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Department of Pure Mathematics Report PM-R8602 January

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Deformations of Connections, the Riemann Hilbert Problem and τ-Functions

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We give sufficient conditions for the existence of integrable deformations of a rational linear ODE on the projective line and we show when the related connection form obtains a reduced form. Certain coefficients in this reduced form admit a meromorphic continuation to the whole parameter space, while their poles coincide with the zero-set of a Fredholm determinant τ. These properties are similar to the ones holding for the solutions of the KP-hierarchy and form a generalization of work by Malgrange.

1980 Mathematics Subject Classification: 35Q15, 35F20.
Key Words & Phrases: Riemann Hilbert problem, integrable connections, isomonodromic deformation, Fredholm determinant, Grasmann manifold.
Note: This paper will appear in the Proceedings of the 7th International Symposium on the Mathematical Theory of Networks and Systems, Stockholm, Sweden, 10-14 juni 1985.

Introduction

At the end of the seventies the Kyoto-school (Sato, Kashiwara, Miwa, Jimbo etc.) introduced so-called τ-functions for several completely integrable systems like
(1) the KP-hierarchy (see [13]) and
(2) the systems which are obtained if one deforms linear ordinary differential systems in such a way that the monodromy of the system is independent of the deformation parameters, see [12].
In both cases "τ-functions" were defined formally as the solutions of $d\log\tau = \omega$, where ω was some specific exact differential form related to the system. In the examples these τ-functions turned out to be equal, up to some invertible factor, to determinants of certain operators. However a general context relating the two cases remained obscure.

Segal and Wilson showed in [16] how to obtain from each point of the Grassmann manifold of a separable Hilbert space a solution of the KP-hierarchy and to each such a point they associated a τ-function which is in fact a Fredholm determinant. Moreover their functions satisfy a characteristic equation of the Japanese τ-functions. Their construction of solutions of the KP-hierarchy also shows that the zero-set of the τ-function forms an obstruction to an analytical continuation of the solutions to the whole parameter space.

Here we want to discuss similar properties for the analogues in the second example of the equations of the KP-hierarchy. The organization of this paper is as follows: the first paragraph contains a description of the Riemann Hilbert problem and its relation to the kind of questions treated here. In the second paragraph we introduce integrable deformations of rational linear ordinary differential equations on $\mathbb{P}^1(\mathbb{C})$. In particular we give a sufficient condition for the existence of such a deformation and we show under which condition the connection form of the connection, which describes the deformation, can be brought in a reduced form. Certain coefficients of this reduced form are solutions of non-linear partial differential equations, which are the analogues of the equations of the KP-hierarchy. The main result of the next paragraph is the meromorphic continuation of these coefficients to the whole parameter space. In particular they are holomorphic on the complement of an analytic subvariety Θ, which is locally described by the vanishing of a Fredholm determinant τ. Under special conditions τ can be defined globally. Finally we give in the last paragraph an interpretation of these τ-functions in the context of the Grassmann manifold of $l^2(\mathbb{Z})$.

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1. Preliminaries

1.1. Before recalling the classical Riemann Hilbert problem, we introduce first some notations. Throughout this paper we denote the standard coordinate function which identifies $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C}\cup\{\infty\}$ by x. Further, if Z is any complex connected variety then we write $\mathfrak{R}(Z)$ for the universal covering space of Z. Since $\mathfrak{R}(Z)$ is a fiberbundle over Z with fiber the fundamental group $\pi_1(Z)$ of Z, there is a left action of $\pi_1(Z)$ on $\mathfrak{R}(Z)$, which is denoted by a ".".

For the moment let X be $\mathbb{P}^1(\mathbb{C})$ and $Y=\{b_i\,|\,b_i\in X,\ 1\leq i\leq r,\ b_i\neq b_j \text{ for } i\neq j\}$. If ρ is any p-dimensional complex representation of $\pi_1(X-Y)$, then we introduce the following notion

1.2. Definition A holomorphic function Φ: $\mathfrak{R}(X\text{-}Y)\to Gl_p(\mathbb{C})$ is called *of type* ρ if we have for all $g\in\pi_1(X-Y)$ and all $\tilde{x}$ in $\mathfrak{R}(X\text{-}Y)$

$$\Phi(g.\tilde{x})=\Phi(\tilde{x})\rho(g)^{-1}.$$

1.3. Remark To such a Φ one can associate the differential form $\Omega=d\Phi\Phi^{-1}$ and the differential equation

$$d\vec{f} = \Omega(\vec{f}).$$

One easily checks that Ω is defined over X-Y and we call ρ the *monodromy* of this differential equation.

Next we consider a certain growth condition for functions of type ρ around a singular point y in Y. Let $B(y,\epsilon)$ denote an open ball around y with radius ϵ, containing no other points of Y.

1.4. Definition A function Ω of type ρ is called *of moderate growth around y*, if there is a $B(y,\epsilon)$ as above such that on each strip $|\arg(t)|\leq N$ in $\mathfrak{R}(B(y,\epsilon)\text{-}\{y\})$ we have an estimate

$$|\Omega(t)|\leq K|t|^M,$$

for certain constants K and M, depending of N.

Remark The differential equation associated to Φ is called *Pfaffian* or is said to have *regular singularities along Y* if Φ is of moderate growth around each point of Y.

Example Examples of differential equations with regular singularities along $\{a_1,\ldots,a_n,\infty\}$ are the ones with a differential form looking like

$$\sum_{i=1}^{n}\frac{A_i}{x-a_i}dx.$$

However not all of them have this form, see e.g.[11].

1.5. Classical Riemann Hilbert problem : For any finite-dimensional complex representation of $\pi_1(X-Y)$, find a Φ of type ρ, which is of moderate growth around each point of Y.

Remark It is possible to translate this problem into a boundary value problem on a closed contour in $\mathbb{P}^1(\mathbb{C})$, see e.g. [1].

1.6. Next we describe how this problem can be generalized to several complex variables. We take for X now a complex connected variety and for Y a complex smooth subvariety of X of codimension one. If ρ is a p-dimensional complex representation of $\pi_1(X-Y)$, then one introduces for holomorphic functions : $\Re(X\text{-}Y)\to Gl_p(\mathbb{C})$ the notion of being of type ρ as above. As in (1.3) one associates to such a Φ a differential form and a set of differential equations of which Φ forms a fundamental solution matrix. The conditions under which this system is Pfaffian are more involved and we refer to [5] for more details. Once more we can pose the question if there exists for each complex representation ρ of $\pi_1(X-Y)$ a Φ of type ρ with a Pfaffian associated system of differential equations. This is the Riemann Hilbert problem such as it was formulated and solved for certain X in [6].

1.7. To get an idea of the crucial matters involved it is good to consider the problem from a wider perspective. This was done by Deligne in [3]. There he gave a base-free description of ordinary linear differential equations on a complex connected variety Z which possess locally at each point a fundamental solution matrix. He considers pairs (E,∇), consisting of a flat holomorphic vectorbundle E over Z, of rank p, and an integrable connection ∇ of the sheaf of holomorphic sections E of E, which we call shortly an integrable connection of E. Next he observes that there is an equivalence between the following categories

1.8.
$$\left\{\begin{array}{c}\text{pairs } (E,\nabla)\text{, with E a flat holomorphic}\\ \text{vectorbundle over Z of rank p and}\\ \nabla \text{ an integrable connection of E}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\text{sheaves over Z, which are locally}\\ \text{isomorphic to the constant}\\ \text{sheaf } \underline{\mathbb{C}}^p\end{array}\right\}$$

This correspondence is obtained by taking the subsheaf of E consisting of the horizontal sections of ∇, which is locally constant of rank p thanks to the Cauchy theorem. Further we have an equivalence between the categories

1.9.
$$\left\{\begin{array}{c}\text{sheaves over Z, which are locally isomorphic}\\ \text{to the constant sheaf } \underline{\mathbb{C}}^p\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}p-\text{dimensional complex}\\ \text{representations of } \pi_1(Z)\end{array}\right\}$$

The correspondence takes here the following form : given a sheaf F over Z, which is locally isomorphic to $\mathbb{C}^p$, then one can show that $\pi_1(Z)$ acts linearly on the fiber over some basepoint in Z and this gives you a p-dimensional complex representation of $\pi_1(Z)$. Deligne also introduced in [3] the notion of an integrable connection with regular singularities . It agrees for the case $X=\mathbb{P}^1(\mathbb{C})$ with the one introduced above. Notations being as in (1.6), then one formulates the Riemann Hilbert problem in terms of connections as follows :

1.10. The Riemann Hilbert problem : given a finite-dimensional complex representation ρ of $\pi_1(X-Y)$, find a trivial holomorphic vectorbundle E over X and a connection ∇ of E|X-Y such that the monodromy of the correponding system of differential equations is ρ and that it has regular singularities along Y.

In view of (1.8) and (1.9) we see that in order to solve it we have to look for pairs (E,∇) as in (1.8) which extend suitably. This kind of question one naturally meets in the next paragraph. We end this paragraph with a few definitions. Let X and Y be as in (1.6) and (E,∇) as in (1.8).

1.11. Definitions (i) We say that ∇ is *meromorphic over X* if there exists for each point of X a neighbourhood U and a trivializing basis of sections of $\underline{E}|U$ such that the connection form of ∇ with respect to this basis is meromorphic on U.
(ii) We say that ∇ has a *logarithmic pole over Y* if there exists at each point of Y a set of local coordinates $(x_1, \ldots, x_n)$ such that Y is locally given by the equation $x_1=0$ and such that the connection

form Ω of ∇ with respect to the $\{x_i \mid 1 \leq i \leq n\}$ has the form

$$\Omega = A_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} A_i dx_i ,$$

where the A_i are all holomorphic on the whole coordinate patch.

2. Integrable deformations

2.1. We start with an ordinary meromorphic linear differential equation over $\mathbb{P}^1(\mathbb{C})$, i.e. a system of the form

2.2.
$$dY = \left\{ \left[\sum_{1 \leq k \leq m} \left[\sum_{l \geq 1} \frac{A^0_{kl}}{(x-a^0_k)^l} \right] + \sum_{l \geq 0} A^0_{\infty l} x^l \right] dx \right\} Y,$$

where A^0_{kl} and $A^0_{\infty l}$ are complex $p \times p$- matrices and $A^0_{kl} = A^0_{\infty l} = 0$ for sufficiently large l. In deforming this differential equation one arrives naturally at systems of differential equations in several complex variables. As we have said in the preliminaries a proper way to formulate them if one wants to make global statements is the framework of connections. Hence we formulate our starting point also in these terms and then it comes down to the following data:
(i) A trivial holomorphic vectorbundle E^0 over $\mathbb{P}^1(\mathbb{C})$ of rank p,
(ii) m different points $a^0_1, \ldots, a^0_m$ in $\mathbb{C}$ and
(iii) a connection ∇^0 of $E^0 | \mathbb{P}^1(\mathbb{C})$-$\{a^0_1, \ldots, a^0_m, \infty\}$, which is meromorphic over $\mathbb{P}^1(\mathbb{C})$.

Now the kind of deformations of the pair (E^0, ∇^0) that we are going to consider are intuitively speaking based on the idea to "move" in some way or another the poles $(a^0_1, \ldots, a^0_m)$ of ∇^0. To be more precise,

2.3. Definition An *integrable deformation* (E, ∇) of the pair (E^0, ∇^0) consists of the following ingredients:
(i) the *deformation space* T, a complex connected paracompact variety,
(ii) the *deformation functions*, which are holomorphic maps $a_i : T \to \mathbb{C}$, $1 \leq i \leq m$, such that $a_i(t_0) = a^0_i$ for some *base point* t_0 in T and for all i,
(iii) a holomorphic vectorbundle E over $X = \mathbb{P}^1(\mathbb{C}) \times T$ of rank p and
(iv) an integrable connection ∇ of $E|X$-Y, meromorphic over X, such that the restriction of (E, ∇) to $\mathbb{P}^1(\mathbb{C}) \times \{t_0\}$ is isomorphic to (E^0, ∇^0). Here Y is the singular set associated to the moving of the poles $\{a^0_1, \ldots, a^0_m\}$. More precisely, $Y = T_1 \cup \cdots \cup T_m \cup T_\infty$, where the T_i and T_∞ are given by

$$T_i = \{(x, t) \mid x \in \mathbb{C},\ t \in T,\ x = a_i(t)\} \text{ and } T_\infty = \{(\infty, t) \mid t \in T\}.$$

The reason that we consider only integrable ∇ is that the analogue in the present situation of the equations of the KP-hierarchy is the set of integrability conditions for the connection form Ω of ∇. Moreover in order that (E, ∇), with ∇ not necessarily integrable, could be a solution to a Riemann-Hilbert problem as desribed in (1.10), it is necessary that the local differential equations corresponding to it, are solvable, in other words that ∇ is integrable.

Let i be the embedding of $\mathbb{P}^1(\mathbb{C})$ into X given by $x \mapsto (x, t_0)$. By restricting it to $\mathbb{P}^1(\mathbb{C})$-$\{a^0_1, \ldots, a^0_m, \infty\}$, it induces a natural homomorphism i^* from $\pi_1(\mathbb{P}^1(\mathbb{C})$ -$\{a^0_1, \ldots, a^0_m, \infty\})$ to $\pi_1(X-Y)$. The deformation will be called *isomonodromic* if i^* is an isomorphism.

Next we consider the question, given T, t_0, and the a_i, does there exist any integrable deformation of (E^0, ∇^0) with these data? If one tries to construct a desired deformation, one can meet topological obstructions. For, according to (1.8) and (1.9) $(E|X$-$Y, \nabla)$ would correspond to a p-dimensional representation of $\pi_1(X-Y)$ and since its restriction to $\mathbb{P}^1(\mathbb{C}) \times \{t_0\}$ has to be isomorphic to (E^0, ∇^0), the composition of this representation with i^* should give you a representation isomorphic to the one corresponding to $(E^0 \mid \mathbb{P}^1(\mathbb{C})-\{a_1^0,\ldots,a_m^0,\infty\},\nabla^0)$. Already in one of the simplest examples this can go wrong. Let me illustrate this: take $T=\mathbb{C}^m$ and for a_i the projection on the i^{th}-coordinate. Write p_1 for the projection: $X\text{-}Y\to\mathbb{C}$, given by $p_1((x, t))=x$. Then the fiber over x is equal to $(\mathbb{C}-\{x\})^m$. Hence $(X\text{-}Y, \mathbb{C}, p_1)$ is a fiberbundle and since the base space is contractible, the long exact sequence of homotopy groups tells us that $\pi_1(X-Y)$ is isomorphic to $\mathbb{Z}^m$. Since $\pi_1(\mathbb{P}^1(\mathbb{C})-\{a_1^0,\ldots,a_m^0,\infty\})$ is equal to the free group on m generators, any (E^0,∇^0) corresponding to a non-commutative representation of $\pi_1(\mathbb{P}^1(\mathbb{C})-\{a_1^0,\ldots,a_m^0,\infty\})$ does not permit a deformation by $\mathbb{C}^m$ of the desired kind.

From the foregoing example it will be clear that in general one cannot let happen such catastrophes like coinciding of the poles. Hence it is natural to require from now on that the deformation functions satisfy the following

2.4. Assumption : $a_i(t)\neq a_j(t)$, for all i and j, $i\neq j$, and for all t in T.

However, under this assumption the same phenomenon can happen still. Consider namely the projection p_2:$X\text{-}Y\to T$, given by $p_2((x,t))=t$. The fiber over t is equal to $\mathbb{C}-\{a_1(t),\ldots,a_m(t)\}$, hence $(X\text{-}Y, T, p_2)$ is a fiberbundle and we have the following long exact sequence

2.5.
$$\cdots\to\pi_2(T)\to\pi_1(\mathbb{P}^1(\mathbb{C})-\{a_1^0,\ldots,a_m^0,\infty\})\xrightarrow{i^*}\pi_1(X-Y)\to\pi_1(T)\to 1$$

Hence, if $\pi_2(T)\neq 1$, the deformation does not have to exist. If this condition is fulfilled, then the next question that arises is: can the p-dimensional representation of $\pi_1(\mathbb{P}^1(\mathbb{C})-\{a_1^0,\ldots,a_m^0,\infty\})$ be extended to a p-dimensional representation of $\pi_1(X-Y)$? This is for sure possible if the sequence (2.5) splits at the end, a case which occurs if you take T of the form $T_0\times T_1$, with T_0 simply-connected and the a_i of the form $b_i\circ p_0$, where p_0 is the projection onto T_0 given by $p_0((t_0, t_1))=t_0$ and the b_i functions on T_0. Hence we avoid this kind of obstructions if we make the following

2.6. Assumption : the deformation space T satisfies $\pi_2(T)=\pi_1(T)=1$.

2.7. Example In this section we give an example of a non-trivial deformation space, which satisfies both assumptions and which occurs in [14] and [15]. Let Z be the space

$$\mathbb{C}^m-\bigcup_{\substack{i\neq j\\ 1\leq i,j\leq m}} D_{ij},\text{ with } D_{ij}=\{(x_k)\in\mathbb{C}^m, x_i=x_j\}.$$

We take as a deformation space the universal covering $\mathfrak{R}(Z)$ of Z and as deformation function a_i the composition of the natural map $\mathfrak{R}(Z)\to Z$ and the projection on the i^{th}-coordinate. Clearly $\pi_1(\mathfrak{R}(Z))=1$. Since $\mathfrak{R}(Z)$, together with the natural projection: $\mathfrak{R}(Z)\to Z$, is a fiberbundle over Z with fiber $\pi_1(Z)$, the long exact sequence of homotopy groups tells you that $\pi_k(\mathfrak{R}(Z))$ is isomorphic to $\pi_k(Z)$ for all $k\geq 2$. Now Faddell and Neuwirth have shown in [4] that for all $k\geq 2$

$$\pi_k(Z)=\bigoplus_{1\leq r\leq m}\pi_k((S^1)^r)=1,$$

where $(S^1)^r$ denotes the disjoint union of r copies of the unit circle S^1. Hence $\mathfrak{R}(Z)$ is contractible. The space $\mathfrak{R}(Z)$ possesses still another useful property, namely it is Steinian. For Z is the complement of the zero-set of an analytic function on $\mathbb{C}^m$, so it is Steinian (see [7]) and according to [17] every unramified covering of a Stein space is again a Stein space. By a theorem of Grauert [8] every holomorphic vectorbundle over a Stein space is trivial if and only if it is topologically trivial, hence all holomorphic vectorbundles over $\mathfrak{R}(Z)$ are trivial.

2.8. Remark In [12] they consider deformations which preserve the monodromy, but they work in local coordinates. For simplicity, I will only discuss their deformations of the poles over an open subset T of $\mathbb{C}^m$. The only condition they put on T is assumption (2.4) and one does not meet any of the conditions $\pi_2(T)=1$ or $\pi_1(T)=1$ in their work. The outcome of their manipulations reveals the drawback of working in local coordinates, for a requirement to get a monodromy-preserving deformation is its integrability! Therefore a part of their deformations simply do not exist.

2.9. Next we show that the conditions (2.4) and (2.6) are sufficient to be able to construct integrable deformations. Let T, a_i and t_0 be as in (2.3)(i) and (ii), then we have

2.10. Theorem *Under the assumptions (2.4) and (2.6), there exists an integrable deformation* (E,∇) *of* (E^0,∇^0) *with T as its deformation space, with the* a_i *as deformation functions and* t_0 *as base point.*
Proof First one constructs (E,∇) on X-Y. From (2.5) we know that i^* is an isomorphism. Hence the p-dimensional complex representation of $\pi_1(\mathbb{P}^1(\mathbb{C})-\{a_1^0,...,a_m^0,\infty\})$ corresponding to the pair $(E^0|\mathbb{P}^1(\mathbb{C})-\{a_1^0,...,a_m^0,\infty\},\nabla^0)$ induces one of $\pi_1(X-Y)$. According to (1.8) and (1.9), it determines a holomorphic flat vectorbundle $\tilde{E}$ over X-Y and an integrable connection ∇ of E, such that the horizontal sections of ∇ correspond to the locally constant sections of $\tilde{E}$. As in the case of the Riemann-Hilbert problem, the crucial point is now the construction of a suitable extension of (E,∇). This will be done on disjoint open neighbourhoods of the components of Y. First we introduce some notations. For each t in T, we set

$$D(t)=\{x\in\mathbb{C},\ |x|<\min_{\substack{1\leq i,j\leq m\\ i\neq j}}\frac{|a_i(t)-a_j(t)|}{4}\}\ \text{and}\ D_\infty(t)=\{x\in\mathbb{P}^1(\mathbb{C}),\ |x|>4\max_{1\leq i\leq m}|a_i(t)|\}.$$

Now let V(i) resp. $V(\infty)$ be the neighbourhoods of T_i resp. T_∞ defined by

$$V(i)=\{(x,t)\in\mathbb{P}^1(\mathbb{C})\times T,\ x\in a_i(t)+D(t)\cap D(t_0)\}\ \text{and}\ V(\infty)=\{(x,t)\in\mathbb{P}^1(\mathbb{C})\times T,\ x\in D_\infty(t)\cap D_\infty(t_0)\}.$$

Clearly they are disjoint and under the projection $(x,t)\to t$ the $V(i)-T_i$ and $V(\infty)-T_\infty$ map onto T, with the fiber of t being equal to resp. $a_i(t)+\{(D(t)\cap D(t_0))-\{0\}\}$ and $\{(D_\infty(t)\cap D_\infty(t_0))-\{0\}\}$. Hence w.r.t. this map they are fiberbundles with base space T and fiber $D(t_0)-\{0\}$ and from the long exact sequence of homotopy groups and (2.4) and (2.5) we conclude that i^* induces an isomorphism between $\pi_1(a_i(t_0)+(D(t_0)-\{0\})$ resp. $\pi_1(D_\infty(t_0)-\{0\})$ and $\pi_1(V(i)-T_i)$ resp. $\pi_1(V(\infty)-T_\infty)$.

For $1\leq i\leq m$, let $p_i:V(i)\to a_i^0+D(t_0)$ be defined by $p_i(x,t)=x-a_i(t)+a_i^0$. Consider now on V(i) the trivial holomorphic vectorbundle E_i obtained by "pulling back" through p_i the bundle $E^0|a_i^0+D(t_0)$. Let Ω^0 be the connection form of ∇^0 w.r.t. the trivializing sections $(e_1,...,e_p)$. One defines the connection ∇_i of $E_i|V(i)-T_i$ as the connection whose connection form w.r.t. the $(p_i^*(e_1),...,p_i^*(e_p))$ is $p_i^*(\Omega^0)$. Clearly ∇_i is integrable, it is meromorphic over V(i) and the restriction of (E_i,∇_i) to $a_i^0+D(t_0)$ through i is isomorphic to $(E^0|a_i^0+D(t_0),\nabla^0)$. Since $p_i\circ i$ is the identity on $a_i^0+D(t_0)$, one easily sees that the monodromy representations of $(E|V(i)-T_i,\nabla)$ and $(E_i|V(i)-T_i,\nabla_i)$ are the same. Hence the pairs $(\tilde{E}|V(i)-T_i,\nabla)$ and $(E_i|V(i)-T_i,\nabla_i)$ are isomorphic and in this way we have extended $(\tilde{E},\nabla)$ to $X-T_\infty$.

Finally we take the projection p_∞: $V(\infty)\to D_\infty(t_0)$ defined by $p_\infty(x,t)=x$ and consider as above the pull-back (E_∞,∇_∞) of $(E^0|D_\infty(t_0),\nabla^0)$. Then ∇_∞ is integrable, it is meromorphic over $V(\infty)$ and the restriction of (E_∞,∇_∞) to $D_\infty(t_0)$ through i is isomorphic to $(E^0|D_\infty(t_0),\nabla^0)$. Moreover the same argument as above shows that $(E|V(\infty)-T_\infty,\nabla)$ is isomorphic to $(E_\infty|V(\infty)-T_\infty,\nabla_\infty)$. This completes the proof of the theorem.

2.11. Remarks(i) From the construction given above one deduces directly that if ∇^0 had a logarithmic pole at infinity, i.e. $A^0_{\infty l}=0$ for all $l\geq 0$, then the constructed integrable deformation has a logarithmic pole along T_∞.
(ii) Let (E,∇) be an integrable deformation of (E^0,∇^0) as constructed in theorem (2.10) and let U be an open non-empty subset of T, containing t_0, such that $E|\mathbb{P}^1(\mathbb{C})\times U$ is trivial and such that $\pi_1(U)=1$. As we will see in (3.7)(i) such sets exist. We denote a set of trivializing sections of $E|\mathbb{P}^1(\mathbb{C})\times U$ by $(f_1,...,f_p)$ and for the rest we will use the notations of the proof of theorem (2.10) for the deformation space U. The construction given above shows that the connection form Ω of ∇ w.r.t. the $(f_1,...,f_p)$ has on each V(i) the form

$$\Omega=\sum_{l\geq 1}\left\{\frac{A_{il}(t)}{(x-a_i(t))^l}\right\}d(x-a_i)+\omega_i,$$

where ω_i is holomorphic on V(i) and where the p×p-matrices A_{il} are holomorphic on U, satisfy $A_{il}=0$ for sufficiently large l and $A_{il}(t_0)=A^0_{il}$ for all l (see (2.2)). If we introduce the matrix-valued differential form Ω_f by

$$\Omega_f=\sum_{1\leq i\leq m}\left\{\sum_{l\geq 1}\left\{\frac{A_{il}(t)}{(x-a_i(t))^l}\right\}d(x-a_i)\right\},$$

then $\Omega_\infty=\Omega-\Omega_f$ is holomorphic on $\mathbb{C}\times U$ and meromorphic over $\mathbb{P}^1(\mathbb{C})\times U$. Hence Ω_∞ depends polynomially on x and is holomorphic in t. In particular, if ∇ has a logarithmic pole along T_∞, then Ω_∞ does not depend on x. In this last case Ω_∞ is the connection form w.r.t. $(f_1\,|\,U,..., f_p\,|\,U)$ of an integrable connection $\nabla(\infty)$ of $E|U$. To see that $\nabla(\infty)$ is integrable we first note that it is a local question. Hence we may assume that we have local coordinates $t_1,\dots,t_k$ on U. For uniformity sake we denote the coordinate x by t_0 and we write ∂_i for the differentiation w.r.t. t_i. Then we have for Ω_f and Ω_∞ the decompositions

$$\Omega_f=\sum_{j=0}^{k}F_j(t_0,\dots,t_k)dt_j \text{ and } \Omega_\infty=\sum_{j=0}^{k}C_j(t_1,\dots,t_k)dt_j,$$

where the C_j are holomorphic on U, $C_0\equiv 0$ and the F_j have the property that for all $j\geq 0$ and all $l\geq 1$

$$\lim_{t_0\to\infty}F_j(t_0,\dots,t_k)=\lim_{t_0\to\infty}\partial_l F_j(t_0,\dots,t_k)=0.$$

Let [A,B] be the commutator of two matrices A and B. Then the integrability conditions for Ω are given by the system of equations

2.12. $$\partial_i(F_j+C_j)-\partial_j(F_i+C_i)=[F_j+C_j,\ F_i+C_i],$$

with $0\leq i,j\leq k$. Consider now those equations with $i\geq 1$ and $j\geq 1$ and let t_0 tend to infinity, then the limit behaviour of the F_j stated above gives us the integrability conditions for Ω_∞.

Because of the integrability of $\nabla(\infty)$ the system $dS=-S\Omega_\infty$ is locally solvable and since $\pi_1(U)=\{1\}$, there exists a holomorphic $S:U\to Gl_p(\mathbb{C})$ such that $dS=-S\Omega_\infty$. Moreover we may assume that $S(t_0)=id$. With respect to the basis of trivializing sections $(S(f_1),...,S(f_p))$ the connection form of $\nabla(\infty)$ gets the following form

2.13. $$(dS)S^{-1} + S\Omega S^{-1} = S\Omega_f S^{-1} = \sum_{1\leq i\leq m}\left\{\sum_{l\geq 1}\left\{\frac{B_{il}(t)}{(x-a_i(t))^l}\right\}d(x-a_i)\right\}.$$

It will be shown in (3.12) that the B_{il} extend meromorphically to T. The integrability conditions for this reduced form are a system of non-linear differential equations in the unknown B_{il}. These equations depend rationally on x and by equating the coefficients of equal powers of $(x-a_i(t))^{-1}$ on both sides of (2.12) one arrives at a set of x-independent equations for the B_{il}. They form the analogue of the non-linear equations, which form the KP-hierarchy. To give some idea of what kind of equations one gets in this way, we illustrate this with an example. We take T and the a_i as in example (2.7) and we assume that ∇ has a logarithmic pole along Y, i.e. $B_{il}=0$ for all $l\geq 2$ and for all i. Then the integrability conditions reduce to the so-called "Schlesinger-equations" :

$$dB_{i1} = -\sum_{j\neq i}[B_{i1},B_{j1}]\frac{d(a_i-a_j)}{a_i-a_j}.$$

(iii) The integrable connection $\nabla(\infty)$ constructed in the preceding remark exists under weaker assumptions than the ones required over there. I want to discuss that in this remark. First we assume that the deformation space T is such that there is an open neighbourhood $W(\infty)$ of T_∞, with $W(\infty)\cap Y=T_\infty$, satisfying $E|W(\infty)$ is trivial. Such a $W(\infty)$ always exists if $E|U\times T$ is trivial, where U is an open neighbourhood of ∞ in $\mathbb{P}^1(\mathbb{C})$. This condition is surely fulfilled if T is a contractible Stein space. In $W(\infty)$ we consider open sets of the form $B(\infty)\times W$ with $B(\infty)$ an open neighbourhood of ∞ in $\mathbb{P}^1(\mathbb{C})$ and W an open part of T with the coordinates $t_1,\ldots,t_n$. Assume now that on each $B(\infty)$-$\{\infty\}\times W$ the connection form of ∇ has the form

$$M_0(x,t_1,\ldots,t_n)dx + \sum_{j=1}^{n}M_j(x,t_1,\ldots,t_n)dt_j,$$

with all the M_j, $j\geq 1$, holomorphic on $B(\infty)\times W$. As we have seen above this is the case if (E,∇) is an integrable deformation as constructed in theorem (2.10) with a logarithmic pole along T_∞. To the differential form

$$\sum_{j=1}^{n}M_j(\infty,t_1,\ldots,t_n)dt_j$$

corresponds a connection $\nabla(\infty,W)$ of $E|\{\infty\}\times W$ and as in remark (2.11)(ii) one shows that it is integrable. Finally all the $\nabla(\infty,W)$ compose to an integrable connection $\nabla(\infty)$ of $E|\{\infty\}\times T$.

3. τ-functions

3.1. Let (E, ∇) be an integrable deformation of (E^0,∇^0). Following [14], we consider the subset Θ of T, where the deformed bundle is no longer trivial, i.e.

$$\Theta=\{t|t\in T,\ E|\mathbb{P}^1(\mathbb{C})\times\{t\}\text{ is non-trivial }\}$$

The properties of Θ which we will need are gathered in the following

3.2. Proposition (i) *Θ is an analytic subvariety of T.*
(ii) *Let $\mathcal{O}^*$ be the sheaf of holomorphic invertible functions on* T. *Assume that there are open sets O_1 and O_2 in $\mathbb{P}^1(\mathbb{C})$ such that $E|O_1\times T$ and $E|O_2\times T$ are trivial, where $O_1\cup O_2=\mathbb{P}^1(\mathbb{C})$ and $O_1\cap O_2\supset\{x\in\mathbb{P}^1(\mathbb{C}),\ |x|=1\}$. If moreover $H^1(T,\mathcal{O}^*)=0$, then there exists a holomorphic map τ:T$\to\mathbb{C}$ such that the zero-set of τ is equal to Θ. In particular τ can be chosen such that it is a Fredholm*

determinant.
(iii) *$E|\mathbb{P}^1(\mathbb{C})\times T\text{-}\Theta$ is trivial if and only if $E|\{\infty\}\times T\text{-}\Theta$ is trivial.*

Proof of (i) We first reduce the triviality condition to the solvability of a Riemann-Hilbert boundary value problem. Choose ρ_1 and ρ_2 such that $0<\rho_1<1<\rho_2$ and set

$$D_1=\{x\,|\,x\in\mathbb{P}^1(\mathbb{C}),\ |x|>\rho_1\}\text{ and }D_2=\{x\,|\,x\in\mathbb{P}^1(\mathbb{C}),\ |x|<\rho_2\}.$$

Let U be any non-empty open subset of T such that $E|D_1\times U$ and $E|D_2\times U$ are trivial. Such subsets exist since $E|\mathbb{P}^1(\mathbb{C})\times\{t_0\}$ is trivial. Take sections $(f_1,\ldots,f_p)$ of $E|D_1\times U$ and $(g_1,\ldots,g_p)$ of $E|D_2\times U$, which trivialize these bundles. There is a holomorphic map S from $D_1\cap D_2\times U$ to $Gl_p(\mathbb{C})$ which expresses the $\{f_i\}$ into the $\{g_j\}$. Now $E|\mathbb{P}^1(\mathbb{C})\times\{t\}$, $t\in U$, is trivial if and only if there are holomorphic maps $S_+(t):D_2\to Gl_p(\mathbb{C})$ and $S_-(t):D_1\to Gl_p(\mathbb{C})$ such that for all $x\in D_1\cap D_2$

3.3. $$S(x,t)=\left[S_-(t)(x)\right]^{-1}S_+(t)(x)$$

For convenience sake, we may moreover assume that $S_-(t)(\infty)=id$. Our next step will be to give an analytical description of the solvability of this boundary value problem.

Let $\mathcal{H}$ be the Hilbert space consisting of the direct sum of p copies of $L^2(S^1)$ and write $\mathcal{H}_0$ for the subspace

$$\left\{(d_j)_{1\leq j\leq p}\,\middle|\,d_j(\phi)=\sum_{k\geq 0}a_{kj}e^{2\pi ik\phi}\text{ for all }j\right\}$$

For fixed t, multiplication by S(-,t) determines a bounded isomorphism $\mathbb{S}(t)$ of $\mathcal{H}$, whose decomposition with respect to $\mathcal{H}_0\oplus\mathcal{H}_0^\perp$ is denoted by

3.4. $$\mathbb{S}(t)=\begin{bmatrix}a(t) & b(t)\\ c(t) & d(t)\end{bmatrix}.$$

The triviality condition can be expressed in term of these operators as follows

3.5. Lemma *S(-,t) has a decomposition (3.3) if and only if a(t) is invertible.*

Proof of the lemma: From [2] page 61, we know that S(-,t) possesses a so-called Wiener-Hopf factorization

3.6. $$S(x,t)=\left[S_-(t)(x)\right]^{-1}\mathrm{diag}(x^{m_1},\ldots,x^{m_p})S_+(t)(x),$$

where $S_-(t)$ and $S_+(t)$ are holomorphic maps from D_1 resp. D_2 to $Gl_p(\mathbb{C})$ and $\mathrm{diag}(x^{m_1},\ldots,x^{m_p})$ is a diagonal matrix with entries $(\delta_{ij}x^{m_j})$, $m_j\in\mathbb{Z}$, $m_{j+1}\leq m_j$. The factors in the right hand side of (3.6) determine each a bounded automorphism of $\mathcal{H}$. Their decomposition with respect to $\mathcal{H}_0\oplus\mathcal{H}_0^\perp$ is given by respectively

$$\begin{bmatrix}a_- & 0\\ c_- & d_-\end{bmatrix},\ \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\text{ and }\begin{bmatrix}a_+ & b_+\\ 0 & d_+\end{bmatrix}.$$

Hence a(t) is equal to $a_-\alpha a_+$ and the invertibility of a(t) is equivalent to that of α. It will be clear that this last fact holds if and only if $m_i=0$ for all i and the lemma is proved.

3.7. Remarks(i) Since S(x,t) depends holomorphically of t, the same holds for a(t). Now a(t) has a convergent power series development around any point and the invertibility of such a series is determined by that of its constant term, hence the invertibility of a(t) is an open condition and if U is an open subset of T as in the beginning of the proof of the proposition then we denote the non-empty open set $\{u|u\in U,\ a(u) \text{ is invertible }\}$ by $\tilde{U}$.

(ii) One has a concrete expression for $S_+(t)(x)$ in terms of a(t). Namely, if we define $\Sigma_+(t)$ by

$$\Sigma_+(t)(x)=a(t)^{-1}(\delta_{ij}),$$

where $a(t)^{-1}$ acts on the p×p-identity matrix δ_{ij} by its natural action on the columns, then $\Sigma_+(t)$ is holomorphic on a neighbourhood of $\{x|x\in\mathbb{P}^1(\mathbb{C}),\ |x|\leq 1\}$. By definition we have

$$S(t,x)\Sigma_+(t)(x)=id+\Sigma_-(t)(x),$$

where $\Sigma_-(t)$ is holomorphic on a neighbourhood of $\{x|x\in\mathbb{P}^1(\mathbb{C}),\ |x|\geq 1\}$ with $\Sigma_-(t)(\infty)=0$. Since constants are the only holomorphic functions on $\mathbb{P}^1(\mathbb{C})$ we get

$$S_+(t)\Sigma_+(t)=S_-(t)(id+\Sigma_-(t))=S_-(t)(\infty)(id+\Sigma_-(t)(\infty))=id.$$

(iii) The decomposition (3.3) furnishes us a set of sections $\{h_1,\ldots,h_p\}$ that trivialize $E|\mathbb{P}^1(\mathbb{C})\times\tilde{U}$. We can take namely for all i, $1\leq i\leq p$,

$$h_i=\sum_{j=1}^{p}S_-(t)_{ij}f_j=\sum_{j=1}^{p}S_+(t)_{ij}g_j.$$

(iv) From (3.8) one sees directly that a(t) is a Fredholm operator of index $\sum_{i=1}^{p}m_i$. By applying a Wiener-Hopf factorization to the transposed of S(x,t) one gets that d(t) is Fredholm too. Later on we will need that b(t) is a nuclear operator. Since β is finite-dimensional, it will be sufficient to prove this for b_+.

(v) If U is connected then the index of a(t) is constant on U and from the first remark we may conclude that it is zero everywhere.

After these remarks we continue the proof of proposition (3.2). Let u be any point in U. First we note that on a neighbourhood V of u the following holds

3.8. There is a holomorphic map q from V to the bounded invertible operators on $\mathcal{H}_0$ such that for all t in V, $a(t)q(t)^{-1}$-id is finite-dimensional.

This is easily seen as follows: since the index of a(u) is zero, there exists a finite-dimensional operator k in $End(\mathcal{H}^0)$ such that a(u)+k is invertible. One can take e.g. an isomorphism between the kernel of a(u) and some complementary subspace of the image of a(u) and compose it with the projection on the kernel of a(u). For all t sufficiently close to u, we have then

$$a(t)(a(t)+k)^{-1}=id-k(a(t)+k)^{-1},$$

what shows the assertion. According to [9] any operator of the form id+A, with A a finite-dimensional operator in $End(\mathcal{H}_0)$, is invertible if and only if its determinant is non-zero. Here the determinant of id+A is defined by

3.9. $$\det(\mathrm{id}+A)=1+\sum_{r=1}^{\infty}\mathrm{Trace}(\overset{r}{\wedge}A).$$

Therefore, locally, the points where a(t) is invertible are completely characterized by $\det(\mathrm{id}-k(a(t)+k)^{-1})\neq 0$ and this shows that Θ is an analytic subvariety of T.

3.10. Remark It is possible to define the determinant for a wider class of operators, namely for those of the form "identity + trace-class". This is done by first giving sense to all the factors occurring in the sum in the righthand side of (3.9) and by showing next its convergence. Details can be found e.g. in [9].

Proof of (ii) From the first assumption we conclude that one can choose D_1 and D_2 as in the proof of (i) such that $E|D_i\times T$ is trivial. Therefore we may assume that $\mathbb{S}$, as introduced in (3.4), is defined on the whole of T. In particular we have a globally defined collection of Fredholm operators $\{a(t)|t\in T\}$. Let $\{V_i|i\in I\}$ be a locally finite open covering of T such that on each V_i we have a q_i satisfying (3.8). For each i and j in I such that $V_i\cap V_j$ is non-empty, we denote $q_iq_j^{-1}$ by ϕ_{ij}. Note that each $\phi_{ij}(t)$ is an operator of the form "identity + finite-dimensional". Clearly the $\{\det(\phi_{ij})\}$ determine a holomorphic line-bundle over T and since $H^1(T,\mathcal{O}^*)=0$, it is trivial, i.e. on each V_i there is an analytic map τ_i from V_i to $\mathbb{C}^*$ such that $\tau_i^{-1}\tau_j=\det(\phi_{ij})$. Choose some non-zero v in $\mathcal{H}_0$ and define for all i in I the automorphism t_i of $\mathcal{H}_0$ by letting it act on v as multiplication by τ_i and on $<v>^{\perp}$ as the identity. If we put $\mathfrak{q}_i=t_iq_i$, then we note directly that the $\mathfrak{q}_i$ have again the property (3.8). Moreover there holds

$$\det(\mathfrak{q}_i(t)\mathfrak{q}_j(t)^{-1})=\det(t_i(t)q_i(t)q_j(t)^{-1}t_i(t)^{-1})\det(t_i(t)t_j(t)^{-1})=1$$

Hence if we define for $t\in V_i$, $\tau(t)=\det(a(t)\mathfrak{q}_i(t)^{-1})$, then τ is well-defined on T and the zero-set of τ is Θ. This is a direct and more general way of introducing τ than that given in [14].

Proof of (iii) The necessity being clear, we show here the sufficiency of the condition. The idea is to glue together the local trivializations obtained in (3.7)(iii). First we fix trivializing sections $(e_1,\ldots,e_p)$ of $E|\{\infty\}\times T\text{-}\Theta$. Let $\{U_j|j\in J\}$ be an open covering of $T\text{-}\Theta$ such that $E|D_1\times U_j$ and $E|D_2\times U_j$ are trivial for all j in J. Now we choose trivializing sections $(f_1^{(j)},\ldots,f_p^{(j)})$ of $E|D_1\times U_j$ such that $f_k^{(j)}|\{\infty\}\times U_j=e_k|U_j$. If we denote the corresponding transfermatrix by $S^{(j)}$, then this last property also holds for

$$\left[\sum_{i=1}^{p}(S^{(j)})_{1i}f_i,\ldots,\sum_{i=1}^{p}(S^{(j)})_{pi}f_i\right],$$

since $S^{(j)}(t)(\infty)=\mathrm{id}$ for all t in U. Thus we obtain a set of trivializing sections $(h_1^{(j)},\ldots,h_p^{(j)})$ of $E|\mathbb{P}^1(\mathbb{C})\times U_j$ satisfying $h_k^{(j)}|\{\infty\}\times U_j=e_k|U_j$. Since the $h_k^{(j)}$ are holomorphic on $\mathbb{P}^1(\mathbb{C})\times U_j$, they are trivial extensions of the $e_k|U_j$ and therefore compose to sections $(h_1,\ldots,h_p)$ trivializing $E|\mathbb{P}^1(\mathbb{C})\times T\text{-}\Theta$. This completes the proof of the proposition.

3.11. Remarks (i) The set Θ clearly forms an obstruction for (E,∇) being a candidate for a solution of the Riemann Hilbert problem corresponding to the representation of $\pi_1(X-Y)$ determined by $(E|X\text{-}Y,\nabla)$.

(ii) From [9], p.336, we know that the inverse of an operator of the form id+A, with A an finite-dimensional operator in $\mathrm{End}(\mathcal{H}_0)$ is given by the formula

$$(\mathrm{id}+A)^{-1}=(\det(\mathrm{id}+A))^{-1}R(A),$$

where R(A) is a finite linear combination of monomial mappings of various degrees from the space of finite-dimensional endomorphisms of $\mathcal{H}_0$ into itself. This implies that $t \mapsto \tau(t)\dot{a}(t)^{-1}$ is a holomorphic map defined on the whole of V. Hence by using (3.7)(ii) one sees that $S_+(t)$ and $S_-(t)$ depend meromorphically on t.

(iii) Let U, $\{f_i\}$, $\{g_j\}$ and $\{h_i\}$ be as in (3.7)(i) and (iii). If T is a contractible Stein space like in example (2.7), then one may take for U the whole deformation space T. Let Ω_1 resp. Ω_2 be the connection forms of ∇ with respect to $(f_1, \ldots, f_p)$ resp. $(g_1, \ldots, g_p)$, then Ω_i is holomorphic on $D_i \times U - Y \cap D_i \times U$ and is meromorphic over $D_i \times U$. The connection form Ω of ∇ with respect to the $\{h_i\}$ is related to the Ω_i by

$$\Omega = S_-\Omega_1(S_-)^{-1} + dS_-(S_-)^{-1} \quad \text{on } D_1 \times \tilde{U},$$

$$\Omega = S_+\Omega_2(S_+)^{-1} + dS_+(S_+)^{-1} \quad \text{on } D_2 \times \tilde{U}.$$

By combining these equations with remark (3.11)(ii), one sees that Ω is meromorphic over $\mathbb{P}^1(\mathbb{C}) \times U$.

3.12. We come now to the meromorphic continuation for the coefficients B_{il} introduced in remark (2.11)(ii). Thus we assume that (E,∇) is an integrable deformation as constructed in theorem (2.10) and that it has a logarithmic pole along T_∞. Further we assume that T is such that $E|D_1 \times T$ and $E|D_2 \times T$ are trivial. According to remark (2.11)(iii), ∇ induces then an integrable connection $\nabla(\infty)$ of $E|\{\infty\} \times T$ and since T is simply connected, there exist a basis $\{e_1, \ldots, e_p\}$ of horizontal sections of $\nabla(\infty)$, which trivialize $E|\{\infty\} \times T$. Take U,$\{f_i\}$, $\{g_i\}$ and the $\{h_i\}$ as in (3.11)(ii) and choose the $\{f_i\}$ moreover such that $f_i \mid \{\infty\} \times U = e_i \mid U$ for all i. Then the h_i are trivial extensions of the $e_i \mid \tilde{U}$. If $U \subseteq T\text{-}\Theta$, then one shows as in remark (2.11)(ii) that the connection form of ∇ w.r.t. the h_i has on $\mathbb{P}^1(\mathbb{C}) \times U$ the form

$$\Omega = \sum_{1 \leq i \leq m} \left\{ \sum_{l \geq 1} \left\{ \frac{C_{il}(t)}{(x - a_i(t))^l} \right\} d(x - a_i) \right\}$$

and in particular if $t_0 \in U$ then $C_{il} = B_{il}$ for all i and l. Hence the B_{il} extend holomorphically to T-Θ. Finally the fact that they are meromorphic over T is a consequence of remark (3.11)(iii). This result generalizes the main theorem in the third section of [14]. Therefore we formulate it as a

3.13. Theorem *Under the assumptions stated above, the solutions* B_{il} *of the integrability equations extend holomorphically to T-*Θ *and meromorphically to T.*

3.14. Remark Finally we give a connection with some of the τ-functions, introduced in [12]. Let (E,∇) be an integrable deformation with deformation space as in example (2.7) and assume that ∇ has a logarithmic pole along Y. Consider the differential form

$$\omega_J = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \mathrm{Trace}(B_{i1} B_{j1}) \frac{d(a_i - a_j)}{a_i - a_j}.$$

From the integrability conditions for the B_{i1} one derives that ω_J is exact. In [Mi] they introduced τ_J by the formula $d\log\tau_J = \omega_J$. Now one shows as in [14] that $d\log\tau - \omega_J$ is a holomorphic differential form on T. Hence, since $\pi_1(T) = 1$, there is a holomorphic ϕ on T such that

$$\omega_J = \frac{d(\tau \exp(\phi))}{\tau \exp(\phi)}.$$

4. The Grassmannian interpretation

4.1. We start from a situation as described in the proof of (3.2)(ii), that is we consider a global map $t \mapsto \Sigma(t)$, a covering $\{V_i \mid i \in I\}$ of T, holomorphic maps q_i from V_i to the automorphisms of $\mathcal{H}_0$, satisfying property (3.8) and $\det(q_i(t)q_j(t)^{-1}) = 1$ for all t in $V_i \cap V_j$ and all i and j in I, and a τ-function defined by $\tau(t) = \det(a(t)q_i(t)^{-1})$. These data have an immediate interpretation in the context of the dual of the determinant bundle over the Grassmann manifold of $\mathcal{H}$. Therefore we recall shortly these ingredients and we start with that of the Grassmann manifold of $\mathcal{H}$. The description given here is in the spirit of [16].

Let H be a Hilbert space with the orthonormal base $\{v_i, i \in \mathbb{Z}\}$. For each $n \in \mathbb{Z}$, H_{-n} denotes the closure of the span of the $\{v_i, i \geqslant n\}$. Further we denote the orthogonal projection $H \mapsto H_0$ by p_0 and we write i_W for the canonical embedding of a subspace W into H. In the case of $\mathcal{H}$ one chooses a base such that $\mathcal{H}_0$ equals the closure of the span of the $\{v_i, i \geqslant 0\}$. To H and the given numbered base one associates

4.2. Definition The *Grassmann manifold* of H, Gr, consists of the closed subspaces W of H such that $p_0 \circ i_W$ is a Fredholm operator of index zero.

Recall that the Grassman manifold, consisting of all n-dimensional subspaces of $\mathbb{C}^{n+m}$, is isomorphic to the quotient of the space of complex $n \times (n+m)$-matrices of rank n under the natural left action of $Gl_n(\mathbb{C})$ or to say it in other words it is the space of all embeddings of $\mathbb{C}^n$ into $\mathbb{C}^{n+m}$ divided by the group of automorphisms of $\mathbb{C}^n$. An analogue of this description exists also for Gr. For each $W \in Gr$, consider

$$S(W) = \left\{ s \;\middle|\; s \in \mathbb{Z}, \text{ there is a w in W of the form } w = v_s + \sum_{k<s} a_k v_k \right\}$$

Since the kernel of $p_0 \circ i_W$ is finite-dimensional, S(W) has a lower bound and we can write $S(W) = \{s_0, s_1, ...\}$ with $s_{i+1} > s_i$ for all $i \geqslant 0$. The fact that the image of $p_0 \circ i_W$ has finite codimension in H_0 implies that there exists a $m > 0$ such that the projection of W onto H_{-m} is surjective. Hence all $n \geqslant m$ belong to S(W). It will be clear that the projection of W onto the closed span of the v_s, $s \in S(W)$, is an isomorphism. From the fact that the index of $p_0 \circ i_W$ is zero, we conclude that for all $n \geqslant m$, $s_n = n$. Let w_i, $i \geqslant 0$, be the inverse image under the isomorphism just described of the element v_{s_i}, $i \geqslant 0$, then we define an embedding $w: H_0 \rightarrow H$ by

$$\sum_{k \geqslant 0} a_k v_k \mapsto \sum_{k \geqslant 0} a_k w_k$$

Clearly W is the image of w and if we use the notations w_+ for $p_0 \circ w$ and w_- for $(id - p_0) \circ w$ then we see that w_+ is of the form "identity + a finite-dimensional operator". It is convenient to see the elements of Gr as images of a wider class of embeddings. If we consider namely a continuous embedding w of H_0 into H such that w_+ is of the form "identity + trace-class", then the image of w belongs also to Gr. The collection of all embeddings of this form we denote by $\mathcal{P}$. Thus the assignment π: w $\mapsto$ the image of w, maps $\mathcal{P}$ onto Gr. If $\pi(w_1) = \pi(w_2)$, then w_1 and w_2 differ necessarily by an automorphism of H_0 of the form "identity + trace-class". Denoting this group by $\mathcal{T}$, we get an identification of $\mathcal{P} / \mathcal{T}$ with Gr. Note that for each t in $\mathcal{T}$ we can speak of the determinant of t, det(t). This is the analogue of the description of the finite-dimensional Grassmann manifold, which we announced above.

Next we will put a topology on Gr and to do so we start with one on $\mathcal{P}$: a basis of neighbourhoods of w in $\mathcal{P}$ consists of

$$\left\{p \;\middle|\; p\in\mathcal{P}, ||p_+ - w_+||_1 < \epsilon \text{ and } ||p_- - w_-|| < \epsilon, \epsilon > 0\right\},$$

where $||-||_1$ is the trace-class norm and $||-||$ the operator norm. $\mathcal{T}$ becomes a topological group if we take as neighbourhood base of the identity $\{t | t\in\mathcal{T}, ||t-\mathrm{id}||_1<\epsilon\}$, $\epsilon>0$. It is ~~not~~ difficult to see that the natural right action of $\mathcal{T}$ on $\mathcal{P}$ is continuous. On Gr we put then the quotient topology.

Apart from the description given above of Gr, we can also see it as an orbit of a certain group of automorphisms of H under its natural action. To be more precise, let $Gl_1(H)$ be the group of bounded automorphisms of H which have with respect to the decomposition $H=H_0\oplus H_0{}^{\perp}$ the form

$$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{array}{c}\text{a and d are Fredholm operators with} \\ \text{index(a)=index(d)=0, c is bounded} \\ \text{and b is a nuclear operator.}\end{array}\right\}$$

$Gl_1(H)$ acts on Gr by its natural action on H and it is not difficult to see that this action is transitive. Further $Gl_1(H)$ is easily seen to act continuously on Gr if we take as a neighbourhood base of the identity in $Gl_1(H)$:

$$\left\{g \;\middle|\; g\in Gl_1(H),\ g=\begin{bmatrix} a & b \\ c & d \end{bmatrix},\ ||a-\mathrm{id}||<\epsilon,\ ||d-\mathrm{id}||<\epsilon,\ ||c||<\epsilon,\ ||b||_1<\epsilon\right\}.$$

Here $||-||_1$ denotes the "Schatten"-norm on the space of nuclear operators, see [9]. Nuclear endomorphisms of a separable Hilbert space are precisely the trace-class operators and then this norm coincides with the trace-class norm.

On Gr we define the determinant bundle Det and its dual Det* as the orbit spaces corresponding respectively to the following actions of $\mathcal{T}$ on $\mathcal{P}\times\mathbb{C}$:

$$(w,\lambda) \mapsto (w\circ t, \lambda\det(t)^{-1})$$
$$(w,\lambda) \mapsto (w\circ t, \lambda\det(t))$$

Here w belongs to $\mathcal{P}$, t to $\mathcal{T}$ and λ lies in $\mathbb{C}$. From the second action it will be clear that we can define a section σ of Det* by

$$\sigma(\pi(w))=(w,\det(w_+)).$$

Since it is not clear how to define an action of $Gl_1(H)$ on Det or Det* we pass to an extension G of $Gl_1(H)$. First we note that for each $g=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $Gl_1(H)$ there exists an automorphism q of H_0 such that aq^{-1}-id is finite-dimensional. For G we take now the collection of pairs (g,q) with g as above in $Gl_1(H)$ and q an automorphism of H_0 such that $aq^{-1}-\mathrm{id}$ is trace-class. This set forms w.r.t. the natural composition a group. Since for each t in $\mathcal{T}$ and for each q in $\mathrm{Aut}(H_0)$ $\det(t)=\det(qtq^{-1})$, we can define an action of G on Det* by

$$(g,q).(w,\lambda)=(g\circ w\circ q^{-1},\lambda)$$

G becomes a continuous transformation group of Det* if we equip it with the topology generated by

$$\left\{\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix},q\right)\in G \;\middle|\; \begin{array}{c}||a-\mathrm{id}||<\epsilon,\ ||d-\mathrm{id}||<\epsilon,\ ||q-\mathrm{id}||<\epsilon, \\ ||aq^{-1}-\mathrm{id}||_1<\epsilon,\ ||b||_1<\epsilon,\ ||c||<\epsilon\end{array}\right\}$$

Let T_1 be the normal subgroup of G given by $\{(\mathrm{id},q)\in G,\ \det(q)=1\}$. Then one checks directly that T_1 consists exactly of all elements in G, which leave all points of Det^* fixed. We denote G/T_1 by $\tilde{G}$ and put the quotient-topology on it.

4.3. As we will show at the end of this paragraph all the $\mathbb{S}(t)$ belong to $Gl_1(H)$. Consider the maps $t\mapsto(\mathbb{S}(t),q_i(t))$. The second property required of the q_i at the beginning of this paragraph implies that they define a global lifting of $\mathbb{S}$ to $\tilde{G}$. As well the τ-functions occurring in the KP-hierarchy as the τ-functions considered here are examples of the following construction : let there be given a collection of elements g(z) in $Gl_1(H)$, depending of a parameter z from some parameterspace Z, and let $z\mapsto(g(z),q(z))$ be a lifting of this map to $\tilde{G}$. Then the failure of σ-equivariance of this lifting at the point W of Gr is described by the function τ_W from Z to $\mathbb{C}$,

$$\tau_W(z)=\det(a(z)\circ w_+\circ q(z)^{-1}+b(z)\circ w_-\circ q(z)^{-1}).$$

Assume that Z is such that one can speak of analytic maps from Z to $Gl_1(H)$ resp. $\tilde{G}$ and that $\mathbb{S}$ and the lifting are analytic, then also all the τ_W are analytic, see e.g. [9]. This is also the case for the KP-hierarchy, where the parameterspace is a certain group of holomorphic functions on the unit circle.

4.4. We conclude this paragraph by showing that the $\mathbb{S}(t)$ belong to $Gl_1(H)$. Thanks to remark (3.7)(iv), it is sufficient to take a holomorphic map S from D_2 to $Gl_p(\mathbb{C})$, to consider, as in lemma (3.5), the automorphism $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ of $\mathcal{H}$, which it defines, and to show that b is nuclear. Put $S(x)=\sum_{i\geq 0} S_i x^i$, then the fact that S is analytic on D_2 implies that for all $k\geq 0$

$$\sum_{i\geq o} i^k\,||S_i||<\infty,$$

where $||S_i||$ is the operator norm of S_i w.r.t. to its natural action on $\mathbb{C}^p$, which we equip with the standard innerproduct $<-,->$. Denote the standard basis of $\mathbb{C}^p$ by $\{e_1,\ldots,e_p\}$. For $\mathcal{H}_0$ we take the orthonormal basis : $e_r e^{2\pi i k\phi}$, $1\leq r\leq p$, $k\geq 0$, and for $\mathcal{H}_0^\perp$: $e_s e^{-2\pi i l\phi}$, $1\leq s\leq p$, $l\geq 1$. The matrix-coefficients of b w.r.t. these bases are the $<S_{k+l}(e_s),e_r>$. For these coefficients we have the estimate

$$\sum_{\substack{1\leq r\leq p\\ 1\leq s\leq p}}\ \sum_{\substack{k\geq 0\\ l\geq 1}}^{\infty} |<S_{k+l}(e_s),\,e_r>| < p^2\sum_{\substack{k\geq 0\\ l\geq 1}}^{\infty} ||S_{k+l}|| < p^2\sum_{r\geq 1} r\,||S_r|| < \infty$$

and according to the characterization of nuclear operators given in [10] this shows that b is nuclear.

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