons publication was delayed. Meanwhile, distribution of a copy of [12] has given rise to a number of papers containing further information on the relativistic Toda systems at issue here [13–19]. (We should add at this point that in this paper we have nothing to say about relativistic Toda type field theories, as introduced and studied in [20, 21].)

2. Discovering the Systems

The relativistic generalizations of the classical Galilei-invariant Calogero–Moser systems are characterized by the time and space translation generators

$$H \equiv \tfrac{1}{2}(S_1 + S_{-1}), \quad P \equiv \tfrac{1}{2}(S_1 - S_{-1}), \tag{2.1}$$

where

$$S_{\pm 1} \equiv \sum_{i=1}^{N} e^{\pm\theta_i} V_i(q_1, \ldots, q_N) \tag{2.2}$$

and the boost generator

$$B \equiv -\sum_{i=1}^{N} q_i. \tag{2.3}$$

(Here, various parameters have been set equal to 1.) One obtains a representation of the Lie algebra of the Poincaré group if and only if $V_i(q)$ satisfies the functional equations

$$V_i \partial_i V_j + V_j \partial_j V_i = 0, \quad i \neq j \tag{2.4}$$

$$\sum_{i=1}^{N} \partial_i V_i^2 = 0 \tag{2.5}$$

as is readily verified.

The solution of these equations generalizing the Calogero–Moser systems reads

$$V_i(q_1, \ldots, q_N) = \prod_{j \neq i} f(q_i - q_j), \quad f^2(q) \equiv a + b\mathscr{P}(q), \tag{2.6}$$

where $\mathscr{P}$ is the Weierstrass function. The fact that (2.4) holds is immediate, whereas (2.5) is not obvious, but true, when V_i is given by (2.6) [1].

The natural Ansatz for a relativistic generalization of the classical nonperiodic and periodic Galilei-invariant Toda systems is to keep the above form of H, P and B, and to require that V_i involve the exponential function and have a nearest neighbor structure instead of the mean field structure (2.6). Specifically, one can take

$$V_i(q_1, \ldots, q_N) \equiv f(q_{i-1} - q_i) f(q_i - q_{i+1}), \quad i = 1, \ldots, N, \tag{2.7}$$

where

$$f(q) \equiv (1 + g^2 e^q)^{1/2}, \quad g \in \mathbf{R} \tag{2.8}$$

and where

$$q_0 \equiv \begin{cases} q_N \\ -\infty \end{cases}, \quad q_{N+1} \equiv \begin{cases} q_1 & \text{(periodic case)} \\ \infty & \text{(nonperiodic case)} \end{cases} \tag{2.9}$$

As phase space we may and will choose

$$\Omega \equiv \{(q, \theta) \in \mathbf{R}^{2N}\}, \quad \omega \equiv \sum_{i=1}^{N} dq_i \wedge d\theta_i \tag{2.10}$$

in both cases.

There is no difficulty in verifying that V_i as defined by (2.7) satisfies the functional equations (2.4) and (2.5) when f is given by (2.8). Thus, the systems just defined are indeed Poincaré-invariant. It is also quite easy to see that the "obvious" guess for the integrals is correct: Setting

$$\theta_I \equiv \sum_{i\in I} \theta_i, \quad I \subset \{1,\ldots,N\} \tag{2.11}$$

the functions

$$S_k \equiv \sum_{|I|=k} e^{\theta_I} \prod_{\substack{i\in I \\ i-1\notin I}} f(q_{i-1}-q_i) \prod_{\substack{i\in I \\ i+1\notin I}} f(q_i - q_{i+1}), \quad k=1,\ldots,N. \tag{2.12}$$

Poisson commute with $S_{\pm 1}$, and hence are conserved under the H flow. (This fact reduces in essence to the functional equation (2.5).) However, to prove that the S_k are in involution involves more work. We shall prove the functional equations that imply classical commutativity in Appendix A by taking Planck's constant to 0 in the functional equations that express the commutativity of operators obtained by quantizing the S_k. In the nonperiodic case we shall obtain two other proofs of classical commutativity along the way in the next section. However, in the classical periodic case the indirect proof in Appendix A is the only one we have found for general N. (The first proof of complete integrability in the classical periodic general N case was obtained by Bruschi and Ragnisco, by exploiting a Lax pair formulation for the periodic S_1 flow [14].)

A quantization of the relativistic Calogero–Moser systems preserving integrability was first obtained in [2] by splitting the potential in a suitable way. Similarly, an appropriate splitting works for the relativistic Toda systems. Specifically, the operators

$$\hat{S}_k \equiv \sum_{|I|=k} \prod_{\substack{i\in I \\ i-1\notin I}} f(q_{i-1}-q_i) e^{\beta\hat{\theta}_I} \prod_{\substack{i\in I \\ i+1\notin I}} f(q_i - q_{i+1}), \quad k=1,\ldots,N \tag{2.13}$$

where, e.g.,

$$(e^{\beta\hat{\theta}_1}\psi)(q_1,\ldots,q_N) \equiv \psi(q_1 - i\beta,\ldots,q_N) \tag{2.14}$$

commute, as is proved in Appendix A. Here, β may be physically interpreted as $\hbar/mc$, where $\hbar$ denotes Planck's constant, m the particle mass and c the speed of light [2].

From the arguments in Appendix A it is readily seen that the infinite relativistic Toda lattice is also formally integrable, both at the classical and at the quantum level. However, in the remainder of the main text we shall restrict ourselves to the finite classical nonperiodic systems. (Cf. [16, 19] and [14, 15] for information on the classical infinite and periodic cases, respectively.)

3. The Lax Matrix

The key to finding "the" Lax matrix for the (nonperiodic) relativistic Toda systems is the fact that these systems may be obtained as limits of relativistic Calogero–Moser systems. To prove this claim, we introduce

$$q_j^\varepsilon \equiv q_j - 2j \ln \varepsilon, \quad j=1,\ldots,N \tag{3.1}$$

and note that when (q, θ) varies over a compact K in the Toda phase space $\Omega \simeq \mathbf{R}^{2N}$, then (q^ε, θ) varies over a compact K^ε in the phase space [1, Eq. (1.8)], provided $\varepsilon \in (0, \delta_1(K))$. Let us now substitute

$$\mu \to 1, \quad \alpha \to g/2\varepsilon, \quad q_j \to q_j^\varepsilon \tag{3.2}$$

in the functions given by [1, Eqs. (2.27), (2.29)] and denote the resulting functions by $S_k^\varepsilon(q, \theta)$. Then it is readily verified that the S_k^ε and their (q, θ)-partials admit analytic continuations to $\{|\varepsilon| < \delta_2(K)\}$ that equal the S_k (2.12) and their partials for $\varepsilon = 0$. Thus we obtain not only a proof of our claim, but also a second proof of the involutivity of the S_k. (Indeed, the S_k^ε Poisson commute, as proved in [1].) Note that the above limit amounts to simultaneously taking the interparticle distances and the coupling constant to ∞.

If one makes the substitution (3.2) in the Lax matrix [1, Eq. (4.8)], then one finds that the "Cauchy matrix" C_{jk} [1, Eq. (4.11)] has no finite limit. However, if we first make the similarity transformation

$$L^\varepsilon(q, \theta)_{jk} \equiv \exp(q_j^\varepsilon/2) L(q^\varepsilon, \theta)_{jk} \exp(-q_k^\varepsilon/2) \tag{3.3}$$

and then take ε to 0, we do get a finite limit. It reads

$$L^0 = DAD, \tag{3.4}$$

where

$$D \equiv \operatorname{diag}(d_1, \ldots, d_N), \quad d_j \equiv e^{\theta_j/2} V_j(q)^{1/2}, \tag{3.5}$$

$$A \equiv \begin{pmatrix} 1 & a_1 & & & \\ 1 & 1 & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & 1 & a_{N-1} \\ 1 & \cdot & \cdot & 1 & 1 \end{pmatrix}, \tag{3.6}$$

$$a_i \equiv 1 - (1 + g^2 e^{q_i - q_{i+1}})^{-1}, \quad i = 1, \ldots, N-1. \tag{3.7}$$

(We have taken $\beta = 1$ in [1, Eq. (4.8)].)

It is not hard to verify directly that L^0 as defined by (3.4)–(3.7) has symmetric functions given by (2.12). Indeed, this readily follows by using

$$\begin{aligned} \det \begin{pmatrix} 1 & e_1 & & & \\ \cdot & 1 & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & e_l \\ 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix} &\equiv (e_1, \ldots, e_l) = (e_2, \ldots, e_l) - e_1(e_2, \ldots, e_l). \\ &= (1 - e_1)(e_2, \ldots, e_l) = \cdots \\ &= \prod_{j=1}^{l} (1 - e_j). \end{aligned} \tag{3.8}$$

Note also that when the point (q, θ) varies over $\mathbf{R}^{2N}$, the matrix $D^2 A = D L^0 D^{-1}$

varies over the set

$$\mathscr{L} \equiv \{\mathrm{diag}(b_1,\ldots,b_N)A \,|\, b_1,\ldots,b_N>0, a_1,\ldots,a_{N-1}\in(0,1)\}. \tag{3.9}$$

We continue by proving that any matrix in $\mathscr{L}$ has positive and simple spectrum. Presumably, this can be shown directly, but we have not found such a proof. Our proof hinges on exploiting asymptotic properties of the flow generated by S_1. (Positivity of $\sigma(L^0)$ also follows from the fact that $\sigma(L^\varepsilon)$ is positive [1], but the simplicity of $\sigma(L^\varepsilon)$ (also proved in [1]) might a priori break down for $\varepsilon \to 0$.)

Theorem 3.1. *For any* $(q,\theta)\in\mathbf{R}^{2N}$ *the matrix* L^0 *has positive and simple spectrum.*

Proof. We use an argument due to Moser [6], as adapted to the relativistic context in [1]. Specifically, we consider Hamilton's equations

$$\dot{q}_j = e^{\theta_j} V_j, \tag{3.10}$$

$$\dot{\theta}_j = -\sum_k e^{\theta_k}\partial_j V_k \tag{3.11}$$

for the flow generated by S_1. In the case at hand, [1, Eqs. (3.12)–(3.15)] still hold, but here no restriction on the q_j is needed in l.c. (3.15). From the arguments spelled out below l.c. (3.15) one now infers the existence of $q_j^\pm$, $\theta_j^\pm\in\mathbf{R}$ such that

$$\theta_1^+ < \cdots < \theta_N^+, \quad \theta_j^- = \theta_{N-j+1}^+, \tag{3.12}$$

$$\lim_{t\to\pm\infty} \theta_j(t) = \theta_j^\pm, \tag{3.13}$$

$$\lim_{t\to\pm\infty} (q_j(t) - te^{\theta_j^\pm}) = q_j^\pm. \tag{3.14}$$

Thus one obtains

$$\lim_{t\to\pm\infty} L^0(t)_{kl} = \begin{cases} 0 & k<l \\ \exp\left[\tfrac{1}{2}(\theta_k^\pm + \theta_l^\pm)\right] & k\geqq l \end{cases}. \tag{3.15}$$

Since the symmetric functions S_k of L^0 commute with S_1, they are conserved under the S_1 flow. Hence, the spectrum of $L^0(t)$ is time-independent. Combining this with (3.12) and (3.15) the assertion follows. □

In the above proof one only needs $\{S_1, H\}=0$ (to prove that the S_1 flow is complete) and $\{S_1, S_k\}=0$ (to prove isospectrality of the family $L^0(t)$). As already pointed out in Sect. 2, it is quite easy to verify that these Poisson brackets vanish. A third proof of the involutivity of the S_k now follows as a corollary: by Jacobi's identity $\{S_k, S_l\}$ is a constant of the motion with limit 0 for $t\to\infty$, and hence vanishes identically.

Just as in [1], another conclusion that can be drawn from the above is that the scattering transformation has a soliton structure, with two-particle phase shift obtained by solving the reduced $N=2$ Hamilton equations. As in [1], this last conclusion involves some tacit assumptions that are hard to prove directly. However, we shall rigorously reobtain the same conclusion in Sect. 5.

4. An Explicit Description of a Special Flow

We continue by solving the Hamilton equations for S_1 explicitly. Denote by M_j the $j \times j$ matrix obtained from $M \in M_N(\mathbf{C})$ by retaining only the last j rows and columns and set

$$m_N \equiv |M_1|, \quad m_{N-1} \equiv |M_2|/|M_1|, \ldots, m_1 \equiv |M_N|/|M_{N-1}|. \tag{4.1}$$

Our claim is that the solution reads

$$q_j(t) = q_j + \ln[m_j(e^{tL^0})], \tag{4.2}$$

$$\theta_j(t) = \ln[\dot{q}_j(t)/V_j(q(t))]. \tag{4.3}$$

Thus the flow behaves just as in the nonrelativistic case, cf. e.g. [4]. (Note (4.3) follows from (3.10).) We shall presently prove the claim just made by using a Lax pair formulation of the Hamilton equations (3.10), (3.11). However, it is illuminating to see how one can prove (4.2) for $|t|$ small by exploiting the explicit solution to the case II S_1 flow of [1], and we shall first detail this.

To this end we begin by recalling that Hamilton's equations for the latter flow are solved by the logarithms of the (ordered) eigenvalues of the matrix

$$e^{Q/2}e^{tL}e^{Q/2}, \quad Q \equiv \mathrm{diag}(q_1, \ldots, q_N), \tag{4.4}$$

cf. [1, Sect. 5, Appendix B]. Let us now fix $(q, \theta) \in \Omega$ and choose $\delta > 0$ such that $q_1^\delta < \cdots < q_N^\delta$ and such that the matrix elements of L^ε are bounded for $\varepsilon \in [0, \delta]$, cf. the first two paragraphs of Sect. 3. Then the logarithms $q_j^\varepsilon(t)$ of the eigenvalues of the matrix

$$E \equiv e^{tL^\varepsilon}e^{Q^\varepsilon}, \quad Q^\varepsilon \equiv \mathrm{diag}(q_1^\varepsilon, \ldots, q_N^\varepsilon) \tag{4.5}$$

constitute the position part of the S_1^ε flow with initial value (q^ε, θ), provided $\varepsilon \in (0, \delta]$. The crux is now that one can invoke [22, Theorem A2] (with an obvious change in ordering) to handle the $\varepsilon \to 0$ asymptotics of the spectrum of E, for t varying over a closed disc D_r with radius r around $0 \in \mathbf{C}$.

Indeed, let us substitute

$$t \to -2\ln\varepsilon, \quad D \to \mathrm{diag}(1, \ldots, N), \quad M \to e^{tL^\varepsilon}e^Q \tag{4.6}$$

in [22, Eq. (A30)]. Now choose $r > 0$ such that the minors $|M_j|$ stay at a finite distance from the origin when ε varies over $[0, \delta]$ and t over D_r. Then it follows from l.c. Th. A2 that one can find $\tilde{\delta} \in (0, \delta]$ such that E has simple spectrum for $(\varepsilon, t) \in (0, \tilde{\delta}] \times D_r$. Thus the eigenvalues of E are analytic on D_r. But then the functions $q_j^\varepsilon(t)$ admit an analytic continuation to D_r. Moreover, invoking l.c. Th. A2 once more, we infer

$$\exp(q_j^\varepsilon(t)) = m_j(e^{tL^\varepsilon}e^Q)\exp(-2j\ln\varepsilon)[1 + \rho_j^\varepsilon(t)], \tag{4.7}$$

where the error term ρ_j^ε goes to 0 for $\varepsilon \to 0$, uniformly on D_r. If we now set

$$\tilde{q}_j^\varepsilon(t) \equiv q_j^\varepsilon(t) + 2j\ln\varepsilon, \tag{4.8}$$

then we may conclude that

$$q_j(t) \equiv \lim_{\varepsilon \to 0} \tilde{q}_j^\varepsilon(t) = q_j + \ln[m_j(e^{tL^0})] \tag{4.9}$$

in the sense of uniform convergence of analytic functions on D_r. Thus we obtain (4.2) for $t\in[-r,r]$, and since we may interchange t-derivatives with the $\varepsilon\to 0$ limit, $q_j(t)$ is indeed the position part of the S_1 flow when $|t|\leqslant r$.

More generally, this argument applies to any Hamiltonian of the form considered in [22, Theorem 2.7]. The fact that the representation of the position part of these flows is not only locally, but also globally valid, is an obvious consequence of the results obtained in Subsect. 5.3. However, we do not have sufficient control over the $\varepsilon\to 0$ limit to derive this from the results obtained in [1, 22]. In fact, from now on we shall have no occasion to view the Toda systems as a limit of Calogero–Moser systems.

We now return to the S_1 flow for which (4.2) can be proved for any $t\in\mathbf{R}$ by verifying the assumptions of a general result, which will be obtained next. We first introduce some notation. For $M\in M_N(\mathbf{C})$ we denote by M^+/M^- the matrices obtained from M by putting all elements on the diagonal and below/above the diagonal equal to 0. Hence, M^+/M^- belong to the Lie algebra of the group N^+/N^- of upper/lower triangular matrices with ones on the diagonal. Using from now on the notation

$$A\sim B\Leftrightarrow|A_j|=|B_j|,\quad j=1,\dots,N, \tag{4.10}$$

the relation

$$N_uMN_l\sim M,\quad \forall(N_u,M,N_l)\in N^+\times M_N(\mathbf{C})\times N^- \tag{4.11}$$

is readily verified. This relation is crucial for the remainder of this paper.

In order to state the general result from which (4.2) follows by specializing to the case at hand, we now assume that an $N\times N$ matrix-valued function $X(q,\theta)$ and a Hamiltonian $\mathscr{H}(q,\theta)$ on a $2N$-dimensional phase space Ω are given and that $\mathscr{H}$ generates a complete flow $(q(t),\theta(t))$ on Ω. From now on we shall denote evaluation of functions on Ω along the flow by using a subscript t.

Theorem 4.1. *Suppose that*

$$\{q_j,\mathscr{H}\}=X_{jj},\quad j=1,\dots,N, \tag{4.12}$$

$$\{X,\mathscr{H}\}=[X,X^+]. \tag{4.13}$$

Then one has

$$m_j(e^{tX})>0 \tag{4.14}$$

and the flow satisfies

$$q_j(t)=q_j+\ln[m_j(e^{tX})] \tag{4.15}$$

for any $j\in\{1,\dots,N\}$, $t\in\mathbf{R}$ *and* $(q,\theta)\in\Omega$.

Proof. Fix $(q,\theta)\in\Omega$ and consider the ODE systems

$$\dot{Z}_u(t)=Z_u(t)X_t^+,\quad Z_u(0)=1_N, \tag{4.16}$$

$$\dot{Z}_l(t)=\tilde{X}_t^-Z_l(t),\quad Z_l(0)=1_N \tag{4.17}$$

where

$$\tilde{X}\equiv e^{-Q}Xe^{Q}. \tag{4.18}$$

Evidently, the unique solutions to these systems satisfy

$$Z_u(t)\in N^+, \quad Z_l(t)\in N^-, \quad \forall t\in\mathbf{R}. \tag{4.19}$$

Next, introduce

$$E(t)\equiv Z_u(t)e^{Q_t}Z_l(t). \tag{4.20}$$

From the first assumption (4.12) and from (4.16), (4.17) one then concludes

$$\dot{E}(t)=Z_u(t)X_t e^{Q_t}Z_l(t). \tag{4.21}$$

Using the second assumption (4.13) and (4.16) one now gets

$$(\dot{E}(t)E(t)^{-1})^{\cdot}=(Z_u(t)X_tZ_u(t)^{-1})^{\cdot}=Z_u(t)(\dot{X}_t+[X_t^+,X_t])Z_u(t)^{-1}=0. \tag{4.22}$$

As a consequence we must have

$$E(t)=e^{tC}E(0), \quad C\equiv \dot{E}(0)E(0)^{-1}. \tag{4.23}$$

Evaluating (4.20), (4.21) for $t=0$, it follows that

$$E(t)=e^{tX}e^{Q}. \tag{4.24}$$

Finally, we use (4.20) and (4.11) to infer

$$m_j(E(t))=e^{q_j(t)}>0, \quad \forall t\in\mathbf{R}. \tag{4.25}$$

In view of (4.24) this implies (4.14) and (4.15). □

This theorem is inspired by [4, Proposition 8.2]. However, the proof given in [4] appears inconclusive to us, inasmuch as it is a priori unclear that any geodesics exist for which the matrix $\tilde{M}(t)$ in [4, Eq. (8.10)] is equal to $\tilde{M}(q(t),p(t))$, with $\tilde{M}(q,p)$ a function on phase space and $(q(t),p(t))$ a Hamiltonian flow. Our proof proceeds the other way around, so that this problem does not arise.

We continue by noting that the assumptions (4.12), (4.13) are equivalent to

$$\{q_j,\mathscr{H}\}=\tilde{X}_{jj}, \quad j=1,\ldots,N, \tag{4.26}$$

$$\{\tilde{X},\mathscr{H}\}=[\tilde{X}^-,\tilde{X}] \tag{4.27}$$

(with $\tilde{X}$ defined by (4.18)). Let us now set $\mathscr{H}\equiv S_1$. When $X\equiv L^0$, then (4.12) is satisfied, but (4.13) is not. Similarly, (4.27) is false when $\tilde{X}\equiv L^0$. However, it turns out to be possible to make a diagonal similarity transformation that turns L^0 into a matrix L for which $X\equiv L$ satisfies (4.12), (4.13) (so that $\tilde{X}\equiv e^{-Q}Le^{Q}$ satisfies (4.26), (4.27)). We shall skip the tedious analysis via which this transformation can be obtained in a systematic way [12]. The result reads

$$L\equiv D_lAD_r, \tag{4.28}$$

where

$$D_l\equiv \mathrm{diag}(e^{-\theta_2\cdots-\theta_N}(1+g^2e^{q_1-q_2})^{1/2},e^{-\theta_3\cdots-\theta_N}(1+g^2e^{q_2-q_3})^{1/2},\ldots,1), \tag{4.29}$$

$$D_r\equiv \mathrm{diag}(e^{\theta_1\cdots+\theta_N},e^{\theta_2\cdots+\theta_N}(1+g^2e^{q_1-q_2})^{1/2},\ldots,e^{\theta_N}(1+g^2e^{q_{N-1}-q_N})^{1/2}). \tag{4.30}$$

Then one has

$$L_{jk}^+=\begin{cases}0 & k\neq j+1\\ g^2e^{q_j-q_{j+1}} & k=j+1\end{cases} \tag{4.31}$$

and it is straightforward to verify that $D_r D_l = D^2$ and that

$$\{q_j, S_1\} = L_{jj}, \quad j = 1, \ldots, N, \tag{4.32}$$

$$\{L, S_1\} = [L, L^+] \tag{4.33}$$

as announced. Thus the following result is a corollary of Theorem 4.1.

Theorem 4.2. *The flow generated by*

$$S_1 \equiv \sum_{i=1}^{N} e^{\theta_i}(1 + g^2 e^{q_{i-1} - q_i})^{1/2}(1 + g^2 e^{q_i - q_{i+1}})^{1/2} = \mathrm{Tr}\, L \tag{4.34}$$

on the phase space $\Omega \equiv \mathbf{R}^{2N}$ *is given by* (4.2), (4.3) *with* m_j *defined by* (4.1).

In the next section we shall also consider the nonrelativistic (nonperiodic) Toda systems. As is well known [23, 7], one can take

$$L_{nr} \equiv \begin{pmatrix} \theta_1 & e^{q_1 - q_2} & & & 0 \\ 1 & \theta_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \theta_{N-1} & e^{q_{N-1} - q_N} \\ 0 & & & 1 & \theta_N \end{pmatrix}, \tag{4.35}$$

$$H_2 \equiv \frac{1}{2} \sum_{i=1}^{N} \theta_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}} = \frac{1}{2} \mathrm{Tr}\, L_{nr}^2 \tag{4.36}$$

and, as before, $\Omega = \mathbf{R}^{2N}$. Then it is easy to verify

$$\{q_j, H_2\} = L_{nrjj}, \quad j = 1, \ldots, N, \tag{4.37}$$

$$\{L_{nr}, H_2\} = [L_{nr}, L_{nr}^+], \tag{4.38}$$

so that the assumptions of Theorem 4.1 are satisfied. Hence:

Theorem 4.3. *The* H_2 *flow is given by*

$$q_j(t) = q_j + \ln [m_j(e^{tL_{nr}})], \tag{4.39}$$

$$\theta_j(t) = \dot{q}_j(t), \tag{4.40}$$

where m_j *is defined by* (4.1).

5. Action-Angle Transformations

5.1. Generalities. For certain systems of Calogero–Moser type action-angle transformations can be constructed "kinematically." More precisely, a real-analytic diffeomorphism Φ from the given phase space Ω onto another phase space $\hat{\Omega}$ can be defined without invoking any dynamics, cf. [22, Sect. 2], but a special dynamics does enter in an essential way to prove that the Φ thus obtained is symplectic, and hence may be regarded as an action-angle transformation, cf. [22, Appendix C]. Both in the nonrelativistic and in the relativistic Toda case considered here we have found no way to avoid extensive use of a special dynamics already at the level of defining the diffeomorphism; more specifically, we need the results of Theorems 3.1 and 4.2 and their nonrelativistic counterparts.

In this subsection we sketch our construction in general terms. While we proceed, we shall make certain assumptions that will be verified in the special contexts of Subsects. 5.2 and 5.3. The reader might skip this subsection at first reading and refer back to it when needed.

We start from functions X and $\mathscr{H}$ on Ω as considered in Theorem 4.1, and will make free use of the matrices introduced in the proof of that theorem. The following properties of X are assumed to hold true for any $(q,\theta)\in\Omega$: First, one has

$$\sigma(X)=\{\lambda_1,\ldots,\lambda_N\},\quad \lambda_1<\cdots<\lambda_N. \tag{5.1}$$

Second, X_t^+ converges to 0 sufficiently fast so that

$$\lim_{t\to\infty} Z_u(t)\equiv Z_u \tag{5.2}$$

exists, and

$$\lim_{t\to\infty} X_t\equiv X_\infty=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ * & & \lambda_N\end{pmatrix}. \tag{5.3}$$

Third, $\tilde{X}_t^-$ converges to 0 sufficiently fast so that

$$\lim_{t\to\infty} Z_l(t)\equiv Z_l \tag{5.4}$$

exists, and

$$\lim_{t\to\infty} \tilde{X}_t\equiv \tilde{X}_\infty=\begin{pmatrix}\lambda_1 & & *\\ & \ddots & \\ 0 & & \lambda_N\end{pmatrix}. \tag{5.5}$$

Let us now calculate $\dot{E}(t)E(t)^{-1}$ and $E(t)^{-1}\dot{E}(t)$ by using (4.20), (4.21) and (4.24). This yields

$$X=Z_u(t)X_tZ_u(t)^{-1}, \tag{5.6}$$

$$\tilde{X}=Z_l(t)^{-1}\tilde{X}_tZ_l(t), \tag{5.7}$$

so taking $t\to\infty$ we obtain

$$XZ_u=Z_uX_\infty, \tag{5.8}$$

$$Z_l\tilde{X}=\tilde{X}_\infty Z_l. \tag{5.9}$$

Next, set

$$\hat{X}\equiv \operatorname{diag}(\lambda_1,\ldots,\lambda_N). \tag{5.10}$$

Using the assumptions (5.1), (5.3), (5.5) it is not hard to check that unique $F_l\in N^-$ and $F_u\in N^+$ exist such that

$$F_l\hat{X}=X_\infty F_l, \tag{5.11}$$

$$\hat{X}F_u=F_u\tilde{X}_\infty. \tag{5.12}$$

where

$$\delta(\theta) \equiv \ln(1/\theta^2). \tag{5.36}$$

Next, we set

$$\hat{q}_j \equiv q_j^+ + \tfrac{1}{2}\Delta_j(\hat{\theta}) \tag{5.37}$$

and observe that the above arguments give rise to a map Φ from $\Omega \simeq \mathbf{R}^{2N}$ into the action-angle phase space

$$\hat{\Omega} \equiv \{(\hat{q}, \hat{\theta}) \in \mathbf{R}^{2N} | \hat{\theta}_1 < \cdots < \hat{\theta}_N\}, \quad \hat{\omega} \equiv \sum_{i=1}^{N} d\hat{q}_i \wedge d\hat{\theta}_i. \tag{5.38}$$

We now introduce a map

$$\mathscr{E}: \hat{\Omega} \to \Omega, \quad (\hat{q}, \hat{\theta}) \mapsto (q, \theta) \tag{5.39}$$

by setting

$$q_j \equiv \ln(\Sigma_{N-j+1}/\Sigma_{N-j}), \tag{5.40}$$

$$\theta_j \equiv \dot{\Sigma}_{N-j+1}/\Sigma_{N-j+1} - \dot{\Sigma}_{N-j}/\Sigma_{N-j}. \tag{5.41}$$

Here, $\Sigma_k(\hat{q}, \hat{\theta})$ and $\dot{\Sigma}_k(\hat{q}, \hat{\theta})$ are defined by

$$\Sigma_k \equiv \sum_{|I|=k} e^{\hat{q}_I} V_I, \quad k = 1, \ldots, N, \quad \Sigma_0 \equiv 1, \tag{5.42}$$

$$\dot{\Sigma}_k \equiv \sum_{|I|=k} \hat{\theta}_I e^{\hat{q}_I} V_I, \quad k = 1, \ldots, N, \quad \dot{\Sigma}_0 \equiv 0, \tag{5.43}$$

where

$$V_I \equiv \prod_{i \in I, j \notin I} |\hat{\theta}_i - \hat{\theta}_j|^{-1}. \tag{5.44}$$

Theorem 5.1. *The maps Φ and $\mathscr{E}$ are real-analytic and symplectic diffeomorphisms from Ω onto $\hat{\Omega}$ and from $\hat{\Omega}$ onto Ω, respectively, and one has $\mathscr{E} = \Phi^{-1}$.*

Proof. Combining (5.30) with (5.32) and (5.37) one obtains

$$\exp\left(\sum_{j=N-k+1}^{N} q_j(t) \right) = \Sigma_k(\hat{q} + t\hat{\theta}, \hat{\theta}). \tag{5.45}$$

This implies that the H_2 flow reads

$$q_j(t) = \ln[(\Sigma_{N-j+1}/\Sigma_{N-j})(\hat{q} + t\hat{\theta}, \hat{\theta})], \tag{5.46}$$

$$\theta_j(t) = \dot{q}_j(t). \tag{5.47}$$

Evaluating this at $t = 0$ yields (5.40), (5.41). Thus, $\mathscr{E}$ satisfies (5.17), so that Φ is a bijection with inverse $\mathscr{E}$. Real-analyticity and canonicity of Φ and $\mathscr{E}$ follow from the arguments in Appendix C. □

We proceed by deriving a corollary. Let us set

$$H_h \equiv \mathrm{Tr}\, h(L), \quad h \in C^\infty_{\mathbf{R}}(\mathbf{R}), \tag{5.48}$$

$$\hat{H}_h \equiv H_h \circ \mathscr{E}. \tag{5.49}$$

Here, $h(L)$ is defined by the functional calculus, i.e.,

$$h(L) \equiv Z_u F_l \operatorname{diag}(h(\hat{\theta}_1), \ldots, h(\hat{\theta}_N)) F_l^{-1} Z_u^{-1} \tag{5.50}$$

(recall (5.28)). Thus one has

$$\hat{H}_h(\hat{q}, \hat{\theta}) = \sum_{i=1}^{N} h(\hat{\theta}_i). \tag{5.51}$$

We denote by $(q(t), \theta(t))$ the (a priori local) flow generated by H_h.

Corollary 5.2. *The H_h flow is complete and its position part is given by*

$$q_j(t) = \ln[(\Sigma_{N-j+1}/\Sigma_{N-j})(y(t))], \tag{5.52}$$

where

$$y_j(t) \equiv \hat{q}_j + th'(\hat{\theta}_j), \quad y_{N+j}(t) \equiv \hat{\theta}_j, \quad j = 1, \ldots, N. \tag{5.53}$$

Proof. Due to (5.51) the flow generated by $\hat{H}_h$ is given by (5.53) and is manifestly complete. Since $\mathscr{E}$ is a symplectic diffeomorphism, (5.49) implies

$$e^{tH_h}\mathscr{E} = \mathscr{E}e^{t\hat{H}_h}. \tag{5.54}$$

Hence e^{tH_h} is complete, too. Moreover, (5.52) is an obvious consequence of (5.54) and the definition (5.40) of the position part of $\mathscr{E}(\hat{q}, \hat{\theta})$. □

5.3. The Relativistic Case. To ease the notation we have thus far not made use of the freedom to introduce scale parameters. As regards q we shall continue to do so, but in this subsection we replace θ by $\beta\theta$ with $\beta \in (0, \infty)$. This will enable us to clarify how various objects of interest are related to their nonrelativistic counterparts in the nonrelativistic limit $\beta \to 0$. First of all, we shall from now on work with the Lax matrix

$$L_{jk} \equiv \begin{cases} 0 & k > j+1 \\ \beta e^{q_j - q_{j+1}} & k = j+1 \\ \beta^{j-k} \exp\left(\beta \sum_{i=k}^{j} \theta_i\right)(1 + \beta^2 e^{q_{j-1} - q_j})^{1/2}(1 + \beta^2 e^{q_{k-1} - q_k})^{1/2} & k < j+1 \end{cases} \tag{5.55}$$

This L is obtained from the previous L (4.28) by substituting $\theta \to \beta\theta$, taking the coupling constant g equal to β, and making a β-dependent similarity transformation. Clearly, L is holomorphic in $|\beta| < \varepsilon(K)$ when (q, θ) varies over a compact $K \subset \Omega$ and one has

$$L = \mathbf{1}_N + \beta L_{nr} + O(\beta^2), \quad \beta \to 0. \tag{5.56}$$

This implies in particular that the complete integrability of the nonrelativistic Toda systems follows from the integrability of the relativistic ones (cf. [1, Eqs. (4.17)–(4.20)] for the relevant argument).

Next, we note that the choice

$$X = L, \mathscr{H} = \beta^{-1} S_1(q, \beta\theta) \tag{5.57}$$

ensures that the assumptions of Theorem 4.1 are satisfied. Now we get from (the obvious generalization of) Theorem 3.1 and its proof

$$\sigma(L)=\{e^{\beta\hat{\theta}_1},\ldots,e^{\beta\hat{\theta}_N}\},\quad \hat{\theta}_1<\cdots<\hat{\theta}_N,\quad \hat{\theta}_j\equiv\theta_j^+, \tag{5.58}$$

$$L_\infty=\beta^{-1}\Lambda_r(\beta e^{\beta\hat{\theta}_1},\ldots,\beta e^{\beta\hat{\theta}_N}), \tag{5.59}$$

$$\tilde{L}_\infty=\beta\Lambda_n(\beta^{-1}e^{\beta\hat{\theta}_1},\ldots,\beta^{-1}e^{\beta\hat{\theta}_N})^T, \tag{5.60}$$

cf. (B3), (B2). Also, the matrix $F_l\in N^-$ such that

$$F_l\hat{L}=L_\infty F_l,\quad \hat{L}\equiv\operatorname{diag}(e^{\beta\hat{\theta}_1},\ldots,e^{\beta\hat{\theta}_N}) \tag{5.61}$$

is given by

$$F_l=\Gamma(-\beta^{-1}e^{-\beta\hat{\theta}_1},\ldots,-\beta^{-1}e^{-\beta\hat{\theta}_N}) \tag{5.62}$$

in view of (B6) and (B7). Finally, the matrix $F_u\in N^+$ such that (5.26) holds true (with $\hat{L},\tilde{L}_\infty$ as just specified, of course) is given by

$$F_u=\Gamma(\beta^{-1}e^{\beta\hat{\theta}_1},\ldots,\beta^{-1}e^{\beta\hat{\theta}_N})^T \tag{5.63}$$

in view of (B6). Note that F_l and F_u reduce to their nonrelativistic counterparts (5.25), (5.27) for $\beta\to 0$, whereas L_∞ and $\tilde{L}_\infty$ satisfy analogs of (5.56).

Proceeding now as in the previous subsections, we infer that (5.29) holds true, and using (the generalization of) Theorem 4.2 in combination with Lemmas B2 and B3 we obtain

$$\exp\left(\sum_{j=N-k+1}^{N}q_j(t)\right)=\sum_{|I|=k}\left(e^{\beta\theta_I}\prod_{j\in I}G_{jj}\right)\prod_{\substack{i\in I,j\notin I\\ j>i}}\left(\beta/2\operatorname{sh}\frac{\beta}{2}(\hat{\theta}_i-\hat{\theta}_j)\right)^2. \tag{5.64}$$

Then (5.31) follows as before, so that G is positive and (5.32) holds true. Moreover, (5.34)–(5.35) follow, with

$$\delta(\theta)\equiv\ln[\beta^2/4\operatorname{sh}^2(\beta\theta/2)]. \tag{5.65}$$

Introducing $\hat{q}_j$ by (5.37) we obtain again a map Φ from Ω into the action-angle phase space (5.38). The generalization of $\mathscr{E}$ is now defined through

$$q_j\equiv\ln(\Sigma_{N-j+1}/\Sigma_{N-j}), \tag{5.66}$$

$$\theta_j\equiv\beta^{-1}\ln([\Sigma^+_{N-j+1}/\Sigma_{N-j+1}-\Sigma^+_{N-j}/\Sigma_{N-j}]/\hat{V}_j), \tag{5.67}$$

where

$$\hat{V}_j\equiv[1+\beta^2\Sigma_{N-j+1}\Sigma_{N-j-1}/\Sigma^2_{N-j}]^{1/2}[1+\beta^2\Sigma_{N-j+2}\Sigma_{N-j}/\Sigma^2_{N-j+1}]^{1/2}, \tag{5.68}$$

$$\Sigma_k\equiv\sum_{|I|=k}e^{\hat{q}_I}V_I,\quad k=1,\ldots,N,\quad \Sigma_{-1}\equiv 0,\quad \Sigma_0\equiv 1,\quad \Sigma_{N+1}\equiv 0, \tag{5.69}$$

$$\Sigma^+_k\equiv\sum_{|I|=k}\sum_{j\in I}e^{\beta\hat{\theta}_j}e^{\hat{q}_I}V_I,\quad k=1,\ldots,N,\Sigma^+_0\equiv 0, \tag{5.70}$$

$$V_I\equiv\prod_{i\in I,j\notin I}\left|\beta/2\operatorname{sh}\frac{\beta}{2}(\hat{\theta}_i-\hat{\theta}_j)\right|. \tag{5.71}$$

Note that Σ_k and Σ^+_k are analytic in $|\beta|<\varepsilon(\hat{K})$ when $(\hat{q},\hat{\theta})$ varies over a compact

$\hat{K} \subset \hat{\Omega}$. Also, Σ_k reduces to (5.42) for $\beta = 0$. In contrast, one has

$$\Sigma_k^+ = k\Sigma_{k,nr} + \beta\dot{\Sigma}_k + O(\beta^2), \quad \beta \to 0 \tag{5.72}$$

which can be understood from the relation

$$\beta^{-1}[S_1(q, \beta\theta) - N] = \sum_{i=1}^{N} \theta_i + \beta H_2(q, \theta) + O(\beta^2), \quad \beta \to 0. \tag{5.73}$$

As a consequence, $\mathscr{E}$ is analytic in $|\beta| < \tilde{\varepsilon}(\hat{K}) \leqslant \varepsilon(\hat{K})$ and reduces to (5.40), (5.41) for $\beta = 0$.

Theorem 5.3. *The assertion of Theorem 5.1 holds true.*

Proof. As the generalizations of (5.46), (5.47) we obtain

$$q_j(t) = \ln[(\Sigma_{N-j+1}/\Sigma_{N-j})(\hat{q}_1 + te^{\beta\hat{\theta}_1}, \ldots, \hat{q}_N + te^{\beta\hat{\theta}_N}, \hat{\theta})], \tag{5.74}$$

$$\theta_j(t) = \beta^{-1} \ln[\dot{q}_j(t)/V_j(q(t))]. \tag{5.75}$$

From this it follows that $\mathscr{E}$ satisfies (5.17). Real-analyticity and canonicity are proved in Lemmas C1 and C2, respectively. □

To prove the generalization of Corollary 5.2 we set

$$H_h \equiv \operatorname{Tr} h(\beta^{-1} \ln L), \quad h \in C_{\mathbf{R}}^{\infty}(\mathbf{R}), \tag{5.76}$$

where $h(\beta^{-1} \ln L)$ is defined via the right-hand side of (5.50). Hence, (5.51) remains valid when $\hat{H}_h$ is defined by (5.49).

Corollary 5.4. *The assertion of Corollary 5.2 holds true, with Σ_k defined by* (5.69).

Proof. This follows as before. □

We can use this corollary to obtain two further representations for the θ part (5.67) of $\mathscr{E}(\hat{q}, \hat{\theta})$. (The last one will be used in the next section.) First, let us recall that the Hamiltonian

$$\tilde{\mathscr{H}} = \beta^{-1} S_{-1}(q, \beta\theta) = \beta^{-1} \sum_j e^{-\beta\theta_j} V_j(q) \tag{5.77}$$

commutes with $\mathscr{H}$ (cf. Sect. 2) and hence is conserved under $e^{t\mathscr{H}}$. Taking $t \to \infty$ in $\tilde{\mathscr{H}} \circ e^{t\mathscr{H}}$ we conclude

$$\tilde{\mathscr{H}} \circ \mathscr{E} = \beta^{-1} \sum_j e^{-\beta\hat{\theta}_j} = \beta^{-1} \operatorname{Tr} \hat{L}^{-1} \tag{5.78}$$

or equivalently

$$\tilde{\mathscr{H}} = H_{\tilde{h}} = \beta^{-1} \operatorname{Tr} L^{-1}, \quad \tilde{h}(x) \equiv \beta^{-1} e^{-\beta x}. \tag{5.79}$$

(This can also be verified directly, since $(L^{-1})_{jj}$ can be calculated by using (3.8).)

Next, we note that

$$\{q_j, \tilde{\mathscr{H}}\} = -e^{-\beta\theta_j} V_j. \tag{5.80}$$

Hence we have

$$\theta_j(t) = -\beta^{-1} \ln[-\dot{\tilde{q}}_j(t)/V_j(q(t))], \tag{5.81}$$

where the t-dependence refers to $e^{t\tilde{\mathscr{H}}}$. But we can read off the position part of $e^{t\tilde{\mathscr{H}}}$ from Corollary 5.4. Doing so, we obtain

$$\theta_j = -\beta^{-1}\ln([-\Sigma_{N-j+1}^{-}/\Sigma_{N-j+1} + \Sigma_{N-j}^{-}/\Sigma_{N-j}]/\hat{V}_j), \tag{5.82}$$

where

$$\Sigma_k^{-} \equiv -\sum_{|I|=k}\sum_{j\in I} e^{-\beta\hat{\theta}_j}e^{\hat{q}_I}V_I, \quad k=1,\dots,N, \quad \Sigma_0^{-}\equiv 0. \tag{5.83}$$

This amounts to a second representation for θ_j.

We may argue in the same way for the Hamiltonian

$$H \equiv \tfrac{1}{2}(\mathscr{H}+\tilde{\mathscr{H}}) = \beta^{-1}\sum_j \mathrm{ch}(\beta\theta_j)V_j(q). \tag{5.84}$$

Then the analog of (5.81) reads

$$\theta_j(t) = \beta^{-1}\mathrm{arsh}\,[\dot{q}_j(t)/V_j(q(t))]. \tag{5.85}$$

Using the formula for $q(t)$ that follows from Corollary 5.4 we now obtain the third representation

$$\theta_j = \beta^{-1}\,\mathrm{arsh}([\Sigma_{N-j+1}^{0}/\Sigma_{N-j+1} - \Sigma_{N-j}^{0}/\Sigma_{N-j}]/\hat{V}_j), \tag{5.86}$$

where

$$\Sigma_k^{0} \equiv \sum_{|I|=k}\sum_{j\in I}\mathrm{sh}(\beta\hat{\theta}_j)e^{\hat{q}_I}V_I, \quad k=1,\dots,N, \quad \Sigma_0^{0}\equiv 0. \tag{5.87}$$

6. Further Developments

(i) (*Invariance Principle*). Combining the proof of Theorem 3.1 (and its nonrelativistic counterpart) with (5.34)–(5.36) and (5.65) one obtains

$$q_{N-j+1}^{+} = q_j^{-} + \sum_{k>j}\delta(\theta_j^{-}-\theta_k^{-}) - \sum_{k<j}\delta(\theta_j^{-}-\theta_k^{-}), \tag{6.1}$$

$$\theta_{N-j+1}^{+} = \theta_j^{-}, \tag{6.2}$$

$$\delta(\theta) = \begin{cases}\ln[\beta^2/4\,\mathrm{sh}^2(\beta\theta/2)] & (\mathrm{rel})\\ \ln(1/\theta^2) & (\mathrm{nr})\end{cases}. \tag{6.3}$$

Together with the bijectivity of Φ this amounts to an explicit description of the scattering corresponding to the Hamiltonians $\beta^{-1}S_1(q,\beta\theta)$ and $H_2(q,\theta)$, respectively. Just as for the Calogero–Moser type systems studied in [22], this scattering behavior is shared by a large class of independent Hamiltonians. The precise statement of this invariance principle can be readily obtained from l.c. pp. 145–146, and the proof is quite simple in the case at hand due to the explicit formulas (5.66)–(5.71) and (5.40)–(5.44) for $\mathscr{E}$. More generally, it is equally easy to prove "asymptotic constancy" of $\mathscr{E}$ in the sense of [22, Theorem 5.1]. We leave the details to the interested reader.

(ii) (*Dual Systems*). The functions $\Sigma_k(\hat{q},\hat{\theta})$ of Subsects. 5.2/5.3 may be viewed as limits of the symmetric functions of the dual Lax matrix $\hat{A}$ of the $I_{\mathrm{rel}}/II_{\mathrm{rel}}$ case of

[22]. More precisely, as one takes $\varepsilon \to 0$ with $\hat{q}$, $\hat{\theta}$ fixed (recall the beginning of Sect. 3), one needs to multiply the latter functions by $\varepsilon^{k(N-k)}$ to obtain the Σ_k. The fact that the $\varepsilon \to 0$ limit amounts to taking interparticle distances to ∞ is reflected in the formula

$$(\Sigma_k \circ \Phi)(q, \theta) = \exp\left(\sum_{j=N-k+1}^{N} q_j \right). \tag{6.4}$$

Indeed, as one takes $\varepsilon \to 0$ with q, θ fixed, one obtains (6.4) when one multiplies $S_k(e^{Q(\varepsilon)})$ by $\varepsilon^{k(2N-k+1)}$, cf. (3.1). The different powers of ε in these two limits are compatible when one has

$$\hat{q}_j(q(\varepsilon), \theta) + (N+1)\ln \varepsilon \to \hat{q}_j(q, \theta), \quad \varepsilon \to 0. \tag{6.5}$$

(The functions $\hat{q}_j$ at the left hand side are defined in [22].)

The long time behavior of the dual positions $\hat{\theta}_j$ under the Σ_k flow amounts to finding the spectral asymptotics of $L(q, \theta_k(t))$, where $\theta_k(t)$ can be read off from (6.4). The results in [22, Appendix A] are not applicable to this problem, since the nondegeneracy assumption l.c. (A3) is violated. However, for $N = 2$ one readily sees that the Σ_1 flow is not asymptotically free in the usual scattering theory sense. Indeed, for $t \to \pm\infty$ not only one of the $\hat{\theta}_j$ diverges, but also $\hat{q}_1$ and $\hat{q}_2$ diverge, in agreement with constancy of Σ_1 and Σ_2. Most likely, a similar behavior occurs for $N > 2$.

On the other hand, the results of [22, Appendix A] can be used to find the $\hat{\theta}$-asymptotics of the point

$$(\hat{q}(t), \hat{\theta}(t)) \equiv \Phi(q, \theta(t)), \tag{6.6}$$

where

$$\theta_j(t) \equiv \theta_j + t c_{\tau(j)}, \quad c_1 < \cdots < c_N, \quad \tau \in S_N. \tag{6.7}$$

Indeed, from [22, Theorem A1] one obtains

$$\hat{\theta}_{\tau(j)}(t) - \theta_j(t) \to 0, \quad t \to \infty \quad (\text{nr}). \tag{6.8}$$

In the relativistic case one infers from [22, Theorem A2] (using also (3.8)) that the limit of the left-hand side of (6.8) for $t \to \infty$ exists as well. However, now the limit depends on τ, due to the nearest neighbor structure of $V_j(q)$. For instance, when τ is the identity permutation one obtains

$$\hat{\theta}_j(t) - \theta_j(t) \to (2\beta)^{-1} \ln\left[(1 + \beta^2 e^{q_{j-1} - q_j})/(1 + \beta^2 e^{q_j - q_{j+1}})\right], \quad t \to \infty\, (\text{rel}), \tag{6.9}$$

and when τ is the reversal permutation one gets

$$\hat{\theta}_{N-j}(t) - \theta_j(t) \to (2\beta)^{-1} \ln\left[(1 + \beta^2 e^{q_j - q_{j+1}})/(1 + \beta^2 e^{q_{j-1} - q_j})\right], \quad t \to \infty\, (\text{rel}). \tag{6.10}$$

The limits just discussed can be used to determine the $\hat{\theta}$-asymptotics of the flow generated by any Hamiltonian D on $\hat{\Omega}$ satisfying

$$(D \circ \Phi)(q, \theta) = \sum_{j=1}^{N} d(q_j), \quad d \in C^\infty_{\mathbf{R}}(\mathbf{R}), \quad d'' > 0. \tag{6.11}$$

(More precisely, the $\hat{\theta}$-asymptotics can be calculated when the conserved vector q takes values in a wedge $q_{\tau^{-1}(1)} < \cdots < q_{\tau^{-1}(N)}$, $\tau \in S_N$.) However, since the functions

$\Sigma_k(\hat{q}(q,\theta(t)),\hat{\theta}(q,\theta(t)))$ do not depend on t, some of the quantities $\hat{q}_j(q,\theta(t))$ must diverge. Therefore, such Hamiltonians do not give rise to a clearcut scattering theory, just as the functions Σ_k.

(iii) (*Lax Pairs*). The relation

$$F_l G F_u = Z_u^{-1} e^Q Z_l^{-1} \tag{6.12}$$

following from (5.14) may be viewed as an equality of matrix-valued functions on Ω. Using this equality along the e^{tH_h} flow $(q(t),\theta(t))$ (cf. Corollaries 5.2, 5.4) we now have

$$\begin{aligned}(Z_u^{-1}e^Q Z_l^{-1})^{\cdot} &= (F_l G F_u)^{\cdot} = F_l \dot{G} F_u = F_l h'(\hat{L}) G F_u \\ &= h'(L_\infty) F_l G F_u = h'(L_\infty) Z_u^{-1} e^Q Z_l^{-1} \\ &= Z_u^{-1} h'(L) e^Q Z_l^{-1} \quad (\mathrm{nr}),\end{aligned} \tag{6.13}$$

where we used time-independence of F_l and F_u, (5.24) and (5.22). Similarly, we obtain

$$(Z_u^{-1}e^Q Z_l^{-1})^{\cdot} = Z_u^{-1} h'(\beta^{-1}\ln L) e^Q Z_l^{-1} \quad (\mathrm{rel}). \tag{6.14}$$

On the other hand we may introduce matrices $M^\delta = M^\delta(q(t),\theta(t))$, $\delta = +, -$, by setting

$$M^+ \equiv Z_u (Z_u^{-1})^{\cdot}, \quad M^- \equiv (Z_l^{-1})^{\cdot} Z_l, \tag{6.15}$$

and then we get

$$(Z_u^{-1}e^Q Z_l^{-1})^{\cdot} = Z_u^{-1}(M^+ + \dot{Q} + e^Q M^- e^{-Q}) e^Q Z_l^{-1}. \tag{6.16}$$

Comparing with (6.14) and (6.13) we infer

$$M^+ + \dot{Q} + e^Q M^- e^{-Q} = \begin{cases} h'(\beta^{-1}\ln L) & (\mathrm{rel}) \\ h'(L) & (\mathrm{nr}) \end{cases}. \tag{6.17}$$

(Equations (6.15) should not be confused with the ODE systems (4.16), (4.17); the solutions to the latter are not functions on Ω evaluated along a flow.)

Next, we evaluate the equality $L = Z_u L_\infty Z_u^{-1}$ (cf. (5.22)) along the flow and use (6.15) to conclude

$$\dot{L} = [L, M^+]. \tag{6.18}$$

In view of (6.17) this can be rewritten

$$\{L, \mathrm{Tr}\, h(\beta^{-1}\ln L)\} = [L, h'(\beta^{-1}\ln L)^+] \quad (\mathrm{rel}), \tag{6.19}$$

$$\{L, \mathrm{Tr}\, h(L)\} = [L, h'(L)^+] \quad (\mathrm{nr}). \tag{6.20}$$

From this it follows in particular that

$$\left\{L^j, \frac{1}{\beta k}\mathrm{Tr}\, L^k\right\} = [L^j, (L^k)^+] \quad (\mathrm{rel}), \tag{6.21}$$

$$\left\{L^j, \frac{1}{k+1}\mathrm{Tr}\, L^{k+1}\right\} = [L^j, (L^k)^+] \quad (\mathrm{nr}) \tag{6.22}$$

for any $j,k = 1,2,\ldots$.

(iv) (*Generalized Toda Systems Associated with C_l and BC_l*). In [22, Sect. 5B] we have introduced generalized Calogero–Moser systems associated to the root systems C_l and BC_l via restriction of relativistic Calogero–Moser systems to submanifolds of phase space characterized by a symmetry property. These restrictions are such as to preserve complete integrability. An analogous result holds true for the relativistic Toda case. This can be proved along the same lines as in [22], so that we only detail the changes.

First of all, the spaces Ω_r, $\hat{\Omega}_r$ should be replaced by

$$\Omega_r \equiv \mathbf{R}^{2l}, \quad \hat{\Omega}_r \equiv \{(\hat{q},\hat{\theta}) \in \mathbf{R}^{2l} | \hat{\theta}_1 < \cdots < \hat{\theta}_l < 0\} \tag{6.23}$$

and the Hamiltonians H_r^e, H_r^0 by

$$H_r^e \equiv H_{l-1} + \mathrm{ch}(\beta\theta_l)(1+\beta^2 e^{q_{l-1}-q_l})^{1/2}(1+\beta^2 e^{2q_l})^{1/2}, \tag{6.24}$$

$$H_r^0 \equiv H_{l-1} + \mathrm{ch}(\beta\theta_l)(1+\beta^2 e^{q_{l-1}-q_l})^{1/2}(1+\beta^2 e^{q_l})^{1/2} + \tfrac{1}{2}(1+\beta^2 e^{q_l}), \tag{6.25}$$

where

$$H_{l-1} \equiv \sum_{j=1}^{l-1} \mathrm{ch}(\beta\theta_j)(1+\beta^2 e^{q_{j-1}-q_j})^{1/2}(1+\beta^2 e^{q_j-q_{j+1}})^{1/2}, \tag{6.26}$$

cf. [22, p. 151]. Just as in l.c., it is by no means clear from the construction of Φ that one has $\Phi(\Omega^s) \subset \hat{\Omega}^s$. However, this follows by using the Hamiltonian βH (with H defined by (5.84)) in the same way as the Hamiltonian P_0 is used in the proof of [22, Theorem 5.2]. Therefore, it remains to show $\mathscr{E}(\hat{\Omega}^s) \subset \Omega^s$.

In the present case this is not easily concluded from a consideration of the dual systems, but now we have the explicit formulas (5.66)–(5.71) defining $\mathscr{E}$ available. From these it is easy to see that q has the required symmetry property if $(\hat{q},\hat{\theta})$ does. However, from the representation (5.67) it is very far from obvious that θ has the required symmetry, too. But we may also invoke the representation (5.86) of θ, and from the latter the symmetry property is readily verified. Thus one obtains invariant submanifolds and corresponding integrable systems that amount to a one-parameter generalization of the (nonperiodic) Toda systems associated with C_l and BC_l [4].

Appendix A. Commutativity and Functional Equations

In this appendix the commutativity assertions made in Sect. 2 are proved. In Theorem A1 we show that the functions S_k defined by (2.12) Poisson commute if and only if the function $f(q)$ satisfies certain functional equations. In Theorem A2 an analogous result is proved for the operators $\hat{S}_k$ defined by (2.13). Finally, Theorem A3 shows that these quantum and classical functional equations are satisfied when f is defined by (2.8).

It should be emphasized that we are handling the periodic and nonperiodic cases simultaneously by using (2.9) and mod N addition in the former case.

Theorem A1. *One has*

$$\{S_k, S_l\} = 0, \quad \forall (k,l) \in \{1,\ldots,N\}^2, \quad \forall N > 1 \tag{A1}$$

if and only if

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=k}} \partial_I \prod_{\substack{i\in I\\i-1\notin I}} f^2(q_{i-1}-q_i) \prod_{\substack{i\in I\\i+1\notin I}} f^2(q_i-q_{i+1})=0, \quad \forall k\in\{1,\ldots,N\}, \forall N>1. \tag{A2}$$

Proof. If one replaces (A1) in Appendix A of [1] by

$$(IJ)\equiv \prod_{\substack{i\in I,i+1\in J\\i\in J,i+1\in I}} f(q_i-q_{i+1}), \quad I,J\subset\{1,\ldots,N\}, \quad I\cap J=\varnothing \tag{A3}$$

then the relations and arguments embodied in l.c. (A1)–(A14) apply verbatim. However, due to the nearest neighbor restriction an additional argument is needed to prove that (A2) implies the functional equations equivalent to (A1), which read

$$\sum_{|C|=m} \partial_c(CD)^2=0, \quad m\equiv k-|A|=l-|B|. \tag{A4}$$

(For the notation used here and the asserted equivalence, see l.c.). Indeed, (A4) amounts to (A2) when $E\equiv C\cup D$ is connected (in the obvious sense), but it is not immediate that (A4) follows from (A2) when E has more than one component.

In order to reduce (A4) to (A2) in the latter case we use induction on the number of components. Thus, assume (A4) holds when E has $M\geqslant 1$ components. Denote one of the components by F and set $G\equiv E\backslash F$. Since F and G are not coupled, we may now write

$$\begin{aligned}\sum_{|C|=m}\partial_c(CD)^2 &= \sum_{n=0}^{m}\Bigg(\sum_{\substack{S\subset F,|S|=n\\T\subset G,|T|=m-n}}(\partial_S+\partial_T)(S,F\backslash S)^2(T,G\backslash T)^2\Bigg)\\ &= \sum_{n=0}^{m}\Bigg(\sum_{T\subset G,|T|=m-n}(T,G\backslash T)^2\Bigg[\sum_{S\subset F,|S|=n}\partial_S(S,F\backslash S)^2\Bigg]\\ &\quad+\sum_{S\subset F,|S|=n}(S,F\backslash S)^2\Bigg[\sum_{T\subset G,|T|=m-n}\partial_T(T,G\backslash T)^2\Bigg]\Bigg).\end{aligned} \tag{A5}$$

Due to the induction assumption the sums in square brackets both vanish, so that the proof is complete. □

Theorem A2. *One has*

$$[\hat{S}_k,\hat{S}_l]=0, \quad \forall(k,l)\in\{1,\ldots,N\}^2, \quad \forall N>1, \quad \forall\beta\in\mathbf{C} \tag{A6}$$

if and only if

$$\begin{aligned}&\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=k}}\Bigg(\prod_{\substack{i\in I\\i-1\notin I}} f^2(q_{i-1}-q_i)\prod_{\substack{i\in I\\i+1\notin I}} f^2(q_i-q_{i+1}+\lambda)\\ &\quad-\prod_{\substack{i\in I\\i-1\notin I}} f^2(q_{i-1}-q_i+\lambda)\prod_{\substack{i\in I\\i+1\notin I}} f^2(q_i-q_{i+1})\Bigg)=0\\ &\forall k\in\{1,\ldots,N\}, \quad \forall N>1, \quad \forall\lambda\in\mathbf{C}.\end{aligned} \tag{A7}$$

Proof. If one replaces (A1) in Appendix A of [2] by

$$(IJ) \equiv \prod_{\substack{i\in I \\ i+1\in J}} f(q_i - q_{i+1}), I, J \subset \{1,\ldots,N\}, I \cap J = \varnothing, \tag{A8}$$

then l.c. (A2)–(A15) can be used. Again, the functional equations

$$\sum_{|C|=m} (DC)^2(C_-D)^2 = \sum_{|C|=m} (D_-C)^2(CD)^2 \tag{A9}$$

that are equivalent to (A6) amount to (A7) when $E \equiv C \cup D$ has one component. To handle the general case we use induction, as in the proof of Theorem A1. Specifically, we now infer, using decoupling of components,

$$\sum_{|C|=m} (DC)^2(C_-D)^2 = \sum_{n=0}^{m} \left[\sum_{S\subset F, |S|=n} (F\backslash S, S)^2(S_-, F\backslash S)^2 \right] \cdot \left[\sum_{T\subset G, |T|=m-n} (G\backslash T, T)^2(T_-, G\backslash T)^2 \right]. \tag{A10}$$

Using the induction hypothesis one may now rewrite the sums in square brackets, and then (A9) results. □

Theorem A3. *The function*

$$f^2(q) \equiv 1 + ae^{cq}, \quad a, c\in \mathbf{C} \tag{A11}$$

satisfies the functional equations (A2) *and* (A7).

Proof. We need only prove the quantum functional equations (A7), since the classical functional equations (A2) then follow when one divides (A7) by λ and sends λ to 0. To this end we set

$$b \equiv ae^{c\lambda}, \quad w_i \equiv e^{c(q_i - q_{i+1})} \tag{A12}$$

and rewrite (A7) as

$$\sum_{|I|=k} \prod_{\substack{i+1\in I \\ i\notin I}} (1 + aw_i) \prod_{\substack{i\in I \\ i+1\notin I}} (1 + bw_i) \doteq (a \leftrightarrow b). \tag{A13}$$

After expanding the products and resumming, the left-hand side can be written

$$\sum_{l,m=0}^{k} a^l b^m P_{lm}. \tag{A14}$$

Here, P_{lm} is a (possibly empty) sum of monomials in the w_i of degree $l+m$. Thus (A13) is equivalent to

$$P_{lm} = P_{ml}, \quad l < m. \tag{A15}$$

In order to prove (A15) we pair off equal contributions $w_{i_1} \cdots w_{i_{l+m}}$ to P_{lm} and P_{ml}, which arise by expanding products at the left-hand side of (A13) corresponding to different index sets, as will be detailed now. First, picture a given I as a chain of sites $1,\ldots,N$ with colors 1 or 0, depending on whether the site i belongs to I

or not. (In the periodic case the chain should be vizualized as points on a circle.) Thus, the chain has k ones and $N/N-1$ pairs of adjacent sites in the periodic/nonperiodic case. If the ith pair equals 01 or 10 we either connect it by a line or not; drawing the line codes the choice of aw_i and bw_i, respectively, in the expansion of the product at the left-hand side of (A13), whereas unconnected pairs code a factor 1 in the product. In this way a 1–1 correspondence between two-colored graphs and terms in the sum is obtained.

Next, fix an index set I with $|I| = k$ and consider a graph G arising from I that contributes to P_{lm} with $l < m$. Then G must have l lines connecting 01 pairs and m lines connecting 10 pairs. Denote the components of G by $C_1, \ldots, C_{n(G)}$. (Of course, "component" refers here to the lines and not to the colors.) Since one has $l \neq m$ by assumption, the set S of all components that contain an odd number of lines is not empty. (Note that $2n$ connected lines yield a factor $a^n b^n$.) Now any $C_i \in S$ contains an even number of sites and hence an equal number of zeros and ones. Therefore, replacing every 1/0 in all $C_i \in S$ by 0/1 leads to a graph that arises from a different index set with cardinality k. Since the two-colored graph correspondence just defined is involutive and since the contributions to P_{lm} and P_{ml} are manifestly equal, (A15) follows. □

Appendix B. Some Algebraic Lemmas

This appendix concerns certain $N \times N$ matrix-valued functions on $\mathbf{C}^N$. Specifically, we introduce

$$\Lambda \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_N), \tag{B1}$$

$$\Lambda_n \equiv \begin{pmatrix} \lambda_1 & & & \\ 1 & \lambda_2 & & O \\ & \ddots & \ddots & \\ O & & 1 & \lambda_N \end{pmatrix}, \tag{B2}$$

$$\Lambda_r \equiv \begin{pmatrix} \lambda_1 & & & O \\ \lambda_1\lambda_2 & \lambda_2 & & \\ \vdots & \vdots & \ddots & \\ \lambda_1 \cdots \lambda_N & \lambda_2 \cdots \lambda_N & \cdots & \lambda_N \end{pmatrix}, \tag{B3}$$

$$\Gamma \equiv \begin{pmatrix} 1 & & & O \\ (12) & 1 & & \\ (12)(13) & (23) & \ddots & \\ \vdots & \vdots & & \ddots \\ (12)\cdots(1N) & (23)\cdots(2N) & \cdots & (N-1,N) \quad 1 \end{pmatrix}, \tag{B4}$$

where

$$(ij) \equiv (\lambda_i - \lambda_j)^{-1}. \tag{B5}$$

Lemma B1. *For any $\lambda \in \mathbf{C}^N$ with $\lambda_i \neq \lambda_j$ one has*

$$\Gamma\Lambda = \Lambda_n \Gamma. \tag{B6}$$

For any $\lambda \in \mathbf{C}^N$ *with* $\lambda_i \neq 0$ *one has*

$$\Lambda_r(\lambda_1, \ldots, \lambda_N)^{-1} = -\Lambda_n(-1/\lambda_1, \ldots, -1/\lambda_N). \tag{B7}$$

Proof. Clearly, $\Gamma\Lambda - \Lambda_n\Gamma$ is strictly lower triangular, so that (B6) amounts to the relation

$$\Gamma_{kl} = (lk)\Gamma_{k-1,l} \quad k > l. \tag{B8}$$

This is indeed satisfied, since (B4) says

$$\Gamma_{kl} = \prod_{j=l+1}^{k} (lj), \quad k \geqq l. \tag{B9}$$

To verify (B7), note that

$$\Lambda_{rkl} = \prod_{j=l}^{k} \lambda_j, \quad k \geqq l \tag{B10}$$

and that the matrix R at the right-hand side has elements

$$R_{kl} = \frac{1}{\lambda_k}\delta_{kl} - \delta_{k,l+1}. \tag{B11}$$

Hence one gets $R\Lambda_r = \mathbf{1}_N$. □

In the main text we need to know the lower corner principal minors $|(AXB)_l|$, $l = 1, \ldots, N$, where A is defined in terms of Γ, B is defined in terms of Γ^T, and X is of the form

$$X = \operatorname{diag}(x_1, \ldots, x_N). \tag{B12}$$

We shall first derive a general formula and then calculate the relevant minors of Γ explicitly. For $M \in M_N(\mathbf{C})$ we denote by $M\begin{pmatrix} i_1, \ldots, i_l \\ j_1, \ldots, j_l \end{pmatrix}$ the determinant of the $l \times l$ matrix obtained from M by retaining only rows $i_1, \ldots, i_l$ and columns $j_1, \ldots, j_l$.

Lemma B2. *For any* $A, B \in M_N(\mathbf{C})$ *and* X *given by* (B12) *one has*

$$|(AXB)_l| = \sum_{1 \leqq i_1 < \cdots < i_l \leqq N} A\begin{pmatrix} N-l+1, \ldots, N \\ i_1, \ldots, i_l \end{pmatrix} B\begin{pmatrix} i_1, \ldots, i_l \\ N-l+1, \ldots, N \end{pmatrix} x_{i_1} \cdots x_{i_l}. \tag{B13}$$

Proof. Using obvious notation one has

$$(AXB)_l = (A_{-+}A_{--})\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix}\begin{pmatrix} B_{+-} \\ B_{--} \end{pmatrix}. \tag{B14}$$

Since the elements of the $l \times l$ matrix at the right-hand side are linear in the x_j, its determinant is a homogeneous polynomial $P_l(x_1, \ldots, x_N)$ of degree l. Thus, a monomial in P_l that is of degree greater than 1 in the x_j must contain fewer than l of the x_j. Its coefficient is unchanged if we put the remaining x_j equal to 0. But then the resulting $l \times l$ matrix has rank smaller than l, so that its determinant vanishes. Hence, no such monomials occur. Similarly, the coefficient of the monomial $x_{i_1}\cdots x_{i_l}$ with $i_1<\cdots<i_l$ can be obtained by putting the remaining x_j equal to 0, so that (B13) follows. □

Lemma B3. *Let*

$$I\equiv\{i_1,\ldots,i_l\},\quad 1\leqq i_1<\cdots<i_l\leqq N. \tag{B15}$$

For any $\lambda\in\mathbf{C}^N$ with $\lambda_i\neq\lambda_j$ one has

$$\Gamma\begin{pmatrix}N-l+1,\ldots,N\\ i_1,\ldots,i_l\end{pmatrix}=\prod_{\substack{i\in I,j\notin I\\ j>i}}(\lambda_i-\lambda_j)^{-1}. \tag{B16}$$

Proof. Using (B9) we obtain

$$\text{lhs}=\begin{vmatrix}\prod\limits_{j=i_1+1}^{N-l+1}{}'(i_1j)\cdots\prod\limits_{j=i_l+1}^{N-l+1}{}'(i_lj)\\ \vdots \qquad\qquad \vdots\\ \prod\limits_{j=i_1+1}^{N}{}'(i_1j)\cdots\prod\limits_{j=i_l+1}^{N}{}'(i_lj)\end{vmatrix}, \tag{B17}$$

where

$$\prod_{j=i+1}^{k}{}'a_j\equiv\begin{cases}0 & i>k\\ 1 & i=k\\ \prod\limits_{j=i+1}^{k}a_j & i<k\end{cases}. \tag{B18}$$

Now the elements in the last row do not vanish, and if we pull them out of each column we obtain

$$\text{lhs}=\prod_{j=i_1+1}^{N}(i_1j)\cdots\prod_{j=i_l+1}^{N}(i_lj)\cdot V(\lambda_{i_1},\ldots,\lambda_{i_l}), \tag{B19}$$

where

$$V(x_1,\ldots,x_l)\equiv\begin{vmatrix}(x_1-\lambda_N)\cdots(x_1-\lambda_{N-l+2}) & \cdots & (x_l-\lambda_N)\cdots(x_l-\lambda_{N-l+2})\\ \vdots & & \vdots\\ (x_1-\lambda_N) & \cdots & (x_l-\lambda_N)\\ 1 & \cdots & 1\end{vmatrix}. \tag{B20}$$

Thus it remains to show

$$V(x)=\prod_{1\leqq i<j\leqq l}(x_i-x_j). \tag{B21}$$

To prove (B21) we need only reduce $V(x)$ to a Vandermonde determinant. This can be done as follows. First add λ_N times the lth row to the $(l-1)$th row. Then add $-\lambda_N\lambda_{N-1}$ times the lth row plus $(\lambda_N+\lambda_{N-1})$ times the $(l-1)$th row to the $(l-2)$th row, etc. □

Appendix C. Real-Analyticity and Canonicity

In this appendix we complete the proofs of Theorems 5.1 and 5.3.

Lemma C1. *The bijections Φ and $\mathscr{E}$ defined in Subsects. 5.2 and 5.3 are real-analytic maps from Ω onto $\hat{\Omega}$ and from $\hat{\Omega}$ onto Ω, respectively.*

Proof. We only consider the relativistic case, since the nonrelativistic case can be handled in the same way. Real-analyticity of $\mathscr{E}$ on $\hat{\Omega}$ readily follows from the explicit formulas (5.66)–(5.71). Next, consider $\Phi = \mathscr{E}^{-1}$. Since $L(q,\theta)$ is real-analytic (r.a.) on Ω and has simple and positive eigenvalues $e^{\beta\hat{\theta}_1},\ldots,e^{\beta\hat{\theta}_N}$ on Ω by virtue of Theorem 3.1, it follows that the $\hat{\theta}_i$ are r.a. on Ω, too.

To prove that the $\hat{q}_i$ are r.a. involves more work. We first note it suffices to show G is r.a. (Indeed, sufficiency follows by combining positivity of G on Ω, (5.32), (5.37) and real-analyticity of $\Delta_j(\hat{\theta})$ on Ω.) To this end we recall that G is defined by (5.14). Since F_u, e^Q and F_l are r.a. on Ω, we are reduced to proving that Z_δ, $\delta = u, l$, are r.a. We shall show this for $\delta = u$, the proof for $\delta = l$ being analogous.

Consider the relation (5.22) satisfied by Z_u. This may be viewed as a linear system $Ax = b$ of N^2 equations for the $M \equiv N(N-1)/2$ nontrivial matrix elements $x_1,\ldots,x_M$ of $Z_u \in N^+$. We know already that this system has a unique solution, which is moreover non-zero (since $L_{12} \neq 0 = L_{\infty 12}$ for any $(q,\theta)\in\Omega$). Thus there exist M rows in A (possibly depending on $(q,\theta)\in\Omega$) that yield a regular $M \times M$ matrix. But the matrix elements of A and the components of b are all r.a. on Ω (since L and L_∞ are), so by virtue of Cramer's rule this must be true for the matrix elements of Z_u, too. □

Lemma C2. *The real-analytic diffeomorphisms Φ and $\mathscr{E}$ of Lemma C1 are symplectic.*

Proof. Again we only prove this for the relativistic case, the nonrelativistic case being similar, but simpler. Setting

$$q_j^+(q,\theta) \equiv \hat{q}_j - \tfrac{1}{2}\Delta_j(\hat{\theta}), \tag{C1}$$

$$\theta_j^+(q,\theta) \equiv \hat{\theta}_j, \tag{C2}$$

it suffices to show that the transformation $(q,\theta)\to(q^+,\theta^+)$ is canonical. (Indeed, in view of (5.35) and the evenness of $\delta(\theta)$ this entails canonicity of Φ, and hence of $\mathscr{E} = \Phi^{-1}$, too.) To this end we introduce

$$\tilde{q}_j(t,q,\theta) \equiv q_j(t) - t\exp(\beta\theta_j(t)), \tag{C3}$$

$$\tilde{\theta}_j(t,q,\theta) \equiv \theta_j(t), \tag{C4}$$

where the t-dependence refers to the $\mathscr{H}$ flow, cf. (5.57). Since this flow is Hamiltonian and complete, we infer

$$\{\tilde{q}_j,\tilde{q}_k\} = \{\tilde{\theta}_j,\tilde{\theta}_k\} = 0, \tag{C5}$$

$$\{\tilde{q}_j,\tilde{\theta}_k\} = \delta_{jk} \tag{C6}$$

for any $t\in\mathbf{R}$. Recalling now (5.75) and (5.74), one readily verifies that pointwise on

Ω one has

$$\lim_{t\to\infty} \tilde{\theta}(t,q,\theta)=\theta^+(q,\theta), \tag{C7}$$

$$\lim_{t\to\infty} \tilde{q}(t,q,\theta)=q^+(q,\theta). \tag{C8}$$

Therefore, it remains to prove that one may interchange the $t\to\infty$ limit and the differentiations with respect to q_i and θ_i implied in (C5), (C6).

To justify this interchange we exploit the real-analyticity of $(\hat{q},\hat{\theta})$ on Ω proved in Lemma C1. It entails that one can find a closed polydisc $P\subset \mathbf{C}^{2N}$ around a given $(q,\theta)\in\Omega$ such that $\hat{q}, q^+$ and $\hat{\theta}=\theta^+$ extend to holomorphic functions on P. Eventually shrinking P, we can ensure there exists $\varepsilon>0$ such that

$$\mathrm{Re}\,(e^{\beta\hat{\theta}_j}-e^{\beta\hat{\theta}_{j-1}})\geqq\varepsilon, \quad j=2,\ldots,N \tag{C9}$$

on P (since $\hat{\theta}_1(q,\theta)<\cdots<\hat{\theta}_N(q,\theta)$). Consequently, there exists $T\in\mathbf{R}$ such that for $t\geqq T$: (i) the functions Σ_k and Σ_k^+ evaluated in

$$y(t,q,\theta)\equiv(\hat{q}_1+te^{\beta\hat{\theta}_1},\ldots,\hat{q}_N+te^{\beta\hat{\theta}_N},\hat{\theta}) \tag{C10}$$

extend holomorphically to P; (ii) the Σ_k are non-zero on P (since the contribution of $I=\{N-k+1,\ldots,N\}$ dominates the remaining ones). Moreover, eventually increasing T, it now follows from (5.66) and (5.67) that $\tilde{q}(t,q,\theta)$ and $\tilde{\theta}(t,q,\theta)$ have holomorphic extensions to P for $t\geqq T$, which converge uniformly on P to the holomorphic functions $q^+(q,\theta)$ and $\theta^+(q,\theta)$ for $t\to\infty$ by virtue of straightforward estimates. Therefore, the interchange is legitimate. □

Notes added in proof. 1. Further relevant references include [24], [25].
2. (Addendum to Section 5) In [12] we proved bijectivity of Φ without using the formulas (5.41) and (5.67) (which we did not obtain in [12]). We no longer understand why these explicit formulas would entail (5.17) (if we ever did). More precisely, when $\mathscr{E}$ is defined by (5.40), (5.41) and (5.66), (5.67) in the nonrelativistic and relativistic cases, resp., then it does follow from the above proofs of Theorems 5.1 and 5.3 that $\mathscr{E}$ satisfies $\mathscr{E}\circ\Phi=id_\Omega$, so that Φ is injective. However, as we see it now, an additional argument is needed to prove that Φ maps onto $\hat{\Omega}$. (In fact, (5.67) is not even well defined as it stands: it is not obvious that the argument of the logarithm is positive on $\hat{\Omega}$.)

To close the gap, we detail the construction in [12] of a map

$$\mathscr{E}:\hat{\Omega}\to\Omega, \quad (\hat{q},\hat{\theta})\mapsto(q,\theta) \tag{1}$$

that manifestly satisfies (5.17). Once this is done, we may deduce that Φ is a bijection with inverse $\mathscr{E}$, and *then* it follows that $\mathscr{E}$ is actually given by (5.40), (5.41) and (5.66), (5.67). (In particular, it follows that the argument of the logarithm in (5.67) *is* positive on all of $\hat{\Omega}$.)

Turning now to the details, we begin by defining functions $\Sigma_j, \Delta_j, q_j^+$ and matrices $\hat{L}, L_\infty, \tilde{L}_\infty, F_l, F_u, G$ on $\hat{\Omega}$ via the explicit formulas in Sect. 5. Then we define a vector $q\in\mathbb{R}^N$ via (5.40) and (5.66), and matrices $Y_u\in N^+$, $Y_l\in N^-$ by requiring

$$Y_u e^Q Y_l\equiv F_l G F_u, \quad Q\equiv\mathrm{diag}(q_1,\ldots,q_N), \quad Y_u\in N^+, \quad Y_l\in N^- \tag{2}$$

Multiplying out, one readily verifies that such matrices exist, and are uniquely determined and continuous on $\hat{\Omega}$. (Indeed, since the numbers e^{q_j} are non-zero, the Nth row and column of (2) determine the Nth row of Y_l and the Nth column of Y_u, resp.; then the $(N-1)$th row and column of (2) determine the $(N-1)$th row of Y_l and the $(N-1)$th column of Y_u, resp., etc. Recall also that (2) entails (5.40) and (5.66), cf. (4.11).)

We proceed by setting

$$L \equiv Y_u^{-1} L_\infty Y_u \tag{3}$$

or, equivalently,

$$L \equiv e^Q Y_l \tilde{L}_\infty Y_l^{-1} e^{-Q} \tag{4}$$

(To verify that the right hand sides of (3) and (4) are equal, solve (2) for e^Q and use the relations (5.26), (5.24) and (5.61).) To ease the notation, we set from now on $\beta \equiv 1$ in the relativistic case. Then it follows from (4) by using $Y_l \in N^-$ and (5.21), (5.60) that in both cases

$$L_{jk} = \begin{cases} 0, & k > j+1 \\ e^{q_j - q_{j+1}}, & k = j+1 \end{cases} \tag{5}$$

Next, we specialize to the nonrelativistic case. Then one has $F_u = F_l^T$ (cf. (5.25), (5.27)), so that (2) implies $Y_u^T = Y_l$. Since also $L_\infty^T = \tilde{L}_\infty$ (cf. (5.20), (5.21)), it follows from (3) and (4) that $L^T = e^{-Q} L e^Q$. Due to (5) this entails

$$L_{jk} = \begin{cases} 1, & k = j-1 \\ 0, & k < j-1 \end{cases} \tag{6}$$

The upshot is, that when we define $\theta \in \mathbb{R}^N$ by

$$\theta_j \equiv L_{jj}, \quad j = 1, \ldots, N \tag{7}$$

then the map (1) clearly satisfies (5.17). Consequently, the proof of Theorem 5.1 is now complete.

Turning to the relativistic case, we introduce vectors $\hat{d}, \hat{e}, d, e \in \mathbb{R}^N$ by

$$\begin{aligned} &\hat{d}_j \equiv e^{-\hat{\theta}_{j+1} - \cdots - \hat{\theta}_N}, \quad \hat{e}_j \equiv e^{\hat{\theta}_j + \cdots + \hat{\theta}_N} \\ &d \equiv Y_u^{-1} \hat{d}, \quad e \equiv Y_u^T \hat{e} \end{aligned} \tag{8}$$

The point of this is, that (3) then entails

$$L_{jk} = d_j e_k, \quad k \leqq j \tag{9}$$

(To see this, note $L_\infty = \hat{d} \otimes \hat{e} - (\hat{d} \otimes \hat{e})^+$, cf. (5.59), (B3).)

We proceed by deriving information on d and e in several steps. First, using $Y_u \in N^+$, it follows from (8) that

$$d_N = 1, \quad e_1 = e^{\hat{\theta}_1 + \cdots + \hat{\theta}_N} \tag{10}$$

Second, we claim the remaining components of d and e are non-zero, too. To prove this, we first deduce from (3) that

$$(L^{-1})_{kj} = \begin{cases} 0, & k > j+1 \\ -1, & k = j+1 \end{cases} \tag{11}$$

(Recall (5.59) and (B7) to verify this.) Combining this with (5) and (9) yields

$$
\begin{aligned}
1 &= \sum_{k=1}^{N} L_{jk} L_{kj}^{-1} = d_j \sum_{k=1}^{j} e_k L_{kj}^{-1} - e^{q_j - q_{j+1}}, \quad j = 1, \ldots, N-1 \\
1 &= \sum_{j=1}^{N} L_{kj}^{-1} L_{jk} = e_k \sum_{j=k}^{N} L_{kj}^{-1} d_j - e^{q_{k-1} - q_k}, \quad k = 2, \ldots, N
\end{aligned} \tag{12}
$$

From this one reads off $d_1, \ldots, d_{N-1}, e_2, \ldots, e_N \neq 0$, as claimed.

Third, since d_j and e_j are non-zero, we may set

$$a_j \equiv e^{q_j - q_{j+1}} / d_j e_{j+1}, \quad j = 1, \ldots, N-1 \tag{13}$$

Defining a matrix A by (3.6), it then follows from (5) and (9) that L can be written

$$L = \mathrm{diag}(d_1, \ldots, d_N) A \, \mathrm{diag}(e_1, \ldots, e_N) \tag{14}$$

Recalling (3) and (3.8), this equality entails

$$e^{\hat{\theta}_1 + \cdots + \hat{\theta}_N} = d_1 \cdots d_N (1 - a_1) \cdots (1 - a_{N-1}) e_1 \cdots e_N \tag{15}$$

Fourth, we observe that by virtue of (11) and (3) the cofactor C_j of $L_{j,j+1}$ is given by

$$C_j = - e^{\hat{\theta}_1 + \cdots + \hat{\theta}_N}, \quad j = 1, \ldots, N-1 \tag{16}$$

On the other hand, we can also use (14) and (3.8) to calculate C_j. This yields

$$C_j = - d_j \cdots d_j' \cdots d_N (1 - a_1) \cdots (1 - a_j)' \cdots (1 - a_{N-1}) e_1 \cdots e_{j+1}{}' \cdots e_N \tag{17}$$

where the primes signify factors that are to be omitted. Combining (15)–(17) with (13), we deduce

$$d_j e_{j+1} = 1 + e^{q_j - q_{j+1}}, \quad j = 1, \ldots, N-1 \tag{18}$$

Fifth, due to (18) we may introduce $p_N, \ldots, p_2 \neq 0$ such that

$$
\begin{array}{ll}
e_N = p_N (1 + e^{q_{N-1} - q_N})^{1/2}, & d_{N-1} = p_N^{-1} (1 + e^{q_{N-1} - q_N})^{1/2} \\
\vdots & \vdots \\
e_2 = p_N \cdots p_2 (1 + e^{q_1 - q_2})^{1/2}, & d_1 = p_N^{-1} \cdots p_2^{-1} (1 + e^{q_1 - q_2})^{1/2}
\end{array} \tag{19}
$$

These numbers are uniquely determined and continuous on $\hat{\Omega}$, since this holds true for all quantities involved in their definition. Since they are non-zero on $\hat{\Omega}$ and $\hat{\Omega}$ is connected, each of them is either positive or negative on $\hat{\Omega}$.

Sixth, we assert that $p_2, \ldots, p_N$ (and hence d_j and e_j) are actually positive on $\hat{\Omega}$. To prove this, we consider a point $\hat{P}_0$ of the form $\Phi(q_0, \theta_0)$. Then it follows by comparing the various matrices and vectors that the vector q in (2) equals q_0, whereas p_j equals $e^{\theta_{0j}}$. Thus, p_j is positive in $\hat{P}_0$ and hence on $\hat{\Omega}$.

We are now in the position to complete the definition of the map (1): We set

$$
\begin{aligned}
\theta_j &\equiv \ln p_j, \quad j = 2, \ldots, N \\
\theta_1 &\equiv \hat{\theta}_1 + \cdots + \hat{\theta}_N - \ln(p_2 \cdots p_N)
\end{aligned} \tag{20}
$$

and then (5.17) manifestly holds true. Thus, Theorem 5.3 is now proved.

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