STICHTING
MATHEMATISCH CENTRUM
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ZW 1964-003

(TN 36)

On the connection between the arithmetico-geometrical mean and the complete elliptic integral of the first kind

by

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(Voordracht in de serie "Actualiteiten")

February 1964

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Introduction

Subject of this note is the relation between the arithmetico-geometrical mean and the complete elliptic integral of the first kind.

Let a and b be two positive numbers, let the series $\{a_n\}$ and $\{b_n\}$ be defined by the recurrent relations:

$$(1)\qquad a_{n+1}=\frac{a_n+b_n}{2} \quad\text{and}\quad b_{n+1}=\sqrt{a_n b_n},$$

with $a_0=a$, $b_0=b$;
without loss generality it may be assumed that $a \geqq b$.
As can be easily seen, both series converge to the same limit, denoted by $M(a,b)$ and called the arithmetico-geometrical mean of a and b.

The complete elliptic integral of the first kind is defined as:

$$(2)\qquad K(k)=\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-\frac{1}{2}}d\phi.$$

In the first section we derive the relation

$$(3)\qquad \int_0^{\pi/2} (a_0^2\sin^2\phi+b_0^2\cos^2\phi)^{-\frac{1}{2}}d\phi=\int_0^{\pi/2} (a_1^2\sin^2\phi+b_1^2\cos^2\phi)^{-\frac{1}{2}}d\phi$$

by aid of a possibly new method. This method is based on potential theory and is due to the late prof. B. van der Pol. It may be remarked, that this relation is also a direct consequence of Landen's transformation, applied to the left-hand side of (3).

By aid of (3) we can easily establish the relation between the arithmetico-geometrical mean and the complete elliptic integral of the first kind, viz.

$$(4)\qquad \frac{\pi}{2M(a,b)}=\frac{1}{a}K\left(\sqrt{1-\frac{b^2}{a^2}}\right)=\int_0^{\pi/2} (a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{2}}d\phi.$$

An important consequence of this relation is, that the computation of the complete elliptic integral of the first kind can be very easily performed, since the convergence of the series $\{a_n\}$ and $\{b_n\}$

is extremely good (see [2]).

In section 2 we derive by aid of formula (4) the limit expression

$$\lim_{\varepsilon\to 0} M(1,\varepsilon)\ln\frac{4}{\varepsilon} = \frac{\pi}{2}\,. \tag{5}$$

Although all results are well-known (see Gauss [1], and Schlesinger [4]), the treatment may be new.

1. <u>The relation between the arithmetico-geometrical mean and the complete elliptic integral of the first kind.</u>

We consider an infinitely thin circular ring, with a uniform distribution of mass of unit density and lying in the plane z=0 of a Cartesian coordinate system (x,y,z). When the radius of the ring equals R, the points of the ring lie at the circle $x^2+y^2=R^2$.
The potential in an arbitrary point P(x,y,z) is an axially symmetric function and is given by the formula

$$u(r,z)=\int_0^{2\pi} (R^2+r^2+z^2-2Rr\cos\phi)^{-\frac{1}{2}}d\phi, \tag{1.1}$$

with $r=(x^2+y^2)^{\frac{1}{2}}$.
In particular, for the potential at points of the plane z=0 we obtain after some trivial substitutions

$$u(r,0)=4\int_0^{\pi/2}\{(R+r)^2-4rR\sin^2\phi\}^{-\frac{1}{2}}d\phi. \tag{1.2}$$

The potential in P(x,y,z) may be obtained in an alternative way by using the well-known formula:

$$f(r,z)=\frac{1}{\pi}\int_0^{\pi} f(0,z+ir\cos\phi)d\phi$$

valid for axially symmetric harmonic functions (see Whittaker and Watson [3] p. 399).

Therefore we have also:

$$(1.3)\qquad u(r,z)= 2 \int_0^{\pi} \{R^2+(z+ir\cos\phi)^2\}^{-\frac{1}{2}}\, d\phi$$

Taking the special case z=0, $r < R$, we obtain

$$(1.4)\qquad u(r,0)= 4 \int_0^{\pi/2} \{R^2-r^2\cos^2\phi\}^{-\frac{1}{2}} d\phi.$$

The results (1.2) and (1.4) are of course identical for $r < R$, so that an interesting identity is obtained.
By performing the substitutions

$$\alpha=R,\ \beta=r,$$

$$\alpha_1= \frac{\alpha+\beta}{2} \text{ and } \beta_1=\sqrt{\alpha.\beta},$$

we may write this identity in the form:

$$(1.5)\qquad \int_0^{\pi/2} (\alpha^2-\beta^2\sin^2\phi)^{-\frac{1}{2}}d\phi= \tfrac{1}{2}\int_0^{\pi/2} (\alpha_1^2-\beta_1^2\sin^2\phi)^{-\frac{1}{2}}d\phi.$$

Applying the substitutions

$$a= R+r\ ,\ b= R-r,$$

$$a_1= \frac{a+b}{2} \text{ and } b_1=\sqrt{a.b},$$

we obtain a modification of the relation (1.5), namely:

$$(1.6)\qquad \int_0^{\pi/2} (a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{2}}d\phi = \int_0^{\pi/2} (a_1^2\sin^2\phi+b_1^2\cos^2\phi)^{-\frac{1}{2}}d\phi.$$

From (1.6) it follows immediately

$$(1.7)\qquad \int_0^{\pi/2} (a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{2}}d\phi=\lim_{n\to\infty}\int_0^{\pi/2} (a_n^2\sin^2\phi+b_n^2\cos^2\phi)^{-\frac{1}{2}}d\phi.$$

It is easily seen that the left-hand side of (1.7) equals

$\frac{1}{a}K(\sqrt{1-\frac{b^2}{a^2}})$, whereas the right-hand side equals $\frac{\pi}{2M(a,b)}$ and hence

$$(1.8)\qquad \frac{\pi}{2M(a,b)} = \frac{1}{a}K(\sqrt{1-\frac{b^2}{a^2}}) = \int_0^{\pi/2} (a^2\sin^2\phi+b^2\cos^2\phi)^{-\frac{1}{2}}d\phi .$$

2. The limit expression for $M(1,\varepsilon)$

By aid of formula (1.8) we have for any ε, with $0<\varepsilon<1$,

$$\frac{1}{M(1,\varepsilon)} = \frac{1}{2\pi}\int_0^{2\pi}(\sin^2\phi+\varepsilon^2\cos^2\phi)^{-\frac{1}{2}}d\phi =$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{i\phi}[\{(1+\varepsilon)e^{2i\phi}-(1-\varepsilon)\}\{-(1-\varepsilon)e^{2i\phi}+(1+\varepsilon)\}]^{-\frac{1}{2}}d\phi .$$

Substituting $z=e^{i\phi}$, we obtain

$$(2.1)\qquad \frac{1}{M(1,\varepsilon)} = \frac{1}{\pi\sqrt{1-\varepsilon^2}}\oint_C \{(z^2-\frac{1-\varepsilon}{1+\varepsilon})(z^2-\frac{1+\varepsilon}{1-\varepsilon})\}^{-\frac{1}{2}}dz$$

where the integration should be performed in the positive sense along the contour C. C is the unit circle around the origin of the complex plane, which has cuts as shown in fig. 1, where $\xi=\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$ and $\eta=\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$.

We deform this contour into the contour L, which consists of the straight lines L_1 and L_2, parallel to the imaginary axis and intersecting the real axis in the points $z=-(\xi+\eta)/2$ and $z=(\xi+\eta)/2$ resp. This contour is shown in fig. 2.

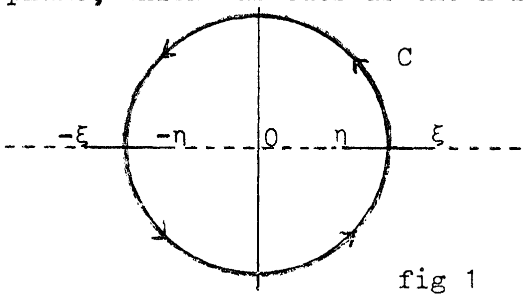

fig 1

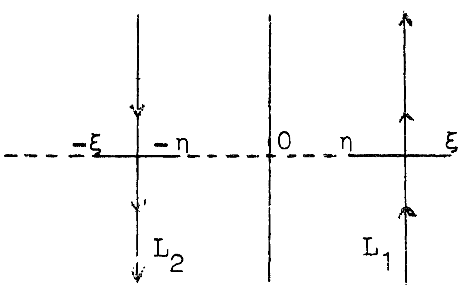

fig 2

After this deformation we may write instead of (2.1):

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = \left\{ \int\limits_{-i\infty+\frac{\xi+\eta}{2}}^{+i\infty+\frac{\xi+\eta}{2}} - \int\limits_{-i\infty-\frac{\xi+\eta}{2}}^{+i\infty-\frac{\xi+\eta}{2}} \right\} \{(z^2-\eta^2)(z^2-\xi^2)\}^{-\frac{1}{2}} dz =$$

$$= \int\limits_{-i\infty}^{+i\infty} \{(z+\tfrac{\xi+\eta}{2})^2-\eta^2\}^{-\frac{1}{2}} \{(z+\tfrac{\xi+\eta}{2})^2-\xi^2\}^{-\frac{1}{2}} dz -$$

$$- \int\limits_{-i\infty}^{+i\infty} \{(z-\tfrac{\xi+\eta}{2})^2-\eta^2\}^{-\frac{1}{2}} \{(z-\tfrac{\xi+\eta}{2})^2-\xi^2\}^{-\frac{1}{2}} dz =$$

$$= \int\limits_{-i\infty}^{+i\infty} (z+\tfrac{\xi+3\eta}{2})^{-\frac{1}{2}} (z+\tfrac{\xi-\eta}{2})^{-\frac{1}{2}} (z+\tfrac{3\xi+\eta}{2})^{-\frac{1}{2}} (z-\tfrac{\xi-\eta}{2})^{-\frac{1}{2}} dz -$$

$$- \int\limits_{-i\infty}^{+i\infty} (z-\tfrac{\xi+3\eta}{2})^{-\frac{1}{2}} (z-\tfrac{\xi-\eta}{2})^{-\frac{1}{2}} (z-\tfrac{3\xi+\eta}{2})^{-\frac{1}{2}} (z+\tfrac{\xi-\eta}{2})^{-\frac{1}{2}} dz .$$

Putting now:

$$\frac{\xi-\eta}{2} = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \gamma,$$

$$\frac{\xi+3\eta}{2} = \frac{2-\varepsilon}{\sqrt{1-\varepsilon^2}} = \alpha,$$

$$\frac{3\xi+\eta}{2} = \frac{2+\varepsilon}{\sqrt{1-\varepsilon^2}} = \beta,$$

we obtain:

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = \int\limits_{-i\infty}^{+i\infty} (z^2-\gamma^2)^{-\frac{1}{2}} (z+\alpha)^{-\frac{1}{2}} (z+\beta)^{-\frac{1}{2}} dz -$$

$$- \int\limits_{-i\infty}^{+i\infty} (z^2-\gamma^2)^{-\frac{1}{2}} (z-\alpha)^{-\frac{1}{2}} (z-\beta)^{-\frac{1}{2}} dz =$$

$$= \int\limits_{-\infty}^{+\infty} \left[\{(i\gamma\ \mathrm{sh}u+\alpha)(i\gamma\ \mathrm{sh}u+\beta)\}^{-\frac{1}{2}} - \{(i\gamma\ \mathrm{sh}u-\alpha)(i\gamma\ \mathrm{sh}u-\beta)\}^{-\frac{1}{2}} \right] du.$$

For small values of ε we have $\alpha = \beta + O(\varepsilon)$ and so we may write

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = \int_{-\infty}^{+\infty} \{(i\gamma\ \mathrm{sh}u+\alpha)^{-1} - (i\gamma\ \mathrm{sh}u-\alpha)^{-1}\}\ du+r(\varepsilon),$$

with $r(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Completing the reduction we obtain finally,

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = 2\alpha \int_{-\infty}^{+\infty} \frac{du}{\gamma^2\mathrm{sh}^2u+\alpha^2} + r(\varepsilon)= \frac{2}{\sqrt{\alpha^2-\gamma^2}}\ \ln \frac{\{\alpha+\sqrt{\alpha^2-\gamma^2}\}^2}{\gamma^2} + r(\varepsilon),$$

or

(2.2) $$\frac{\pi}{2M(1,\varepsilon)} = \ln\frac{4}{\varepsilon} + r_1(\varepsilon),\ \text{where } r_1(\varepsilon) \to 0 \text{ for } \varepsilon \to 0.$$

Hence we arrive at the desired result:

(2.3) $$\lim_{\varepsilon\to 0} M(1,\varepsilon)\ \ln\frac{4}{\varepsilon} = \frac{\pi}{2}\ .$$

References

[1] Gauss, C.F: Nachlass zur Theorie des arithmetisch - geometrischen Mittels und der Modulfunktion. Übersetzt und herausgegeben von H. Geppert. Leipzig: Akad. Verlags-gesellschaft 1927

[2] Hofsommer, D.J. and R.P. van de Riet: On the numerical calculation of Elliptic Integrals of the first and second kind and the Elliptic Functions of Jacobi. Numerische Mathematik 5, 291-302 (1963)

[3] Whittaker, E.T. and G.N. Watson: A course of Modern Analysis. London: Cambridge University press (1935)

[4] Schlesinger, L: Handbuch der Theorie der linearen Differentialgleichungen II 2. Leipzig: Teubner verlag 1898.

typ: HvD.