STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1964-010

Valuation Spaces

J. Neggers

October 1964

Printed at the Mathematical Centre at Amsterdam,49,2nd Boerhaavestraat. The Netherlands.

The Mathematical Centre, founded the 11th of February 1946, is a non - profit institution aiming at the promotion of pure mathematics and its applications, and is sponsored by the Netherlands Government through the Netherlands Organization for Pure Scientific Research (Z.W.O.) and the Central National Council for Applied Scientific Research in the Netherlands (T.N.O.), by the Municipality of Amsterdam and by several industries.

Introduction.

The following note is not really a research or progress report as even a casual glance at the material will clearly indicate. It might properly be termed 'an observation' since about the only thing which is accomplished is to make a few definitions, observe a few items and raise some questions. From the observations which have been made, however, it seems that the notions of 'valuation-manifold' and 'algebraically defined valuation-manifold' could perhaps be termed 'useful' in that they include a large class of mathematical objects under one heading and that the objects in this class seem to have enough in common to eventually allow for a homogeneous treatment both along topological and algebraic lines. Just how much of the material included is already known in one form or another I must confess I don't know since the various techniques on which one apparently can draw if desired are not all or not at all, whichever the case may be, common to my native territory of investigation. If thus offer no bibliography since as of yet the only bibliography I have is what I've read and that one can find on most anybody's shelf.

Valuation spaces.

Suppose G is an arbitrary ordered abelian group and suppose X is an arbitrary topological space, we define a valuation V on X as a mapping: $V: X \to G$ such that V is an open map, i.e., if U is an open set in X, then V(U) is an open set in G, where G is assumed to have the valuation topology.

We refine the ordinary topology on X by taking as open sets the sets $V^{-1}(U^*)$, where U^* is open in G. Then the requirement that V be open implies that the new topology is at least as fine as the original topology. We shall refer to this topology on X as the V-topology.

If X possesses a valuation $V: X \to G$ such that V(X) is a homeomorphic image of X with the V-topology, then we shall call X a 'valuation space'. Thus e.g. if $X = E^2$ with the ordinary metric topology and if G is the ordered abelian group of all plane vectors (x,y), where $(x_1,y_1) \leq (x_2,y_2)$ if and only if $y_1 \leq y_2$ or $y_1 = y_2$ and $x_1 \leq x_2$, then the identity mapping becomes a valuation $I: E^2 \to G$ such that under the I-topology $I(E^2) = G$ is a homeomorphic image of E^2.

We shall call a valuation V such that X becomes a 'valuation space' in the V-topology a regular valuation on X.

If V is a regular valuation $V: X \to G$, then the rank of the smallest isolated subgroup Γ of G such that $V(X) \subset \Gamma$ will be called the rank of X with respect to the regular valuation V, denoted by $\text{rank}_V(X)$.

If X possesses a regular valuation V such that $\text{rank}_V(X) < \infty$, then we'll say X has finite rank and we define the rank of X as the minimum of the integers $\text{rank}_V(X)$, where V ranges over all regular valuations on X such that $\text{rank}_V(X) < \infty$.

It is clear that rank(X) is a topological invariant, i.e., if spaces X and Y are homeomorphic and if X is a valuation space with rank(X) = n, then Y is a valuation space and rank(Y) = n.

Notice that rank and dimension are not alwaysthe same. Thus if in E' we consider the integers Z, then Z has dimension 0 in E', however, any regular valuation V: Z → G always has the property that $rank_V(Z) \geq 1$ and thus since I: Z → Z is a rank 1 valuation it follows that rank(Z) = 1.

Proposition 1:

If X and Y are valuation spaces, then X × Y is a valuation space. Furthermore if X and Y have finite rank then X × Y has finite rank and $rank(X \times Y) \leq rank(X) + rank(Y)$.

Proof:

Suppose $V_1: X \to G_1$, $V_2: X \to G_2$ are regular valuations on X and Y respectively. Then construct $G = G_1 \times G_2$ an ordered abelian group with $(\gamma_1,\eta_1) + (\gamma_2,\eta_2) = (\gamma_1+\gamma_2,\eta_1+\eta_2)$ and $(\gamma_1,\eta_1) \leq (_2, _2)$ if and only if $\eta_1 \leq \eta_2$ or $\eta_1 = \eta_2$ and $\gamma_1 \leq \gamma_2$. Then V: X × Y → G defined by $V(x,y) = (V_1(x),V_2(y))$ is a regular valuation and furthermore if G_1 and G_2 are chosen such that rank (G_1) = rank(X), rank(G_2) = rank(Y), then $rank(X \times Y) \leq rank_V(X \times Y) = rank(G) = rank(G_1) + rank(G_2) =$
$= rank(X) + rank(Y)$.

Example:

Suppose X = Y = Z, then rank(X) = rank(Y) = 1 and $rank(X \times Y) \leq 2$.
Let V: Z × Z → Z be defined by
$V(n,m) = 2^n 3^m$ for $n,m \geq 0$, $V(-n,m) = 5^n 3^m$, $V(n,-m) = 2^n 7^m$ and
$V(-n,-m) = 5^n 7^m$, then V is a regular valuation and $rank(Z \times Z) = 1 < 2$.

If X is a 'valuation space' of finite rank, then for any regular valuation V: X → G such that $rank_V(X)$ = rank(X) = rank G . We may have subgroups G^* of G such that $rank(G^*)$ = rank(G) and regular valuations $V^*: X \to G^*$.

We'll say that G is a character-group for X if $G^* \subset G$ and V: X → G, $V^*: X \to G^*$ implies there is a mapping $\sigma: G \to G^*$ such that σ is an order preserving linear homeomorphism.

Example:

The 'valuation spaces' Z and Z x Z have a character-group Z since any non-zero ordered abelian subgroup of Z is an image of Z under an order preserving linear homeomorphism.

Question I:

Are character-groups 'unique', i.e., if G_1 and G_2 are character-groups for X then is there an order preserving linear homeomorphism $\sigma: G_1 \to G_2$?

Question II:

Suppose V: $X \to G$ is a regular valuation such that $\text{rank}_V(X) = \text{rank}(X) = \text{rank}(G)$ and suppose that $[V(X)]$ is the group generated by V(X), is $[V(X)]$ a character-group for X?

Question IIa:

Does every 'valuation space' of finite rank have character-groups?

Remark:

If G is a character-group for X and if V: $X \to G$ is a regular-valuation, then $[V(X)]$ and G have the property that there is an order-preserving linear homeomorphism between $[V(X)]$ and G.

Also in this case it is true that $[V(X)]$ is a character-group since any subgroup of $[V(X)]$ is also a subgroup of G.

Suppose that X is a 'valuation space' and that $X \cong X_1 \times \ldots \times X_r$ where each X_i is a 'valuation space', then

$$\sum_{i=1}^{n} \text{rank}(X_i) - \text{rank}(W) \geq 0 .$$

Suppose that m is the supremum of the values which can so occur, then we define m as the co-rank of X.

We define the dimension of X by the relation:

$$\dim(X) = \text{rank}(X) + \text{co-rank}(X)$$

if both the rank and the co-rank of X are finite.

Example:

1. Suppose $H = \{(x,y) \in E^2 \mid x = \frac{1}{n}, \text{ n an integer} > 0\}$.
   Then rank(H) = 1. Also $H \cong Re \times W$ and rank(X) = rank(Y) = 1. Thus co-rank(H) $\geqslant$ 1.
   Since this is essentially the only direct-product decomposition of H which occurs we have

   $$\dim(H) = \text{rank}(H) + \text{co-rank}(H) = 1 + 1 = 2.$$

2. Suppose we consider E^2 itself, E^2 is a valuation space with $\text{rank}(E^2) = 2$.
   Also $E^2 \cong Re \times Re$ and

   $$\text{rank}(E^2) = 2 = \text{rank}(Re) + \text{rank}(Re)$$

   Hence

   $$\dim(E^2) = \text{rank}(E^2) + \text{co-rank}(E^2) = 2 + 0 = 2.$$

   Here we have made use of the fact that Re is indecomposable in the sense that Re is not the direct product of any other pair of spaces, i.e., $Re \cong X \times Y \Rightarrow X \cong Re, Y = \{a\}$, hence rank(Re) = rank(X) + rank(Y) = 1 + 0 = 1.

We remark in addition that co-rank is a topological invariant and thus dimension is a topological invariant as well.

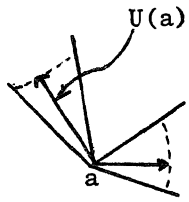
Now say that a topological space X <u>is a valuation-manifold</u> if each point x X is contained in an open neighbourhood U(x) such that U(x) is a 'valuation space'.

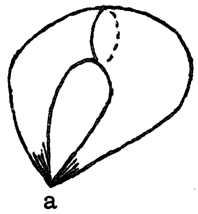
Example:

1. Any n-manifold is a valuation manifold, as a matter of fact a ball neighbourhood or a half-ball neighbourhood is a rank n , co-rank 0 ,

dimension n 'valuation space'.

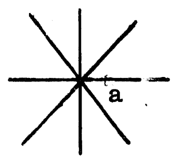
2. The pinched torus



X = 

If we give X a line topology in the neighbourhood of the pinching point, then such a U(a) as indicated above is a rank 1 'valuation space', using the ordinary topology 'away from a', we observe that around every point other than a we have dim 2 neighbourhoods, while 'around a' we only have dim 1 and dim 0 neighbourhoods which are also valuation spaces.

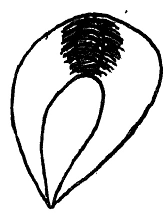
3. The star

X = 

Again if we give X a line topology, then in the neighbourhood of a we obtain both dim 1 and dim 0 neighbourhoods.

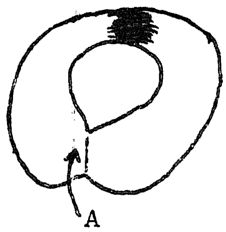
However if we consider $X \setminus a$, then we have $X \setminus a$ a 'valuation space' of dimension 2, hence every point other than a has a dimension 2 neighbourhood.

4. The solid pinched torus

X = 

Here every point different from a has dimension 3 neighbourhoods, while a possesses only dim 1 and dim 0 neighbourhoods.

5. The solid squeezed torus

X = 

Here every point not in A has dimension 3 neighbourhoods while every point in A has dim 2 and dim 1 neighbourhoods.

We note that examples 2-5 are examples of valuation manifolds which are not manifolds.

If X is a valuation manifold and $x \in X$, define:

(i) dim(x) = sup dim(U(x)), U(x) a 'valuation space'.

(ii) dim(X) = sup dim(x), $x \in X$.

(iii) pinching degree at x = dim(X) - dim(x) = p(x).

(iv) pinching degree of X = sup p(x), $x \in X$ = p(X).

Thus the pinched torus, the star, the solid squeezed torus have pinching degree 1, while the solid pinched torus has pinching degree 2.

We may also define:

(v) rank(x) = sup rank(U(x)), U(x) a 'valuation space'.

(vi) rank(X) = sup rank(x), $x \in X$.

(vii) co-rank(x) = sup co-rank(U(x)), U(x) a 'valuation space'.

(viii) co-rank(X) = sup co-rank(x), $x \in X$.

We have the relations: rank(x)   dim(x), rank(X)   dim(X)

and

$$
\begin{aligned}
\dim(X) &= \sup \dim(x) = \sup (\sup \dim U(x)) \\
&= \sup (\sup (\operatorname{rank} U(x) + \text{co-rank}\, U(x))) \\
&\leq \sup (\sup \operatorname{rank} U(x) + \sup \text{co-rank}\, U(x)) \\
&\leq \sup (\operatorname{rank}(x) + \text{co-rank}(x)) \\
&\leq \sup (\operatorname{rank}(x)) + \sup (\text{co-rank}(x)) \\
&\leq \operatorname{rank}(X) + \text{co-rank}(X).
\end{aligned}
$$

Now suppose that question IIa can be answered in the positive. Let X be an arbitrary valuation manifold, $\dim(X) = n < \infty$, and let $x \in X$. If we consider the collection of all 'valuation neighbourhoods' of x such that they have rank equal to the rank at x, then we obtain a family F_x of characteristic groups, one associated with each neighbourhood U(x). Every group in F_x has rank equal to rank(x), if neighbourhoods $U_1(x)$ and $U_2(x)$ are homeomorphic these character-groups can be chosen identical. Say that x is regularly embedded in X if ordering the neighbourhoods U(x) by set-inclusion the family F_x has a direct limit. Let G_x be this direct limit. Then define the characteristic-group at x as G_x, the characteristic-sheaf of X as the collection $\{x; G_x\}_x$ regularly embedded in X.

Example:

1. Let X be the pinched torus, and suppose $x \in X \setminus a$, then F_x consists of essentially but one group, viz., Re $\times$ Re considered as an ordered abelian group. Hence x is regularly embedded and G_x = Re $\times$ Re. If we consider a itself, then we only have line neighbourhoods which are also 'valuation spaces' hence F_a consists essentially of Re and again a is regularly embedded with G_a = Re.

2. In the star we get G_x = Re ($X \setminus a$ has rank 1 !) and G_a = Re.

Thus in particular we have a valuation manifold with a pinch, hence not a manifold, with characteristic sheaf constant.

In the spaces discussed above the different quality of the pinches in each case seem to be reflected in the fact that the pinched torus has co-rank 0 while the star has co-rank 1.

Question III:

Is there a valuation-manifold X with $p(X) > \sigma$ such that co-rank(X) = 0 and X has a constant characteristic sheaf?

Question IV:

If we have a connected rank 1 valuation-manifold X with co-rank 1 then

(a) is G_x = Re for all $x \in X$?

(b) is X locally 'at worst' a star?

If we consider the pinched torus X, then $\partial(X) = \emptyset$, however, we might consider the point a as a sort of quasi-boundary in that if we delete a, then X becomes a manifold and $\partial(X) = a$.
In the same way the solid squeezed torus becomes a 3-manifold if we delete the pinching set A, with boundary the 'outside of the squeezed torus'. Although the pinched torus and the boundary of the solid squeezed torus are not homeomorphic they do have the following in common:

1. Fundamental Group. In both spaces it is true that anly loop around the torus itself can be homotopically contracted to a point on the pinch a, the squeeze A respectively. Hence both have fundamental groups Z.

   They differ in that in the pinched torus any two non-trivial loops have a common point, while in the squeezed torus this is no longer true.

2. Dimension and Rank are in both cases 2, but the solid pinched torus has $p(X) = 1$, while the boundary of the squeezed torus has $p(X) = 0$.

We define the quasi-boundary of X as the set of all points $x \in X$ such that $p(x) > 0$, denoted by $\partial^*(X)$. For the quasi-boundary operator ∂^* we do not in general obtain $\partial^*\partial^* = 0$ but if $p(X) < \infty$ then we do know that $\partial^{*(n)} = 0$ for some integer $n \leq p(X)$.

We observe that since $p(x)$ and $P(X)$ are topological invariants the degree of nilpotency of ∂^* is also an invariant.

Question V:

If X and Y are homeomorphic valuation-manifolds

(a) $\partial^*(X)$ and $\partial^*(Y)$ homeomorphic?

(b) $\partial^*(X)$ and $\partial^*(Y)$ valuation-manifolds?

Question VI:

If X and Y are homeomorphic valuation-manifolds are $X \setminus \partial^*(X)$ and $Y \setminus \partial^*(X)$

(a) homeomorphic?

(b) valuation-manifolds?

We note that $\partial^*(X)$ is not always contained in $\partial(X)$. Indeed if X is the star then $\partial(X) = \emptyset$ and $\partial^*(X)$ is the intersection point of the lines. We note that in all examples given $\partial^*(X)$ is an open set in the local V-topology.

Question VII:

(a) Is $\partial^*(X)$ always open in the local V-topology?

(b) Are there valuation-manifolds X such that X contains pinch points x which are interior points in the original topology?

(c) Are there valuation-manifolds X such that $\partial^*(X)$ is open in the original topology?

Question VIII:

We noticed that the solid pinched torus is a valuation manifold with interior 3-cell. Is the boundary of a 3-cell always a valuation-manifold?

If X is an arbitrary topological space and $V\colon X \to G$ a valuation on X, then we can form a chain of spaces

$\{X_\alpha\}_{\alpha \in G}$, where $X_\alpha = \{x \in X \mid V(x) > \alpha\}$.

In general if we consider $\bigcap_{\alpha \in G} X_\alpha$, then $\bigcap_{\alpha \in G} X_\alpha = \emptyset$, however, suppose we adjoin an ideal element x^* such that $\bigcap_{\alpha \in G} X_\alpha = \{x^*\}$, then for any map $\sigma: X \cup \{x^*\} \to X \cup \{x^*\}$ such that $\sigma(X_\alpha) \subset X_\alpha$ we have immediately that $\sigma(x^*) = x^*$.

Example:

Suppose R^* is a valuation ring, $R = R^* \setminus \{0\}$ becomes a topological space with a valuation and we have that in R^* $\bigcap_{\alpha \in G} R_\alpha = \{0\}$.

Thus we have as immediate consequence that in a valuation ring any mapping which does not decrease the value has a fixed point, viz., $\{0\}$.

On a topological space X with a valuation we can construct an equivalence relation by letting $x \sim y$ if and only if $V(x) - V(y) \in S \subset G$, where S is some subgroup of G.

Then constructing the space X / S, we have that if G / S is an ordered abelian group, then the mapping

$$V / S : X / S \to G / S \quad \text{defined by}$$

$$V / S([x]) = V(x) + S \quad \text{is a valuation.}$$

Furthermore it is true that V / S is a 'regular valuation' for X / S.

Thus if X is any topological space with a valuation V such that $\text{rank}_V(X) \geqslant 1$, then there is an equivalence relation $\sim$ on X with the property that $\text{rank}_{V/\Gamma_1}(X / \Gamma_1) = \text{rank}_V(X) - 1$, and V / Γ' a regular valuation on X / Γ_1 making X / Γ_1 into a 'valuation-space'.

Now on the 'valuation-spaces' we have notions of rank, co-rank and dimension, thus if X is any topological space we get an associated family of 'valuation spaces' $\{X_\lambda\}_{\lambda \in \Lambda}$ through the construction made above. This family of 'valuation spaces' and all the associated paraphernalia discussed above is itself a 'topological invariant' of the space X.

Example:

Suppose R^* is a rank 2 valuation ring with valuation V: $R = R^* \setminus \{0\} \to G$ such that $V(xy) = V(x) + V(y)$, $V(x+y) \leq V(x) + V(y)$. Then if $R_1 = \{x \mid V(x) \in \Gamma_1\}$, it follows immediately that $R_1 \cup \{0\}$ is a subring of R^* with a rank 1 valuation $V \mid R_1$. If we assume that R^* is a domain and extend to its quotient field K^*, then K^* is a purely transcendental extension of the field K_1, the quotient field of $R_1 \cup \{0\}$. Picking a tr. basis of K^* over K_1, then assume K^* is the collection of all rational expressions $p(x_\alpha) / q(x_\alpha)$, $p(x_\alpha) \in R_1[x_\alpha]$, $q(x_\alpha) \in R_1[x_\alpha]$.

Now letting $p_1(x_\alpha) / q_1(x_\alpha) \sim p_2(x_\alpha) / q_2(x_\alpha)$ if and only if

$$p_1(x_\alpha) / q_1(x_\alpha) - p_2(x_\alpha) / q_2(x_\alpha) \in K_1.$$

We have $\quad p_1(x_\alpha)q_2(x_\alpha) - q_1(x_\alpha)p_2(x_\alpha) / q_1(x_\alpha)q_2(x_\alpha) \in K_1.$

Thus $p_1(x_\alpha)q_2(x_\alpha) - q_1(x_\alpha)p_2(x_\alpha) + k_1 q_1(x_\alpha)q_2(x_\alpha) = 0$, for some $k_1 \in K_1$.

In $R_1[x_\alpha]$ we obtain: $\quad p_1(x_\alpha)q_2 - q_1 p_2(x_\alpha) + k_1 q_1 q_2 = 0.$

Thus $p_1(x_\alpha)$ and $p_2(x_\alpha)$ have the property that if $p_1(x_\alpha)$ and $p_2(x_\alpha)$ have the same monomial of minimal value (the $V(x_\alpha)$ are rationally independent over Γ_1) then they are equivalent and conversely.

We have thus obtained a large class of algebraically defined valuation spaces.

Question IX:

Characterize the 'valuation spaces' which can be 'algebraically defined', where with algebraically defined we mean the following:
"A valuation space X is algebraically defined if there is a valuation ring R^* such that on $R^* \setminus \{0\}$ there is a valuation V such that $R^* \setminus \{0\} / \Gamma_1$ is homeomorphic to X '.

Notice that if we have an 'algebraically defined valuation-manifold X ' then at each point $x \in X$ we have neighbourhoods U(x) which are 'algebraically defined valuation spaces', i.e., with each point $x \in X$ we can associate families of local rings, viz., the ones obtained from the 'algebraically defined valuation spaces at x '. Thus in this case we can perhaps get sheaf structures of local rings on these manifolds, which serve as coefficient domains for the elements in the neighbourhood of x which are expressible as polynomials or power-series with coefficients from these local rings.

Thus in an algebraically defined valuation-manifold we have a notion of 'configuration' at a point $x \in X$, where the 'configuration at x ' is the collection of all polynomial or power-series expressions with coefficients in the local-rings obtainable at x.

In particular suppose X is a valuation-manifold with the property that we can obtain a constant sheaf of local rings over X, then we have immediately that every element $x \in X$ possesses a polynomial or power-series expansion with coefficients in a single local-ring. If R^* is this local ring and $\{X_\alpha\}$ is a suitable set of transcendental elements so that x can be expressed as a polynomial or power-series in the $\{X_\alpha\}$, then the analytic automorphisms on $R^*[X_\alpha]$ will introduce a class $\{X\ \}$ of valuation-manifolds homeomorphic to X and we obtain a duality:

$$X \longleftrightarrow R^*[X_\alpha]$$

study of homeomorphism group	Galois theory on $R^*[X_\alpha]$.

<u>Question X</u>:

Can a full dualism between the study of the homeomorphism group and a Galois theory be obtained under these circumstances?

Typ: RMW