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ON M-FUNCTIONS AND NONLINEAR RELAXATION METHODS

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On M-functions and nonlinear relaxation methods<sup>\*)</sup>

by

E.J. van Asselt

ABSTRACT

Globally convergent nonlinear relaxation methods are considered for a class of nonlinear BVPs, where the discretizations are continuous M-functions.

It is shown that the equations with one variable occurring in the nonlinear relaxation methods can always be solved by Newton's method combined with the Bisection method. The nonlinear relaxation methods are used to get an initial approximation in the domain of attraction of Newton's method. Numerical examples are given.

KEY WORDS & PHRASES: *nonlinear relaxation methods, Newton-Bisection method, M-functions*

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\*) This report will be submitted for publication elsewhere.

## 1. INTRODUCTION

In section 2 we introduce nonlinear singularly perturbed elliptic BVP in 2 dimensions, where the discretizations with the first order Osher-Engquist scheme (cf. OSHER [5]) are continuous M-functions. The solutions of the discrete equations are unique, and with the theory of M-functions it follows that the nonlinear Jacobi (NLJAC) and the nonlinear SOR (NLSOR) process are globally convergent (cf. ORTEGA [4]).

In section 3 we show that the equations with one variable occurring in NLJAC and NLSOR can always be solved by Newton's method combined with the bisection method. We will give a 2-D example with an initial approximation for which NLJAC with Newton's method (NLJAC-N) does not converge, whereas NLJAC with Newton's method combined with bisection (NLJAC-NB) does.

In section 4 we use NLSOR-NB to get an initial approximation in the domain of attraction of Newton's method for the whole system of equations (NEWT).

## 2. A CLASS OF NONLINEAR BVPs.

We consider the following class of nonlinear BVPs in two dimensions:

$$(2.1) \quad N_{\varepsilon} u \equiv -\varepsilon_1 \frac{\partial}{\partial x^2} u - \varepsilon_2 \frac{\partial}{\partial y^2} u + a_1(u) \frac{\partial}{\partial x} u + a_2(u) \frac{\partial}{\partial y} u + g(u, x, y) \\ = 0, \text{ on } \Omega = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}, \text{ and}$$

$$(2.2) \quad u(x, y) = b(x, y) \text{ on } \delta\Omega = \bar{\Omega} \setminus \Omega, \text{ with } \varepsilon = (\varepsilon_1, \varepsilon_2), \varepsilon_1, \varepsilon_2 > 0;$$

$$a_1, a_2 \in C^1(\mathbb{R}), g \in C^2(\mathbb{R} \times \Omega), \frac{\partial}{\partial u} g(u, x, y) \geq \mu > 0 \text{ for all } (x, y) \in \Omega,$$

$$u \in C^2(\Omega) \text{ and } b \in C(\delta\Omega).$$

For the discretization of this problem we introduce  $G_h$ , a uniform rectangular grid with mesh-size  $h = (h_1, h_2)$ ,  $h_1 = 1/m$ ,  $h_2 = 1/n$  and meshpoints  $x_{ij}^h = (ih_1, jh_2)$ ,  $0 \leq i \leq m$ ,  $0 \leq j \leq n$ , and use the one sided Osher-Engquist difference scheme (cf. OSHER [5]):

$$\begin{aligned}
(2.3) \quad (N_{\varepsilon, h} u^h)_{ij} &\equiv -\frac{\varepsilon_1}{h_1^2}(u_{i+1j}^h - 2u_{ij}^h + u_{i-1j}^h) - \frac{\varepsilon_2}{h_2^2}(u_{ij+1}^h - 2u_{ij}^h + u_{ij-1}^h) \\
&\quad + \frac{1}{h_1}(\Delta_{x^+} f_{i-}^h(u_{ij}^h) + \Delta_{x^-} f_{i+}^h(u_{ij}^h)) \\
&\quad + \frac{1}{h_2}(\Delta_{y^+} f_{j-}^h(u_{ij}^h) + \Delta_{y^-} f_{j+}^h(u_{ij}^h)) + g(u_{ij}^h, x_{ij}^h) \\
&= 0 \text{ for all } x_{ij}^h \in G_h \cap \Omega,
\end{aligned}$$

$$(2.4) \quad (N_{\varepsilon, h} u^h)_{ij} \equiv u_{ij}^h = b(x_{ij}^h) \text{ for all } x_{ij}^h \in G_h \cap \delta\Omega, \text{ where for } m = 1, 2:$$

$$f_{m^+}^h(u_{ij}^h) = \int_{u_{ij}^h}^{\infty} \max(0, a_m(s)) ds,$$

$$f_{m^-}^h(u_{ij}^h) = \int_{-\infty}^{u_{ij}^h} \min(0, a_m(s)) ds,$$

$$\Delta_{x^+} u_{ij}^h = u_{i+1j}^h - u_{ij}^h,$$

$$\Delta_{x^-} u_{ij}^h = u_{ij}^h - u_{i-1j}^h,$$

$$\Delta_{y^+} u_{ij}^h = u_{ij+1}^h - u_{ij}^h,$$

$$\Delta_{y^-} u_{ij}^h = u_{ij}^h - u_{ij-1}^h.$$

Denoting the space of gridfunctions on  $G_h$  with  $GF_h$ , and extending the results of Lorentz (cf. LORENTZ [3]) to 2 dimensions, we can easily prove that  $N_{\varepsilon, h}$  is a continuous M-function from  $GF_h$  onto  $GF_h$ .

In the following section we use the property that M-functions are strictly diagonally isotone (cf. ORTEGA [4]).

### 3. THE NONLINEAR RELAXATION METHODS.

In this section we consider NLJAC and NLSOR, and give a 2-D example for which NLJAC does not converge when the resulting scalar nonlinear equations are solved with Newton's method only, while NLJAC converges when we use

Newton's method combined with bisection.

Let  $F(x) = 0$  with  $F = (f_1(x), \dots, f_n(x))$ ,  $x = (x_1, \dots, x_n)$  be a system of  $n$  nonlinear equations. Let  $x^{(0)}$  be an initial approximation to the solution, then  $x^{(k+1)}$  is obtained by solving for  $x_i$  the  $i$ -th equation:

$$(3.1) \quad f_i(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k)}, \dots, x_n^{(k)}) = 0 \text{ for NLJAC, or}$$

$$(3.2) \quad f_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i, x_{i+1}^{(k)}, \dots, x_n^{(k)}) = 0 \text{ for NLSOR, and setting}$$

$$(3.3) \quad x_i^{(k+1)} = x_i^{(k)} + \omega(x_i - x_i^{(k)}), \quad \omega \in (0, 1], \quad i = 1, 2, \dots, n.$$

For our class of equations  $F$  is a continuous  $M$ -function, and both NLJAC and NLSOR converge globally to the unique solution of (2.3), (2.4), providing that the (generally nonlinear) equations (3.1), (3.2) for which a unique solution exists can be solved (cf. ORTEGA [4], Theorem 13.5.9).

For  $M$ -functions  $\frac{df_i}{dx_i} > 0$ . This implies that for a positive (negative) function value of an iterant  $x_{i(r)}$  the next Newton iterant  $x_{i(r+1)}$  is always smaller (greater) than  $x_{i(r)}$ , and we can start with only a single initial approximation. When the function values of the subsequent iterants never change sign only Newton iteration is applied, and we have monotone convergence. However if the function value of an iterant  $x_{i(r)}$  changes sign we have an interval round the solution:  $I_{i(r)} = (\min(x_{i(r-1)}, x_{i(r)}), \max(x_{i(r-1)}, x_{i(r)}))$ . This interval is adapted in each iteration step. If  $I_{i(r+1)} \subset I_{i(r)}$  Newton's method is applied, if this is not the case one step of the bisection method is applied.

As the bisection method always converges the above mentioned combination always converges.

This process is described in the following Algol-like procedure NEWTON BISECTION. The procedure takes the current value of  $x$  as an initial value and delivers in  $x$  an approximate solution to  $f(x) = 0$ , so that  $|f(x)| < \text{tol}$ .

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(3.4)      F      : THE FUNCTION FOR WHICH THE ZERO HAS TO BE DETERMINED
           DF      : THE DERIVATIVE OF F WITH THE PROPERTY DF > 0
           TOL     : A GIVEN TOLERANCE
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'PROCEDURE' NEWTON BISECTION:
'BEGIN'
'INT'      SGN      = 'SIGN'(F(X)) ;
'REAL'     XOLD, XNEW, XNEG, XPOS;
           XNEW := X           ;
'WHILE'    'ABS'(F(XNEW)) > TOL 'AND' 'SIGN'(F(XNEW)) = SGN
'DO'       XOLD := XNEW;
           XNEW := XOLD - F(XOLD)/DF(XOLD)

'OD';
'IF'       'SIGN'(F(XNEW)) > 0
'THEN'     XPOS := XNEW; XNEG := XOLD
'ELSE'     XNEG := XNEW; XPOS := XOLD
'FI';

           XOLD := XNEW;
'WHILE'    'ABS'(F(XNEW)) > TOL
'DO'       'WHILE' 'ABS'(XNEW-XOLD) < XPOS-XNEG 'AND'
           'ABS'(F(XNEW)) > TOL
'DO'       'IF'     'SIGN'(F(XNEW)) > 0
           'THEN'   XPOS := XNEW
           'ELSE'   XNEG := XNEW
           'FI';
           XOLD := XNEW;
           XNEW := XOLD - F(XOLD)/DF(XOLD)

'OD';
'IF'       'ABS'(F(XNEW)) > TOL
'THEN'     XNEW := (XNEG + XPOS) * 0.5
'FI'

'OD'; X := XNEW
'END'

```

Both NLJAC-NB and NLSOR-NB converge to the solution and in the course of these processes in each point we will have a better initial approximation for the Newton-Bisection process. This means that for a given tolerance TOL in the course of the NLJAC-NB and NLSOR-NB process the required number of Newton-Bisection iteration steps decreases (cf. TABLE 1, 2, 3). This phenomenon is used in section 4, where NLJAC-NB and NLSOR-NB are used to get an

initial approximation for Newton's method.

In example (3.5) NLJAC-N does not converge, whereas NLJAC-NB does.

(3.5) EXAMPLE.

We consider (2.1)-(2.4) with  $\varepsilon_1 = \varepsilon_2 = 10^{-6}$ ,  $h_1 = h_2 = 1/32$ ,  
 $a_1(u) = a_2(u) \equiv \arctan(4u)$ ,  $g(u,x,y) \equiv u$  and  $b$  defined by

$$\begin{aligned} b(0,y) &= 0 & 0 \leq y \leq 1 & , \\ b(1,y) &= -1 & 0 \leq y \leq 1-10^{-3} & , \\ b(1,y) &= 10^3 y - 10^3 & 1-10^{-3} \leq y \leq 1 & , \\ b(x,0) &= -1 & 10^{-3} \leq x \leq 1 & , \\ b(x,0) &= -10^3 x & 0 \leq x \leq 10^{-3} & , \\ b(x,1) &= 0 & 0 \leq x \leq 1 & . \end{aligned}$$

For the gridfunction  $u^h$  in (2.3), (2.4) we take the lexicographical ordering (i.e.  $u^h = (\dots, u_{ij}^h, u_{ij+1}^h, \dots, u_{i+1j}^h, u_{i+1j+1}^h)$ ). As initial approximation for NLJAC-N ( $\omega=1$ ) and NLJAC-NB ( $\omega=1$ ) we take the gridfunction  $u^{(0)}$  defined by  $u_{ij}^{(0)} \equiv -1$  on  $G_h \cap \Omega$  and  $u_{ij}^{(0)} \equiv b$  on  $G_h \cap \delta\Omega$ .

NLJAC-N does not converge, while we need  $k = 4$  NLJAC-NB steps to obtain  $\|N_{\varepsilon,h} u^{(k)}\| \leq 10^{-12}$  with  $\|\cdot\|$  the maximum norm.

For NLJAC-NB in Table 1 we give the maximum number of Newton-Bisection iterations for  $TOL = 10^{-8}$  (cf. (3.4)).

i	1	2	3	4
N	42	2	1	1

Table 1. The maximum number of Newton-Bisection iterations per NLJAC-NB iteration step  $i$ . ( $TOL = 10^{-8}$ ).

#### 4. NLJAC-NB AND NLSOR-NB COMBINED WITH NEWT.

In this section we use NLJAC-NB and NLSOR-NB to get an initial approximation within the domain of attraction of NEWT.

There are other methods which use time-steps combined with NEWT (cf. ABRAHAMSSON, OSHER [1]). We consider here NLJAC-NB and NLSOR-NB because these can be used as relaxation processes in Multigrid methods (cf. STÜBEN, TROTTEBERG [6]). Moreover the number of Newton-Bisection steps can be used



as a criterion to switch from NLJAC-NB or NLSOR-NB to NEWT, as the number of Newton-Bisection iterations decreases in the course of these processes.

So instead of trying NEWT after each NLJAC-NB or NLSOR-NB iteration step we try it when the maximum number of Newton-Bisection iterations is sufficiently small.

In the following 1-D example we have an initial approximation for which NEWT does not converge while NLJAC-NB and NLSOR-NB do.

We also show that the combination NLSOR-NB with NEWT is cheaper than NLSOR-NB alone.

(4.1) EXAMPLE.

Consider the 1-D problem

$$(4.2) \quad N_{\varepsilon} u \equiv -\varepsilon u'' + (u^2 - 1/4)u' + u = 0 \text{ on } [0,1] \text{ with } \varepsilon = 10^{-6},$$

$u(0) = 1$ ,  $u(1) = -1$ , discretized by the Osher-Engquist scheme (cf. (2.3), (2.4)) on a uniform grid with mesh size  $1/20$  (cf. ABRAHAMSSON, OSHER [1], VAN ASSELT [2]).

NEWT with initial approximation  $u^{(0)}$  defined by  $u_j^{(0)} = 1 - 2x_j$ ,  $j=0, \dots, 20$  does not converge while NLJAC-NB and NLSOR-NB ( $\omega=1$ ) do. We need  $k=58$  NLJAC-NB or  $k=39$  NLSOR-NB steps for  $\|N_{\varepsilon, h} u^{(k)}\| \leq 10^{-6}$ , with  $\|\cdot\|$  the maximum norm.

For NLJAC-NB and NLSOR-NB respectively in Table 2 and Table 3 we give the maximum number of Newton-Bisection iterations with  $TOL = 10^{-8}$  (cf. (3.4)).

i	1	2	3-5	6	7-8	9-17	18-38	39-58
N	7	6	4	5	4	3	2	1

Table 2. The maximum number of Newton-Bisection iterations N per NLJAC-NB iteration step i. ( $TOL = 10^{-8}$ ).

i	1	2-3	4-5	6	7	8	9-12	13-20	21-29	30-39
N	7	5	4	5	4	3	4	3	2	1

Table 3. The maximum number of Newton-Bisection iterations N per NLSOR-NB iteration step i. ( $TOL = 10^{-8}$ ).

When we try NEWT after that the maximum number of Newton-Bisection iterations in NLSOR-NB becomes 2, we need 21 NLSOR-NB steps and 2 NEWT steps for  $\|N_{\epsilon, h} u\| \leq 10^{-6}$ , with  $\|\cdot\|$  the maximum norm.

An operations and function evaluations count shows that for 1 NEWT step the number of operations and function evaluations per gridpoint is less than for 2 NLSOR-NB steps, with one or more Newton-Bisection iterations per gridpoint. Hence for this example the combination NLSOR-NB and NEWT is more than 35% cheaper than NLSOR-NB solely.

## 5. CONCLUSIONS

For continuous M-functions the equations with one variable occurring in the globally convergent nonlinear relaxation methods NLJAC and NLSOR can always be solved by Newton's method combined with the Bisection method, and we need only a single initial approximation. Newton's method alone is not always sufficient.

The nonlinear relaxation methods can be used to get an initial approximation for Newton's method for the whole system of equations, where the decreasing number of Newton-Bisection iterations can be used as switching criterion.

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