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R.J. Chitashvili

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On the Smooth Fit Boundary Conditions in the Optimal Stopping Problem for Semimertingales

R.J. Chitashvili

Department of Probability Theory and Mathematical Statistics
Tbilisi Mathematical Institute, Academy of Sciences of the Georgian SSR
150 a, Plekhanov Avenue, Tbilisi 380012, USSR

The well known smooth fit boundary conditions in the optimal stopping problem of Markov diffusion processes are generalized for the optimal stopping problem of semimartingales.

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1. Let a semimartingale $X=M+A$ be given on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)$, $0 \leq t \leq T$, satisfying usual conditions, where M is a martingale and A is a predictable process with integrable variation.

It is well known that in the optimal stopping problem for X the stopping rule (time change) (τ_t^*), $0 \leq t \leq T$, defined as $\tau_t^* = \inf(s \geq t\colon V_s^* = X_s) \wedge T$ is optimal, where $V_t^* = \sup_{t \leq \tau \leq T} E(X_\tau | \mathcal{F}_t)$ is the value process. In other words, if for each nonnegative semimartingale η, which is right-continuous and admits left-hand limits, we consider the time change $\tau_t^\eta = \inf(s \geq t\colon \eta_s = 0)$ and corresponding expected reward $V_t^\eta = E(X_{\tau_t^\eta} | \mathcal{F}_t)$, then $V^\eta = V^*$ when $\eta = V^* - X$. This is true, in particular, in the regular case, when process X is supposed to be right-continuous and left-quasicontinuous (see [1] for a surway of the theory and further generalizations).

Thus the construction of the optimal stopping rule is connected with the construction of the value process. The methods of constructing the value process, are in turn, based on the, so called, characterization theorems. The most general *supermartingale characterization,* is given in the following proposition. All the semimartingales considered below are right continuous admitting left-hand limits and having special decompositions.

PROPOSITION 1. *The process V is the value process iff*
a) *V is a supermartingale ($V = M - B$, $m \in \mathfrak{M}$, $dB \geq 0$) $V_T = X_T$;*
b) *$V \geq X$ ($V_t \geq X_t$ a.s. $0 \leq t \leq T$);*
c) *V is the minimal process with the properties a) and b).*
Here $\mathcal{K}$ denotes the class of martingales and, for convenience, the fact that (predictable) process B is increasing is expressed as $dB \geq 0$.

This characterization is inconvenient because it is often hard to verify the minimality condition c). The following *variational inequalities* allow us to express property c) directly in terms of the process V.

PROPOSITION 2. *The process V is the value process iff*
a) $V = m - B$, $m \in \mathfrak{M}$, $dB \geq 0$, $V_T = X_T$;
b) $V \geq X$;
c) $dB = I_{[V_- = X_-]} dB$; *i.e. the process B increases only on the set* $(V_- = X_-) = ((t, \omega)\colon V_{t-} = X_{t-})$.

In order to impose additional requirements on this characterization, we represent it now as an optimality test for the given stopping region $S_\eta = (\eta = 0)$ corresponding to some non-negative semimartingale η.

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PROPOSITION 3. *The process V^η is the value process if*
a) $V^\eta = m^\eta - B^\eta$, $m^\eta \in \mathfrak{M}$, $dB^\eta \geq 0$;
b) $V^\eta \geq X$.

Combining this statement with the boundary problem for the process V^η, we can formulate the following *free boundary problem.*

PROPOSITION 4. *The semimartingale $V = m - B$ and the time change (τ_t^η), $0 \leq t \leq T$, are the value process and optimal stopping rule respectively iff*
a) $dB \geq 0$;
b) $V > X$ *on the set* $(\eta > 0)$;
c) V *is a solution of boundary problem*

$$I_{[\eta_- > 0]} dV = I_{[\eta_- > 0]} dm, \quad V = X \text{ on the set } (\eta = 0), \quad V_T = X_T.$$

Obviously condition a) may be written as $I_{[\eta_- = 0]} dB \geq 0$.

The aim of reduction of condition a) to the *smooth fit condition* can be explained as follows. The coincidence $V = X$ on the stopping region S_η does not imply of course the coincidence $I_{[\eta_- = 0]} dB = -I_{[\eta_- = 0]} dA$. However, roughly speaking, on the essential part of S_η it might be possible to verify condition a) directly in terms of the process A given apriority, and hence the verification of a) can be reduced to a certain 'small' subset ∂S_η close to the continuation region $\{\eta > 0\}$.

2. To throw more light to the above reasoning, we present now the example of discrete time case, though the natural generalization of the smooth fit conditions on a boundary for diffusion processes ([2], [3]) will be given in Corollary 2 below.

Let the process X be stepwise constant, $X_t = X_{k\Delta}$, $k\Delta \leq t < (k+1)\Delta$, $k = 0, \dots, T/\Delta$; $\Delta > 0$. Then, of course, all processes S^η will be stepwise constant and can be considered only at points $t = 0, \Delta, \dots, T/\Delta$.

By Proposition 1 the value process V^* is a least solution of the inequality

$$V_{t-1} \geq \max(X_{t-1}, E(V_t | F_{t-1})), \quad V_T = X_T, \tag{1}$$

while Proposition 2 determines it as a unique solution of the Wald-Bellman equation

$$V_{t-1} = \max(X_{t-1}, E(V_t | F_{t-1})), \quad V_T = X_T. \tag{2}$$

In fact, conditions a), b) are equivalent to (1), while condition c) with $\Delta B_t = V_{t-1} - E(V_t | F_{t-1})$ leads to (2).

Propositon 4 tells us that the processes V and η can be found by solving the recurrent equation with boundary conditions

$$\begin{gathered} V_{t-1} = I_{[\eta_{t-1} = 0]} X_{t-1} + I_{[\eta_{t-1} > 0]} E(V_t | F_{t-1}), \quad V_T = X_T; \\ I_{[\eta_{t-1} = 0]}(V_{t-1} - E(V_t | F_{t-1})) \geq 0; \ (V_t - X_t) I_{[\eta_t > 0]} > 0. \end{gathered} \tag{3}$$

It is convenient now to represent the backward recurrent equation (3) as a stochastic difference equation. Denote by ζ the process determined by the jumps

$$\Delta \zeta_t = E(\Delta(V_t - X_t) I_{[\eta_{t-1} = 0, \, \eta_t > 0]} | F_t). \tag{4}$$

We have

$$\begin{gathered} \Delta V_t = I_{[\eta_{t-1} > 0]} \Delta M_t + I_{[\eta_{t-1} = 0]} \Delta A_t + \Delta \zeta_t, \quad V_T = X_T \\ 0 \leq I_{[\eta_{t-1} = 0]} \Delta \zeta_t = 0 \leq -(1 - I_{[\eta_{t-1} > 0, \, V_{t-1} > X_{t-1}]}) \Delta A_t. \end{gathered} \tag{5}$$

Thus for the process V we have obtained the stochastic equation with an 'unknown' martingale part M (which obviously is uniquely determined by the boundary condition $V_T=X_T$ at the right end of the time interval), and with a unknown (predictable) process ζ satisfying the boundary conditions, which can be, in turn rewritten separately

$$\Delta\zeta_t \geqslant 0, \quad I_{[\eta_{t-1}>0,\ V_{t-1}\leqslant X_{t-1}]}\Delta A_t \leqslant 0, \quad \text{(majorising property)},$$
$$\Delta\zeta \leqslant -I_{[\eta_{t-1}=0]}\Delta A_t, \quad \text{(supermartingality)},$$
$$\Delta\zeta_t = I_{[\eta_{t-1}=0]}\Delta\zeta_t, \quad \text{(minimality)}.$$

It is seen from the expression (4) that the verification of boundary conditions can be restricted to the subset of S_η

$$\partial S_\eta = (\eta_{t-1}=0,\ P(\eta_t>0|\eta_{t-1})>0).$$

In fact $\Delta\zeta=I_{\partial S_\eta}\Delta\zeta$, and $\Delta B=-\Delta\zeta-I_{[\eta_-=0]}\Delta A=-\Delta A$ on the set $S_\eta\setminus\partial S_\eta$. In other words it suffieses to verify the conditions

$$(I_{[S_\eta\setminus\partial S_\eta]}+I_{[\eta_->0,\ V_{t-1}\leqslant X_{t-1}]})\Delta A_t\leqslant 0 \quad \text{a.s.,} \quad 0\leqslant t\leqslant T,$$

and

$$0\leqslant\Delta\xi_t\leqslant -\Delta A_t \quad \text{a.s. relative to the measure } \mu^\eta, \tag{6}$$

where μ^η is Dolean's measure associated with the increasing process

$$\mathfrak{L}_t = \sum_{s\leqslant t} I_{[\eta_{s-1}=0,\ \eta_s>0]}.$$

Thus, the problem of obtaining the boundary condition for the general case considered below, can be viewed as a generalization of (6).

3. Note that the process V^η does not depend on the choice of the process η within the class of equivalent processes, i.e. the nonnegative semimartingales $\tilde\eta$ for which $I_{[\tilde\eta_t=0]}=I_{[\eta_t=0]}$, a.s., $0\leqslant t\leqslant T$. Hence we can suppose $|V_t^\eta-X_t|\leqslant\eta_t$, choosing, if nessesary, the new process $\tilde\eta=\eta+|V^\eta-X|$. In the forthcoming we shall assume

$$\sum_{s\leqslant T}|\Delta X_s|<\infty.$$

THEOREM 1. *The process V^η satisfies the equation*

$$dV^\eta = dM^\eta+I_{[\eta_-=0]}dA+d\mathfrak{L}(V^\eta-X,\eta), \quad V_T^\eta=X_T, \tag{7}$$

where

$$\mathfrak{L}(V^\eta-X,\eta) = L(V^\eta-X,\eta)+\mathfrak{L}^d(V^\eta-X,\eta),$$

with

$$L_t(V^\eta-X,\eta) = \lim_{\epsilon\to 0}\epsilon^{-1}\int_0^t I_{[\eta_s\leqslant\epsilon]}d\langle V^\eta-X,\eta\rangle_s^c,$$

and

$$\mathfrak{L}_t^d(V^\eta=X,\eta) = (\Sigma I_{[\eta_{s-}=0,\ \eta_s>0]}\Delta(V_s^\xi-X_s))_t^p.$$

(Here and elsewhere below $\langle V^\eta-X,\eta\rangle^c=\langle V^{\eta c}-X^c,\eta^c\rangle$ is the square mutual characteristic of continuous martingale parts of $V^\eta-X$ and η; $(\ )^p$ denotes the dual predictable projection).

PROOF. For $\epsilon>0$ consider the process $Z_t^\epsilon=\epsilon^{-1}(\epsilon-\eta_t)(V_t^\eta-X_t)$. Evidently $\lim_{\epsilon\to 0}Z_t^\epsilon=I_{[\eta_t=0]}(V_t^\eta-X_t)=0$. We shall apply the integral representation formula for the positive part of the semimartingale Y ([4])

$$y_t^+ = y_0^+ + \int_0^t [I_{[y_{s-}>0]}+\frac{1}{2}I_{[y_{s-}=0]})dy_s + \frac{1}{2}L_t^0(y) + \sum_{s\leq t}(\Delta(y)_s^+ - [I_{[y_{s-}>0]}+\frac{1}{2}I_{[y_{s-}=0]}]\Delta y_s),$$

where $L^0(y)$ is the local time spent at 0.

We have

$$(\epsilon-\eta_t)^+ = (\epsilon-\eta_0)^+ - \int_0^t (I_{[\eta_{s-}<\epsilon]}+\frac{1}{2}I_{[\eta_{s-}=\epsilon]})dy_s + \frac{1}{2}L_t^\epsilon(\eta)$$

$$+\sum_{s\leq t}(\Delta(\epsilon-\eta)_s^+ - (I_{[\eta_{s-}<\epsilon]}+\frac{1}{2}I_{[\eta_{s-}=\epsilon]})\Delta\eta_s).$$

Applying this we obtain

$$Z_t^\epsilon = Z_0^\epsilon + \epsilon^{-1}\int_0^t (\epsilon-\eta_{s-})^+ d(V_s^\eta - X_s) + \sum_{s\leq t}\epsilon^{-1}\Delta(\epsilon-\eta)_s^+ \Delta(V^\eta-X)_s$$

$$-\epsilon^{-1}\int_0^t (I_{[\eta_s<\epsilon]}+\frac{1}{2}I_{[\eta_s=\epsilon]})d<V^\eta-X,\eta>^c - \epsilon^{-1}\int_0^t (V_{s-}^\eta - X_{s-})(I_{[\eta_{s-}<\epsilon]}$$

$$+\frac{1}{2}I_{[\eta_{s-}=\epsilon]})d\eta_s + (2\epsilon)^{-1}\int_0^t (V_s^\eta - X_s)dL_s^\epsilon(\eta) + \epsilon^{-1}\sum_{s\leq t}(V_{s-}^\eta - X_{s-})[\Delta(\epsilon-\eta_s)^+$$

$$-(I_{[\eta_{s-}\leq\epsilon]}+\frac{1}{2}I_{[\eta_{s-}=\epsilon]})\Delta\eta_s].$$

Now using the relations

$$\lim_{\epsilon\to 0}\epsilon^{-1}\eta_t - I_{[\eta_{t-}\leq\epsilon]} = 0, \quad \int_0^t I_{[\eta_s=\epsilon]}d<V^\eta-X,\eta>_s^c = 0,$$

it is easily seen that

$$(\int_0^{\cdot} I_{[\eta_{t-}=0]}d(V^\eta-X)_t)^P = \mathfrak{L}(V^\eta-X,\eta).$$

It remains to show that the process

$$\int_0^t I_{[\eta_{s-}>0]}dV_s^\eta$$

is a martingale. In fact this follows from the general results on balayages ([5]). Indeed

$$V_t^\eta = E(X_{\tau_t^\eta}|F_t) = m_t + E(A_{\tau_t^\eta}|F_t) = m_t + m_t' + A_t^b,$$

where m, m' are martingales, and the predictable projection A^b of the process $A_{\tau_t^\eta}$ has the variation increasing only on the set $((t,\omega)$: $l_t=t)\subset(\eta_-=0)$, where $l_t=\sup(s<t$: $\eta_s=0)$. □

It should be noted also that the process $\tilde{\eta}$ can be chosen so that $I_{[\tilde{\eta}_t=0]}=I_{[\eta_t=0]}$ a.s. $0\leq t\leq T$, and besides

$$(\tilde{\eta}_{t-}=0) = (l_t=t)\cup(\eta_{t-}=0,\ \eta_t=0),$$

by taking

$$\tilde{\eta}_t = E(C_{\tau_t^\eta} - C_t | F_t)$$

with a strongly increasing process C.

4. We shall now present all the above mentioned characterizations of the value process.

THEOREM 2. *The following statements are equivalent*

a) $V_t^* = \sup_{t \leqslant \tau \leqslant T} E(X_\tau | F_t)$;

b) V^* *is the minimal solution of the equation*

$$dV = dm - dB, \quad V \geqslant X, \quad V_T = X_T, \quad m \in \mathfrak{M}, \quad dB \geqslant 0;$$

c) V^* *is the unique solution of the equation*

$$dV = dm - dB, \quad V \geqslant X, \quad V_T = X_T, \quad m \in \mathfrak{M}, \quad 0 \leqslant dB = I_{[V_- = X_-]} dB;$$

d) V^* *is the maximal solution of the equation*

$$dV = I_{[V_- > X_-]} dm + I_{[V_- = X_-]} dA + d\zeta, \quad V_T = X_T, \quad m \in \mathfrak{M}, \quad 0 \leqslant d\zeta = I_{[V_- = X_-]} d\zeta;$$

e) V^* *is the unique solution of the equation*

$$dV = I_{[V_- > X_-]} dm + I_{[V_- = X_-]} dA + d\zeta, \quad V_T = X_T, \quad m \in \mathfrak{M},$$

$$0 \leqslant d\zeta = I_{[V_- = X_-]} d\zeta \leqslant -I_{[V_- = X_-]} dA.$$

PROOF. b) is just a supermartingale characterization. Let us prove c) which expresses the variational inequalities considered for diffusion processes in [6]. As the time change $\tau_t^{V^* - X}$ is optimal we have $V^* = V^{V^* - X}$. By Theorem 1 V^* satisfies the equation c). On the other hand, if V is a solution of c), then V is a supermartingale with $V \geqslant X$. Hence $V \geqslant V^\eta$ for each η. But for the time change

$$\tau_t^\epsilon = \inf(s \geqslant t: V_s \leqslant X_s + \epsilon) \vee T$$

we have

$$V_t = E(V_{\tau_t^\epsilon} | F_t) \leqslant E(X_{\tau_t^\epsilon} | F_t) + \epsilon. \tag{8}$$

Thus $V = V^*$, and c) is proved.

To proved d) note that for each solution of equation d) we have the estimation (8). Thus $V \leqslant V^*$. At the same time $V^* = V^{V^* - X}$ is a solution of d) and, besides, by $V^* \geqslant X$ we have

$$d\zeta -= I_{[V_-^* = X_-]} d(V^* - X) \geqslant 0.$$

Finally V^* satisfies the equation e) (the last inequality in the boundary conditions is connected with a supermartingality property). If V is a solution of e), then the process $V - X$ satisfies the relation

$$d(V - X) = I_{[V_- > X_-]} d(V - X) + d\zeta, \quad V_T = X_T,$$

from which it is easily seen that $V \geqslant X$. Thus by d) we have $V \leqslant V^*$ and hence $V = V^*$. □

The decomposition of the increasing process $B = -V^* + m$ in the form $dB = -I_{[\eta_- = 0]} dA - d\zeta$ with $d\zeta \geqslant 0$ was established in [7] for a more general class of processes X.

5. The connection should be mentioned of this theorem with characterization properties of reflecting processes, especially with Skorohod's (direct) equation.

PROPOSITION 5. *The following statements are equivalent*

a) $\bar{V}_t = \sup_{0 \leqslant s \leqslant t} X_s$;

b) $\overline{V}$ *is the minimal solution of the equation*

$$dV = dB,\ dB \geqslant 0,\ V_0 = X_0,\ V \geqslant X;$$

c) $\overline{V}$ *is the unique solution of the equation*

$$dV = dB,\ V \geqslant X,\ V_0 = X_0,\ 0 \leqslant dB = I_{[V=X]}dB;$$

d) $\overline{V}$ *is the maximal solution of the equation*

$$dV = I_{[V=X]}DX + d\zeta,\ V_0 = X_0,\ 0 \leqslant d\zeta = I_{[V=X]}d\zeta;$$

e) $\overline{V}$ *is the unique solution of the equation*

$$dV = I_{[V=X]}dX + d\zeta,\ V_0 = X_0,\ 0 \leqslant d\zeta,\ I_{[V=X]}dX + d\zeta \geqslant 0.$$

In particular, the process $\overline{X}_t = \overline{V}_t - X_t$ expresses the instantaneous reflection of the process $(-X_t)$ from the zero boundary, and a) is a generalization of Levy's expression for the reflected Wiener process ($X = -W$). For the process $\overline{X}$ c) is Skorokhod's equation ([8], [9])

$$d\overline{X} = dX + dB,\ \overline{X} \geqslant 0,\ \overline{X}_0 = 0,\ 0 \leqslant dB = I_{[\overline{X}=0]}dB. \tag{9}$$

Characterization property d) was noted in [10] for Ito processes.

Using e) we can derive

COROLLARY 1. *For continuous process* $X = M + A$, *if* $varA$ *is dominated by* $<M>$ ($varA << <M>$), *then the process* $\overline{X}$ *is the unique solution of the boundary problem*

$$I_{[\overline{X}>0]}d\overline{X} = I_{[\overline{X}>0]}dm,\ \overline{X} \geqslant 0,\ \overline{X}_0 = 0,\ \int_0^T I_{[\overline{X}=0]}d<M> = 0. \tag{10}$$

In fact, since $\overline{X} \geqslant 0$, we have $d\overline{X} = I_{[\overline{X}>0]}d\overline{X} + dL^0(\overline{X})$. Comparing this with e) and (9), and having in mind that

$$I_{[\overline{X}=0]}dM + I_{[\overline{X}=0]}dA + d\zeta \geqslant 0$$

implies $I_{[\overline{X}=0]}d<M> = 0$, and hence $I_{[\overline{X}=0]}dA = 0$, we obtain $\zeta = L^0(\overline{X})$. □

Thus the process $\overline{X}$ is characterized by the fact that it spends zero time (measured in terms of $<M>$) on the boundary.

6. Returning to the stopping problem expressed by (in some sense dual to Proposition 5) properties of Theorem 2, we shall derive the assertion which can be viewed as a dual analogy of Corollary 1.

We show meanwhile that there is a nonnegative semimartingale η such that

$$var\ \mathfrak{L}(V^\eta - X, \eta) << \mathfrak{L}(\eta, \eta). \tag{11}$$

In fact it suffices to take

$$\tilde{\eta}_t = E(C_{\tau_t^\eta} - C_t | F_t)$$

with a certain increasing predictable process C such that $var\,A << C$. This is easily seen by relating to increasing process B the process

$$\eta_t^B = E(B_{\tau_t^\eta} - B_t | F_t)$$

with the differential $d\eta^B = I_{[\eta^B > 0]}dm^B + d\mathfrak{L}(\eta^B, \eta^B)$, $m^B \in \mathfrak{M}$. For B^1, B^2, with $d(B^1 - B^2) \geqslant 0$, we have $\eta^{B^1} = \eta^{B^2} + \eta^{B^1 - B^2}$ and hence

$$\mathfrak{L}(\eta^{B^1}, \eta^{B^1}) >> \mathfrak{L}(\eta^{B^2}, \eta^{B^2}).$$

The general case $B^2 << B^1$ is treated in the same manner by introducing the decomposition

$$B^2 = \sum_{n=1}^{\infty} I_{[(n+1)^{-1} \leqslant b < n^{-1}]} b \cdot B^1 + I_{[b \geqslant 1]} b \cdot B^1,$$

where $b = dB^2/dB^1$ (i.e. $B^2 = b \cdot B^1$). Obviously, for each n

$$\mathfrak{L}(\eta^{B^n}, \eta^{B^n}) << \mathfrak{L}(\eta^{B^1}, \eta^{B^1}),$$

where $B^n = I_{[(n+1)^{-1} \leqslant b < n^{-1}]} b \cdot B^1$, which leads to (11). Combining the statements e) of Theorems 1 and 2 we obtain

COROLLARY 2. *Let $\eta = \eta^C$ with $\operatorname{var} A << C$ and μ^η, μ^C be Dolean's measures associated with the processes $\mathfrak{L}(\eta, \eta)$ and C respectively. Furthermore, let $\mu^\eta = \mu^\eta_s + \mu^\eta_r$ be the decomposition of the measure μ^η on the singular and regular parts with respect to the measure μ^C. Then the process V^η is the value process iff*

1) $V^\eta > X$ *on the set* $(\eta > 0)$;

2) $d\mathfrak{L}(V^\eta - X, \eta)/d\mathfrak{L}(\eta, \eta) = 0 \quad \mu^\eta_s$ *a.e.* (12)

$$0 \leqslant d\mathfrak{L}(V^\eta - X, \eta)/dC \leqslant -dA/dC, \quad (\eta_- = 0) - \eta^C \text{ a.e.}$$

Thus if $\mu^\eta = \mu^\eta_s$, the boundary conditions reduce to the smooth fit condition (12) and the condition $dA/dC \leqslant 0 \quad (\eta_- = 0) - \mu^C$ a.e.

For continuous processes with the integral representation property for martingales, the connection of condition (12) with classical Stefan's problem becomes more implicit.

Let X, η be continuous and let there exist the continuous mutually orthogonal martingales M^i, $1 \leqslant i \leqslant n$, such that the martingale parts m^η, M, N in the decomposition of semimartingales V^η, X, η are represented as integrals

$$m^\eta = \sum_i \psi^i \cdot M^i, \quad X = \sum_i \phi^i \cdot M^i, \quad N = \sum_i g^i \cdot M^i.$$

For simplicity suppose $<M^i> = C$, $1 \leqslant i \leqslant n$. We need to require below that the processes ψ, ϕ, g have limits on the boundary within the continuation region $(\eta > 0)$.

For some process Z denote by Z^+ and Z^- the upper and lower limits

$$Z_t^+ = \lim_{\epsilon \to 0} \operatorname*{ess\,sup}_{|s-t| \leqslant \epsilon} Z_s, \quad Z_t^- = \lim_{\epsilon \to 0} \operatorname*{ess\,inf}_{|s-t| \leqslant \epsilon} Z_s,$$

where ess sup (ess inf) is taken with respect to the measure μ^C.

COROLLARY 3. *Let*

1) $\psi^+ = \psi^- = \hat{\psi}$, $\phi^+ = \phi^- = \hat{\phi}$, $g^+ = g^- = \hat{g}$ μ^η *a.e.*

2) $\sum_i (\hat{g}^i)^2 > 0$ μ^η *a.e.*

Then the process V^η is the value process if

a) $dA/dC \leqslant 0 \quad (\eta = 0) - \mu^C$ a.e.

b) $V^\eta > X$ *on the set* $(\eta > 0)$ *a.s.*

c) $\sum_i (\hat{\psi}^i - \hat{\phi}^i)\hat{g}^i = 0$ μ^η *a.e.*

PROOF. It is easily verified that μ^C a.e. processes ψ^+, ϕ^+, g^+ (ψ^-, ϕ^-, g^-) are uppersemicontinuous (lowersemicontinuous) in t and hence

$$\lim_{\epsilon \to 0} \epsilon^{-1} \int_0^t I_{[\eta_s < \epsilon]} d < V^\eta - X, \eta >_s \leqslant \lim_{\epsilon \to 0} \epsilon^{-1} \int_0^t I_{[\eta_s < \epsilon]} \Big(\sum_i (\psi^i - \phi^i) g^i / \sum_i (g^i)^\eta\Big)^+ d < \eta >_s$$

$$\leqslant \int_0^t \sum_i ((\psi^i - g^i) g^i / \sum_i (g^i)^\eta)^+ \, dL_s^0(\eta).$$

This and the similar inequality with lower limits, together with assumption 1) implies c). Furthermore, since $I_{[\eta=0]} d\langle N\rangle = 0$, the measure μ^η is singular w.r.t. the measure $\mu\langle N\rangle$ and, hence, by assumption 2) w.r.t. the measure μ^C too. □

The smooth fit condition expressed as $\psi_{\tau_t^\eta} = \phi_{\tau_t^\eta}$ was obtained in [11] for Ito processes X.

7. THE EXAMPLE OF A DIFFUSION PROCESS

Let ξ be a Markov diffusion process with a nonsingular diffusion metrix $B(x)B^*(x)$, $x \in R^{(n)}$, and with a density function $q_x(t,y)$, $x,y \in R^{(n)}$, corresponding to an initial condition $\xi_0 = X$. As for the martingales M^i, $1 \leqslant i \leqslant n$, we can consider the Wiener process $w = B^{-1}(\xi)\cdot\xi$. Moreover, here

$$X_t = \varphi(\xi_t), \quad V_t^\eta = v^\eta(t,\xi_t), \quad \eta_t = G(t,\xi_t),$$

where φ and $G \geqslant 0$ are smooth functions, and

$$v^\eta(t,x) = E_x \varphi(\xi^\eta_{\tau_t}).$$

The stopping region can be defined in terms of the set $\mathfrak{D}_t = (x: G(t,x)=0)$. In fact $S_\eta = ((t,\omega): \xi_t \in \mathfrak{D}_t)$. Let $C_t = t$. Evidently

$$\psi_t = B^*(\xi_t)\nabla v^\eta(t,\xi_t), \quad \phi_t = B^*(\xi_t)\nabla\varphi(\xi_t), \quad g_t = B^*(\xi_t)\nabla G(t,\xi_t).$$

It suffices here to restrict the integration operator corresponding to the measure μ^η to the class of processes represented as $F(t,\xi_t)$ for some measurable function $F(t,x)$, $x \in R^{(n)}$, $0 \leqslant t \leqslant T$. It is easily calculated, for fixed initial conditions $\xi_0 = X$, that

$$\int\int_{[0,T]\times\Omega} F(t,\xi_t)\mu^\eta(dt,d\omega) = E_x \int_0^T F(t,\xi_t) dL_t^0(\eta) = \lim_{\epsilon\to 0} \int_0^T \int_{R^{(n)}} F(t,y).$$

$$\epsilon^{-1} I_{[G(t,y)<\epsilon]} q_x(t,y)(\nabla G(t,y),\ B(y)B^*(y)\nabla G(t,y))dy\,dt =$$

$$= \int_0^T \int_{\partial D_t} F(t,y)(\nabla G(t,s),\ \nabla G(t,y))^{-\frac{1}{2}} (\nabla G(t,s),\ B(y)B^*(y)\nabla G(t,y)) q_x(t,y) d\sigma(y) dt,$$

where ∂D_t is a boundary and the integral with respect to $d\sigma(y)$ is understood as the surface integral over ∂D_t.

Now, assume:

1) The functions B, $\nabla\varphi$, ∇G, ∇v^η are continuous on ∂D_t,
2) $(\nabla G, \nabla G) > 0$ on D_t.

If, in addition, the density function is also positive and continuous on ∂D_t, then the smooth fit condition c) of Corollary 3 reduces to the pointwise equality condition

$$(\nabla(v^\eta - \varphi),\ BB^*\nabla G)(t,x) = 0, \quad x \in \partial D_t, \quad 0 \leqslant t \leqslant T.$$

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