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THE EXIT PROBLEM FOR A STOCHASTIC DYNAMICAL SYSTEM IN A DOMAIN WITH ALMOST EVERYWHERE CHARACTERISTIC BOUNDARIES

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For a dynamical system with a stable equilibrium point the influence of small random perturbations is analyzed with singular perturbation techniques. The WKB ansatz to the asymptotic solution of the exit problem for domains with characteristic boundaries, containing a critical point, is not valid, because of the turning-point behavior of the Fokker-Planck equation near such a point. In this paper this difficulty is resolved by changing the domain for the characteristic exit problem slightly. Explicit computations are carried out for a problem originating from theoretical population biology: the 3-dimensional hypercycle.

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1. INTRODUCTION

In this paper we analyse the influence of small random perturbations acting upon a system having a stable equilibrium Z. With probability 1 such a system will leave the domain of attraction Ω of Z in finite time. The expected time of first exit is a measure of the stochastic stability of the equilibrium state. Applying the method of Matkowsky and Schuss [7] to a class of systems originating from population dynamics we meet serious difficulties. These are due to the turning point behavior of the stationary Fokker-Planck equation near critical points at the boundary $\partial\Omega$.

In the present paper we resolve this difficulty by slightly modifying the domain Ω. The asymptotic solution of the modified problem gives a good approximation to the expected exit time for the full domain of attraction, because it yields more information about paths of exit near critical points at $\partial\Omega$. The result agrees very well with a special case simulated numerically.

In our asymptotic analysis we deal with dynamical systems that model biological populations. Exit at a boundary means extinction of one species. The question of which species will most likely disappear is answered as well. Although in our study the diffusion matrix may depend upon the densities of the interacting species, we do not yet include the more realistic type of diffusion matrices with coefficient that are proportional to the square root of the density of a species (intrinsic stochasticity), see Nisbet and Gurney [9]. In that case the WKB-ansatz for the stationary Fokker-Planck equation does not hold at the boundary, as the deterministic flow in normal direction as well as the diffusion coefficient vanish.

In section 2 we give the stochastic differential equation and its corresponding Fokker-Planck equation; they form the starting point of our analysis. Furthermore, we formulate two singularly perturbed Dirichlet problems from which the expected exit time and the most probable exit boundary can be derived. In section 3 the stationary Fokker-Planck equation is solved with the WKB-method. The equations for the first order approximation can be reformulated as a Hamiltonian system, which is solved by the ray method. In section 4 the asymptotic solutions to the Dirichlet problems of section 2 are given. The material, we present in sections 2, 3 and 4, summarizes the work of Ludwig [6] and Schuss [11] as far as relevant for our investigations.

More about the theory of stochastic processes can also be found in Van Kampen [4], Wentzell and Freidlin [11] and Gardiner [3].

In sections 5 and 6 we carry out all computations for a special case: a 3-dimensional dynamical system. This generalized Volterra-Lotka system, a so-called hypercycle, plays an important role in the theory on the early stage of evolution formulated by Eigen and Schuster [2]. In section 6 the asymptotic results are compared with values obtained from simulations.

2. FORMULATION OF THE EQUATIONS

We consider the dynamical system

$$\frac{dx_i}{dt} = b_i(x), \quad b_i(x) = x_i\, c_i(x), \quad i = 1,2,\ldots,n \tag{2.1}$$

for a domain $\Omega = \{x | x_i > 0\}$, containing a unique asymptotically stable stationary point $\hat{x}$. For any starting value $x_0 \in \Omega$, the solution approaches the point $\hat{x}$ as $t \to \infty$.

Introduction of small random perturbations transforms the system into a stochastic differential equation of the type

$$dX_i = b_i(X)dt + \varepsilon \sum_{k=1}^{n} \sigma_{ik}(X)dW_k, \qquad 0 < \varepsilon << 1, \tag{2.2}$$

$$X_i(0) = x_{i0}, \qquad i = 1,\ldots,n. \tag{2.2b}$$

where W_k denote n independent Brownian motion processes. The dynamics of this system is described by the probability density function $p_\varepsilon(t,x,x_0)$ satisfying the Fokker-Planck equation

$$\frac{\partial p_\varepsilon}{\partial t} = \tfrac{1}{2}\varepsilon^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 a_{ij} p_\varepsilon}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial b_i p_\varepsilon}{\partial x_i}, \tag{2.3}$$

where

$$a_{ij} = \sum_{k=1}^{n} \sigma_{ik}\, \bar{\sigma}_{kj}. \tag{2.4}$$

We write (2.3) with initial values $p_\varepsilon(0,x,x_0) = \delta(x-x_0)$ as

$$\frac{\partial p_\varepsilon}{\partial t} = M_\varepsilon p_\varepsilon. \tag{2.5}$$

Our analysis applies to the case that $a_{ij}(x)$ is bounded away from zero in Ω. In order to concentrate on the essentials of the method we take constant coefficients: $a_{ij} = \delta_{ij}$. In the concluding remarks we discuss extenstions, which might be meaningful in applications to stochastic population dynamics. From probability theory it is known that for any starting value $X(0) = x_0$, the system reaches the boundary $\partial\Omega$ in finite time. The two central questions for this exit problem are the following: which of the boundaries $x_i = 0$ is most probably reached first and, secondly, what is the expected time of first exit? In terms of coexistence of species, the first question can be rephrased as: which species will most likely become extinct? The expected exit time is a measure of stochastic persistence of the ecosystem.

Let L_ε be the formal adjoint of the operator M_ε and let u_ε satisfy the elliptic equation with boundary values

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } \Omega, \tag{2.6a}$$

$$u_\varepsilon = f \quad \text{on } \partial\Omega. \tag{2.6b}$$

Then the first question can be answered by choosing an appropriate function f, because

$$\int_{\partial\Omega} q_\varepsilon(\tilde{x},x) f(\tilde{x}) dS = u_\varepsilon(x), \tag{2.7}$$

where $q_\varepsilon(\tilde{x},x)$ is the probability density of leaving Ω at $\tilde{x} \in \partial\Omega$, given $X(0) = x$. For this starting value the expected exit time $T_\varepsilon(x)$ satisfies the Dynkin equation

$$L_\varepsilon T_\varepsilon = -1 \quad \text{in } \Omega, \tag{2.8a}$$

$$T_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{2.8b}$$

3. THE STATIONARY SOLUTION OF THE FOKKER-PLANCK EQUATION

For the solution of the Dirichlet problems (2.6) and (2.8) we follow a method that requires an asymptotic solution of

$$(3.1) \qquad M_\varepsilon \Psi_\varepsilon = 0 \quad \text{with } \Psi_\varepsilon(\hat{x}) = 1.$$

In the WKB-ansatz to this problem, it is assumed that this solution takes the form

$$(3.2) \qquad \Psi_\varepsilon(x;\varepsilon) = w(x;\varepsilon)e^{-Q(x;\varepsilon)/\varepsilon^2}.$$

Substitution in (3.1) yields after equating the leading order terms

$$(3.3) \qquad \tfrac{1}{2} \sum_{ij} a_{ij} \frac{\partial Q_0}{\partial x_i} \frac{\partial Q_0}{\partial x_j} + \sum_i b_i \frac{\partial Q_0}{\partial x_i} = 0, \quad Q_0(\hat{x}) = 0,$$

the so-called eikonal equation. The next order terms give a transport equation for the function $w(x;\varepsilon)$:

$$(3.4a) \qquad \sum_{ij} (a_{ij}) \frac{\partial Q_0}{\partial x_j} + b_i) \frac{\partial w_0}{\partial x_i} + (\tfrac{1}{2} \sum_{ij} a_{ij} \frac{\partial^2 Q_0}{\partial x_i \partial x_j} + 2 \frac{\partial a_{ij}}{\partial x_i} \frac{\partial Q_0}{\partial x_j} +$$

$$(3.4b) \qquad \sum_i \frac{\partial b_i}{\partial x_i})w_0 = 0, \qquad w_0(\hat{x}) = 1.$$

The left-hand side of the eikonal equation can be interpreted as a Hamiltonian $H(x,p)$ with $p_i = \partial Q_0/\partial x_i$. The associated system of bicharacteristics reads in our case with $a_{ij} = \delta_{ij}$:

$$(3.5a) \qquad \frac{dx_i}{ds} = p_i + b_i,$$

$$(3.5b) \qquad \frac{dp_i}{ds} = -\sum_j p_j \frac{\partial b_j}{\partial x_i}$$

with parameter s defined along the characteristics. The projection of the bicharacteristics on the s-space are called rays.

The function $Q_0(x)$ satisfies the equation

$$\frac{dQ_0}{ds} = \tfrac{1}{2} \sum_i p_i^2 \geq 0. \tag{3.6}$$

Starting on a small sphere around $\hat{x}$ we obtain a bundle of rays. A point in Ω within this bundle is uniquely determined by the value $s = \theta_1$ and the angular variables $\theta_2,\ldots,\theta_n$ denoting a point on the sphere. It can be shown (see [6]), that the Jacobian J of the transformation $x \to \theta$ satisfies

$$\frac{d}{ds}(\log w_0^2 J) = - \operatorname{divb}(x), \tag{3.7}$$

$$J = |q_{ij}|, \quad q_{ij} = \frac{\partial x_i}{\partial \theta_j}. \tag{3.8}$$

Thus, the values of Q_0 and w_0 can be evaluated by integrating the system (3.5) - (3.7) together with the equations for q_{ij}. Starting values for Q_0 and w_0 on the small sphere around $\hat{x}$ follow from substitution of local expansions for Q_0 and w_0 into the eikonal and transport equation.

Carrying out this program of numerical integration for a system with $n = 3$, we met the following difficulties, see [5]:

- It is quite laborious to generate the starting values; for $n > 3$ other methods such as automatic formula manupulation have to be utilized.
- A regular grid of starting values on the sphere yields rays that intersect the boundary $\partial\Omega$ very sparce near singular points of deterministic sytem that are located on $\partial\Omega$. However, as Q_0 takes its minimal value at such a singular point (and values close to this minimum in a neighborhood of it), it is almost impossible to produce accurate values for w_0 and Q_0 at points of $\partial\Omega$, which are of fundamental importance for the asymptotic solution of the Dirichlet problems (2.6) and (2.8).
- The shooting method slightly improved the result at the cost of a considerable amount of computing time.

These difficulties in evaluating numerically values for w_0 and Q_0 at the boundary near a singular point on $\partial\Omega$ are due to the fact that the asymptotic solution (3.2) is not valid at a turning point of the system (3.1). Thus, our method for constructing a solution for the full domain Ω cannot be correct. It is quite well possible that it produces a good asymptotic approximation for ε small, although the limit $\varepsilon \to 0$ cannot be taken. An example of a reasonably good approximation for a similar problem is given

in BOBROVSKY and SCHUSS [1]. MATKOWSKY, SCHUSS and TIER [8] solve the problem by introducing an additional boundary layer solution for Ψ_ε at $\partial\Omega$. In this paper we handle the fundamental difficulty of the turning point behavior modifying the domain Ω slightly.

4. ASYMPTOTIC SOLUTION OF THE DIRICHLET PROBLEMS

For solving asymptotically the Dirichlet problems (1.6) and (2.8) we need the divergence theorem stating that any two functions ϕ and Ψ defined on $\bar{\Omega}$ satisfy the equation

$$(4.1)\qquad \int_\Omega \Psi L_\varepsilon \phi - \phi M_\varepsilon \Psi \, dV =$$

$$\int_{\partial\Omega} \tfrac{1}{2}\varepsilon^2\left(\Psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\Psi}{\partial n}\right) + \Psi\phi \; b.\nu - \varepsilon^2 \Psi\phi \sum_{ij} \frac{\partial a_{ij}}{\partial \nu_i} \, dS,$$

where ν denotes the outward normal on $\partial\Omega$, the conormal derivative is defined by

$$(4.2)\qquad \frac{\partial}{\partial n} = \sum_{ij} a_{ij} \, \nu_i \, \frac{\partial}{\partial x_j}.$$

For the case $b.\nu \le 0$ the singular perturbation method brings about three types of locally valid asymptotic approximations for u_ε and T_ε:
- An outer solution, valid away from the boundaries, with

$$(4.3)\qquad u_\varepsilon \approx C_u, \quad T_\varepsilon \approx C_T.$$

- An ordinary boundary layer solution valied in an ε^2-neighborhood of the boundary, where $b.\nu < 0$, then

$$(4.4a)\qquad u_\varepsilon \approx (f(x_t) - C_u)\, e^{-x_n/\varepsilon^2} + C_u,$$

$$(4.4b)\qquad T_\varepsilon \approx - C_T e^{-x_n/\varepsilon^2} + C_T,$$

where (x_t, x_n) denotes a local coordinate system with the (n-1)-dimensional

vector x_t tangential and x_n perpendicular to the boundary.
- A parabolic boundary layer solution valid in an ε - neighborhood of the boundary, where $b.\nu = 0$, then

$$(4.4a) \qquad u_\varepsilon \approx f(x_t) + (C_u - f(x_t)) \frac{1}{\varepsilon} \sqrt{\frac{2}{\pi}} \int_0^{s(x)} e^{-s^2/2\varepsilon^2} ds$$

$$(4.4b) \qquad T_\varepsilon \approx C_T \frac{1}{\varepsilon} \sqrt{\frac{2}{\pi}} \int_0^{s(x)} e^{-s^2/2\varepsilon^2} ds$$

$$(4.4c) \qquad s(x) = \frac{2}{\sqrt{a_{nn}}} \left\{ \int_0^{x_n} b_n(y, x_t)\, dy \right\}^{1/2}$$

The subscript n to a and b denotes that the component in the direction x_n is taken. Substitution of the asymptotic solution Ψ of section 3 and $\phi = u_\varepsilon$ in (4.1) yields an expression for the unknown constants C_u and C_T. It is noted that for C_T the volume integral vanishes asymptotically. Working out the right hand side one obtains

$$(4.5) \qquad C_u \approx \frac{\frac{2\varepsilon}{\sqrt{\pi}} \int_{\partial\Omega_p} \Psi_\varepsilon \sqrt{c_n a_{nn}}\, f ds + \int_{\partial\Omega_0} \Psi_\varepsilon\, b.\nu\, f\, dS}{\frac{2\varepsilon}{\sqrt{\pi}} \int_{\partial\Omega} \Psi_\varepsilon \sqrt{c_n a_{nn}}\, ds + \int_{\partial\Omega_0} \Psi_\varepsilon b.\, \nu\, dS},$$

where $c_n = b_n(x_n, x_t)/x_n$.
For the nonhomogeneous Dirichlet problem the computations are slightly more complicated, see [11]. It results in

$$(4.6) \qquad C_T \approx \left\{ \frac{\varepsilon}{\sqrt{\pi}} \int_{\partial\Omega_p} \Psi_\varepsilon \sqrt{c_n a_{nn}}\, dS - \tfrac{1}{2} \int_{\partial\Omega_0} \Psi_\varepsilon\, b.\nu\, dS \right\}^{-1} \int_\Omega \Psi_\varepsilon\, dV$$

5. AN EXAMPLE: THE HYPERCYCLE

We consider the dynamical system (2.1) for n = 3 with

$$(5.1) \qquad c_i(x) = k_i x_{i-1} - \sum_{j=1}^{3} k_j x_j x_{j-1}, \quad i = 1,2,3, \quad k = (1,3,5),$$

where the subscripts are taken modulos 3. This generalized Volterra-Lotka system has a cyclic structure with each component being "prey" for one other component. Eigen and Schuster [2] propose this so-called hypercycle as a

canditate for the fundamental process of self-organization as it took place in the early stage of evolution. This system has one stable equilibrium $Z = (z_1,z_2,z_3)$ for $x_i > 0$ with

$$z_i = \frac{1}{k_{i+1}} \Big/ \sum_{j=1}^{3} \frac{1}{k_j} . \tag{5.2}$$

At the boundary there are unstable equilibrium points $X_i = (x_{i1},x_{i2},x_{i3})$ with $x_{ij} = \delta_{ij}$. Globally, a solution with starting value $x_0 = (x_{01},x_{02},x_{03})$, $x_{0i} > 0$ spirals inwards on its way to the equilibrium Z, which is at the invariant surface $\Sigma x_i = 1$.

As in nature many processes are intrinsically stochastic rather than pure deterministic, it is meaningful to consider the influence of small stochastic perturbations. Although the diffusion matrix σ_{ij} of (2.2) may depend upon X, we work out the simplest case with $\sigma_{ij} = \delta_{ij}$. For the domain $\Omega = \{x | x_i > 0\}$ the numerator of (4.6) is evaluated asymptotically:

$$\int_{\Omega} \Psi_\varepsilon dV \approx \frac{(2\pi\varepsilon^2)^{3/2} \Psi_\varepsilon(Z)}{H(Z)^{1/2}} \approx 28.2\,\varepsilon^3, \tag{5.3}$$

where H(Z) denotes the Hessian of $Q_0(x)$ at $x = Z$. In the denominator only the first integral has to be taken. The largest contribution comes from points at the boundary where Q_0 takes the smallest values. They form an ε-neighborhood of the line L connecting $X_1 = (1,0,0)$ with $Z_2 = (z_1,0,z_3)$. At the boundary the minimal value K of $Q_0(x)$ is found at X_1. Along L, Q_0 remains almost at this value until the point Z_2 is reached. Since on L w_0 is zero at X_1 and at its maximum near Z_2, the maximum of Ψ will be on L between Z_2 and X_1 and will shift towards X_1 as $\varepsilon \to 0$. However, since the slope of Q_0 is extremely small, the maximum is still far away from X_1 for ε values of about 10^{-5}. As mentioned in section 3, it is not to be expected that the approximation of Ψ_ε holds near X_1. Let us ignore this fact and carry out the integration

$$C_T(\varepsilon) \approx \frac{28.2\varepsilon^3 e^{K/\varepsilon^2}}{\frac{\varepsilon}{\sqrt{\pi}} \int_0^\infty \int_0^\infty w_0 e^{(K-Q_0)/\varepsilon^2} \sqrt{c_n}\; dx_2 dx_3} \tag{5.4}$$

or

$$(5.5)\qquad C_T(\varepsilon) \approx \frac{28.2\varepsilon^3 e^{K/\varepsilon^2}}{2.6\varepsilon^3 I(\varepsilon)}, \quad K = .0038\ (\pm 2\%),$$

where

$$(5.6)\qquad I(\varepsilon) = \frac{1}{6\varepsilon}\int_{5/23}^{1} w_0 e^{(K-Q_0)/\varepsilon^2}\sqrt{6(1-x)(18-5x)}\,dx$$

Notice that $I(\varepsilon)$ tends to zero for $\varepsilon\to 0$, as Q_0 is minimal in X_1, where w_0 vanishes. Because of the small change in Q_0, $I_0(\varepsilon)$ is close to 1 for $\varepsilon \approx 2^{-5}$, see table I. Writing

$$(5.7)\qquad C_T(\varepsilon) = D_T(\varepsilon)e^{K/\varepsilon^2}$$

we obtain values for $D_T(\varepsilon)$ as given in table I. Comparing this result with Monte Carlo simulations, as carried out in the next section, we observe an excellent agreement for the exponential term. The multiplying constant D_T differs by a factor 3.

ε	$I(\varepsilon)$	$D_T(\varepsilon)$	$\bar{x}_1$
2^{-4}	0.78	14.1	.30
2^{-5}	1.06	10.3	.38
2^{-6}	0.96	11.5	.46

Table I. Values for expected exit times, see (5.7). The last column gives the x_1-coordinate of the point on the boundary where Ψ_ε has its maximum on L.

Because of the restriction in the validity of the asymptotic solution Ψ_ε at X_1, we now consider the exit problem for a domain $\Omega^{(\delta)} \subset \Omega$. This domain is supposed to have the following properties:

(a) $\Omega^{(\delta)} \to \Omega$ as $\delta \to 0$,

(b) $b.\nu \le 0$ on $\partial\Omega^{(\delta)}$,

(c) the x_i axes $(i=1,2,3)$, where Ψ_ε has turning point behavior, are excluded.

Note that in a domain $\Omega^{(\delta)}$ with $b.\nu > 0$ somewhere on $\partial\Omega^{(\delta)}$ the Dirichlet problem (2.8) will have internal boundary layers. We choose a boundary with $b.\nu \leq 0$ such that a good estimate is obtained for the region where the method for the full domain Ω breaks down. The section of $\partial\Omega^{(\delta)}$ with $b.\nu<0$ is chosen near L somewhere between the point where Ψ_ε has its largest value and the singular point X_1. In fig. 1 we sketch the cross-section of $\Omega^{(\delta)}$ with the plane $x_1 = .5$, where locally the boundary with $b.\nu<0$ is formed by the plane $x_3 = .5$. Construction of a domain $\Omega^{(\delta)}$ with the required properties is always possible. The cross-sections with the planes $\Sigma x_i = c$, $0 < c<\infty$ should be taken as depicted in fig. 2. For the region $\Omega^{(\delta)}$ the second integral in the denominator of (4.6) has also to be evaluated to obtain $C_T^{(\delta)}$. One has to integrate in the plane $x_3 = .5$ over an area of order $O(\varepsilon)$ in the x_1-direction and of order $o(\varepsilon^2)$ in the x_2-direction at the nearest point to L. For δ sufficiently small the first integral in the denominator can be approximated by the one over $\partial\Omega$ we evaluated before. Consequently we have

$$C_T^{(\delta)}(\varepsilon) = D_T^{(\delta)}(\varepsilon) e^{K/\varepsilon^2}, \tag{5.8}$$

where $D_T^{(\delta)}(\varepsilon) = 3.1$ with an accuracy of about 25%.

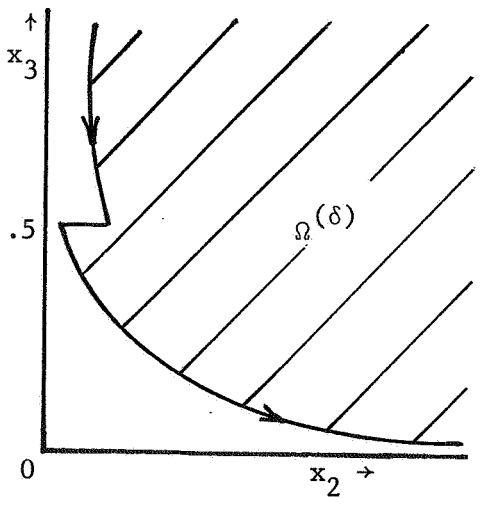

Fig. 1 The cross-section of $\Omega^{(\delta)}$ with the plane $x_1 = .5$.

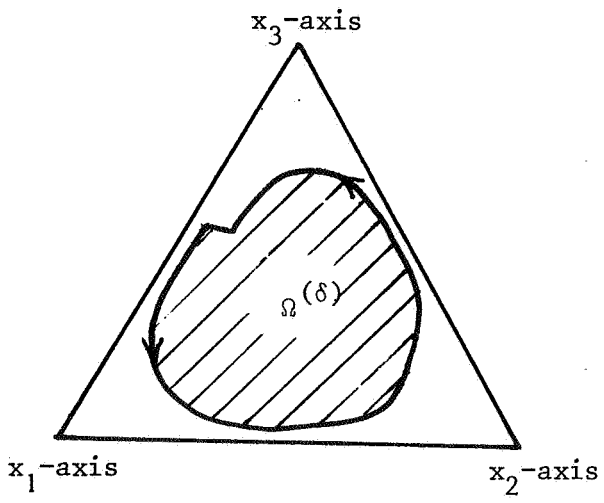

Fig. 2 Cross-section of $\Omega^{(\delta)}$ with a plane $\Sigma x_i = c$, $c > 0$.
i

6. STOCHASTIC DIFFERENCE EQUATIONS

Dynamical systems with small random perturbations can be simulated with the Monte Carlo method. To perform the simulation, the Wiener process W(t) has to be replaced by a pseudo random generator G(t). Euler's method can then be applied to the following stochastic difference equations:

$$(6.1) \qquad x_i(t+h) = x_i(t) + hb_i(x) + \varepsilon\sqrt{h}\, G_i(t), \quad i = 1,\ldots,n.$$

The time step h gives an error in x of order O(h). Define the new stochastic variable $\Delta x_i(t)$, i = 1,...,n:

$$(6.2) \qquad \Delta x_i(t) = x_i(t+h) - x_i(t).$$

This variable has first and second moments

$$(6.3) \qquad E\Delta x_i(t) = hb_i(x) + \varepsilon\sqrt{h}\, EG_i(t) = hb_i(x)$$

$$(6.4) \qquad \text{Var } \Delta x_i = \varepsilon^2 hEG_i^2(t) = \varepsilon^2 h.$$

Consequently in unit time the expectation of Δx_i equals the local vector field b(x) while variance in unit time equals ε^2.For a stepsize h = .03, the hypercycle system of section 5 has been simulated in this way. The average exit times $T_{sim}(Z)$ of 200 runs for different values of ε are given in table II. This average exit time is compared with the asymptotic formula $T_\varepsilon(Z)$ for a domain Ω and $\Omega^{(\delta)}$. It is noted that the expected exit time for the modified domain $\Omega^{(\delta)}$gives a better approximation to the exit problem of the full domain than the asymptotic approximation for the full domain Ω. Fitting the data of the simulations with a curve of the form (5.7) we find D_T = 3.2 and K = .0040, which agrees very well with (5.8).

ε	$T_\varepsilon(Z)$	$T_\varepsilon^{(\delta)}(Z)$	$T_{sim}(Z)$
2^{-1}	-	3.15	.29
2^{-2}	-	3.29	.79
2^{-3}	-	3.47	2.5
2^{-4}	37.3	8.20	10
2^{-5}	504.	152	186

Table II. Values from asymptotic expressions and average values from simulations for the expected exit time.

7. CONCLUDING REMARKS

Asymptotic solution of the exit problem for a domain of attraction Ω with a stationary point of the unperturbed system at its boundary brings about the difficulty of finding an appropriate asymptotic solution for the stationary Fokker-Planck equation. In [8] this problem was handled by introduction of a boundary layer solution. We resolved the difficulty by modifying the domain Ω slightly. From the present result it is concluded that the boundary layer is necessary as the asymptotic solution for Ω without the layer yields asymptotic exit times that differ from the ones for the modified domain $\Omega^{(\delta)}$ as $\delta \to 0$. The latter case agrees quite well with values obtained from simulations.

In this paper we investigated dynamical systems originating from theoretical population biology. In order to study the effect of intrinsic stochastisity one has to include the possibility that the diffusion coefficients vanish at the boundary. This introduces a second limitation in the applicability of the WKB-ansatz for the stationary Fokker-Planck equation. Further investigations should be directed to the solution of this problem. Apart form the boundary layer approach in [8], one should also explore the possibility of taking a function Ψ_ε, that does not necessarily satisfy the Fokker-Planck equation. This function must be chosen such that it is possible to integrate numerically over the volume Ω in the divergence theorem.

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