TW 228/82: H.A. Lauwerier Entire Functions for the Logistic Map I

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ENTIRE FUNCTIONS FOR THE LOGISTIC MAP I

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Entire functions for the logistic map I

by

H.A. Lauwerier

ABSTRACT

A class of entire functions is introduced satisfying a functional equation of the logistic type. Computer plots of the zeros reveal phenomena that can be understood by a perturbation analysis in the "chaotic" regime.

KEY WORDS & PHRASES: *Entire functions, Difference equations, Iterated maps*

1. INTRODUCTION

This report is the first in a series of reports dealing with complex analysis functions and iterated maps. In our opinion the important results obtained recently in connection with simple iterated maps such as the logistic difference equation have their counterpart in the asymptotic behaviour of certain entire functions in the complex plane. The idea can be made clear by considering the following elementary example.

To the special logistic iterated map

$$(1.1) \qquad x_{n+1} = 4x_n(1-x_n)$$

we associate the addition rule

$$(1.2) \qquad F(4z) = 4F(z)(1-F(z))$$

for some analytic function $F(z)$.

Obviously (1.2) is solved by

$$(1.3) \qquad F(z) = \sin^2\sqrt{z},$$

an entire function of fractional order 1/2.

Thus (1.1) can be parametrized by

$$(1.4) \qquad x_n = \sin^2(2^n\theta\pi)$$

The properties of the map (1.1) can be studied using this representation. Moreover (1.4) gives a suitable extension of the difference equation (1.1) to the continuous case in the same sense as the gamma function is the extension of the factorial function. Furthermore the representation (1.4) shows in a simple way what happens if x_n and θ are complex.

The next elementary example is

$$(1.5) \qquad x_{n+1} = 2x_n(1-x_n).$$

The corresponding addition rule

(1.6) $$F(2z) = 2\,F(z)(1-F(z))$$

is solved by

(1.7) $$F(z) = \tfrac{1}{2}(1-e^{-2z}),$$

an entire function of order 1. This gives the parametrization

(1.8) $$x_n = \tfrac{1}{2}\,(1-e^{-2^n\theta\pi}).$$

It will be shown here that the general iterated logistic map

(1.9) $$x_{n+1} = ax_n(1-x_n) \quad , \quad 1<a\leq 4 \;,$$

can be parametrized as

(1.10) $$x_n = F(a^n\theta)\,,$$

where F(z) is a well-defined entire function satisfying the addition rule

(1.11) $$F(az) = a\,F(z)(1-F(z)).$$

In the subsequent sections these functions will be subjected to a careful analysis. Of particular interest are the structure of the zero set of F(z) and the behaviour of F(z) on the positive real asis. A number of interesting results have been obtained using a HP 85 computer and a plotter. It will be shown that the entire functions satisfying (1.11) are of fractional order μ with

(1.12) $$\mu = \frac{\log 2}{\log a}\,, \qquad 2 = a^{\mu}.$$

In the next section we prove the more general result that a functional equation of the kind

(1.13) $\quad F(az) = \phi(F(z))$,

where $\phi(z)$ is holomorphic at $z = 0$ with

(1.14) $\quad \phi(0) = 0\ ,\ \phi'(0) = a\ ,\ |a| > 1$

can be solved by a holomorphic function. Moreover, if $\phi(z)$ is entire - e.g. polynomial - then $F(z)$ is an entire function. In section 3 the parametrizations (1.4) and (1.8) are studied in more detail. The power series expansion of the entire function $F(z)$ defined by (1.11) and normalized by $F'(0) = 1$ is explicitly determined in the form

(1.15) $$F(z) = \sum_{k=1}^{\infty} (-1)^{k-1}\ c_k z^k\ ,$$

where the coefficient satisfy a quadratic recurrent relation. It is proved that $F(z)$ is of the order μ as given by (1.12). For c_k an upper bound has been derived which shows that the type σ of $F(z)$ does not exceed 2. Computer results show that $\sigma = 2$ only for $a = 2$ and $a = 4$ and that $1.826 < \sigma < 2$ in the interval (2.4) with the lower bound at $a = 2.60$ (cf.table 1). In section 4 the zeros of $F(z)$ are studied in dependance of the parameter a, $2 \leq a \leq 4$. Obviously for any zero ζ of $F(z)$ also $a\zeta, a^2\zeta, a^3\zeta\ldots$ are zeros but on each ray there is a first zero ζ such that $F(\zeta/a) = 1$. Therefore we consider the roots of $F(z) = 1$ instead. Figures 1,2 and 3 show that those zeros traverse arcs in the complex plane with a curious way of forming bundles at certain places. A perturbation procedure at $a = 4$ given a complete understanding of this phenomenon. Apparently it has to do with a small value of $\cos(2^{-k}(n+\frac{1}{2})\pi)$ for some $k = 1,2,3\ldots$ and where n numbers the zero. This takes place for $n = 4,8,16,22\ldots$. The first few perturbation terms of the n-th zero are given explicitly in theorem 4.1. They appear to give quite good approximations even when a is appreciably less than 4, say $a = 3.9$.

In a subsequent report we shall consider the asymptotic properties of $F(z)$ in the complex plane as $|z| \to \infty$ and the behaviour of the entire functions on the positive real axis for various values of a. Figures 4 and 5 show typical cases of "strange" behaviour.

Functional equations of the kind (1.11) have been considered in the nineteenth century by a number of authors starting with Abel. The following equation bears his name

$$F(\phi(z)) = F(z) + a \, , \tag{1.16}$$

where $\phi(z)$ is a given function and a a given constant. SCHRÖDER (1870,1871) has given his name to the following equation

$$F(\phi(z)) = a\, F(z) \tag{1.17}$$

which is merely an equivalent from of Abel's equation. The first systematic treatment of the subject was given by KOENIGS (1884,1885). His main result is as follows:

Let the iterated map

$$z_{n+1} = f(z_n) \, , \; z_0 = z \tag{1.18}$$

have the attracting fixed point 0 with $a = f'(0) \neq 0$ and assume that $f(z)$ is holomorphic at $z = 0$ then the limit

$$a^{-n} f(z_n) \to F(z) \tag{1.19}$$

exists as a holomorphic function satisfying the Schröder equation (1.17) with $F(0) = 0$ and $F'(0) = 1$. The work of Koenigs is briefly reviewed by FATOU (1919,1920), KNESER (1950) and DE BRUYN (1961).

If in (1.17) $F(z)$ is replaced by its inverse we obtain the functional equation

$$F(az) = \phi\,(F(z)). \tag{1.20}$$

Our functional equation (1.11) is a special case of this. Equations of the kind (1.20) are mentioned by Fatou. He remarks that a solution of (1.20) gives a parametrization of the map $z \to \phi(z)$ but this possibility is not worked out any further. Functional equations of the kind (1.20) have also been considered by POINCARÉ (1890) in a general setting when considering functions admitting a multiplication rule.

Iterated maps have recently gained an enormous popularity. In a stimulating review the biologist MAY (1976) showed that a simple model such as (1.9) describing a succession of generations of say blowflies has a very complicated intriguing behaviour. A few excellent surveys showing the present state of the art are given in the references. Between 1918 and 1920 Julia and Fatou published comprehensive studies of iterated maps. Their work is of great historic and still of actual interest. Their main result is the description of the closure F of the set of repelling periodic points of the iterated map. They show that F is a perfect set in the complex plane and that there are three possibilities

1. F is the entire plane.
2. F is linearly connected but has no interior points and consists of a collection of arcs.
3. F is disconnnected and is a Cantor set.

Moreover, as a rule F is a curve without tangents such as the snowflake curve of Hedge von Koch, i.e. a fractal in modern language. Although both writers use mainly function theoretic techniques such as conformal mapping and normal families the subject is hardly brought into relation with the Schröder equation and its inverse.

It seems that the time is ripe to revive the subject. Undoubtedly the new tools of modern analysis and the facilities of the computers may enable us to solve old problems and to create new ones. In this connection we draw attention to a very recent note by Sullivan.

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2. FUNCTIONAL EQUATIONS

In order to get some experience we start with the simplest iterative map

(2.1) $$x_{n+1} = ax_n + b \quad , \quad a \neq 1.$$

It is solved by

(2.2) $$x_n = -\frac{b}{a-1} + Ca^n ,$$

where C is a constant.
The correponding functional equation

(2.3) $$F(az) = a\,F(z) + b ,$$

where F(z) is holomorphic at z = 0 is solved by a linear function since

$$F^{(k)}(0) = 0 \qquad \text{for all} \quad k \geq 2 .$$

Hence we must have

(2.4) $$F(z) = -\frac{b}{a-1} + cz ,$$

where c is an arbitrary constant. The solution of (2.3) can be made unique by the additional requirement F'(0) = 1.

The next example is less trivial but still elementary

(2.5) $$x_{n+1} = \frac{ax_n}{1-x_n} .$$

The iteration is linear in x_n^{-1} and it is not difficult to obtain the following solution

(2.6) $$x_n = \frac{(a-1)a^n}{C-a^n} .$$

The corresponding functional equation

$$(2.7) \qquad F(az) = \frac{aF(z)}{1-F(z)}$$

is apparently solved by

$$(2.8) \qquad F(z) = \frac{(a-1)z}{c-z},$$

where c is an arbitrary constant. For

$$F'(0) = 1 \quad , \quad c = a-1$$

we obtain the unique solution

$$(2.8a) \qquad F(z) = \frac{(a-1)z}{a-1-z}.$$

We note that F(z) has a pole at $z = a-1$.

Next we consider the general case

$$(2.9) \qquad x_{n+1} = \phi(x_n) \quad , \quad |\phi'(0)| > 1$$

where $\phi(z)$ is holomorphic at the origin with

$$\phi(0) = 0 \text{ , } a = \phi'(0).$$

This means that $x = 0$ is an unstable fixed point. The corresponding functional equation is

$$(2.10) \qquad F(az) = \phi\,(F(z)).$$

Imposing the additional requirement

$$F'(0) = 1$$

we try a formal power series expansion

$$(2.11) \qquad F(z) = \sum_{k=1}^{\infty} c_k z^k \quad , \quad c_1 = 1.$$

Let $\phi(z)$ be given by its Taylor series

$$(2.12) \qquad \phi(z) = \sum_{k=1}^{\infty} a_k z^k \quad , \quad a_1 = a,$$

then we have the formal identity

$$(2.13) \qquad \sum_{n=1}^{\infty} c_n (az)^n = \sum_{k=1}^{\infty} a_k \left(\sum_{\ell=1}^{\infty} c_\ell z^\ell \right)^k .$$

Equating coefficients of equal powers of z we obtain

$$a^2 c_2 = ac_2 + a_2,$$

$$a^3 c_3 = ac_3 + 2a_2 c_2 + a_3.$$

Generally $(a^k - a)c_k$ can be obtained from the c_j with indices up to $k-1$ by a polynomial expression with positive coefficients. There exists a positive constant r such that

$$|a_k| \leq \frac{|a|}{r^{k-1}}$$

for all k.

If in the relations from which the c_k are to be determined a is replaced by $|a|$ and $|a_k|$ is replaced by $|a|r^{-k}$ we obtain upper bounds for the original c_k. What we have done is the special choice

$$\tilde{\phi}(z) = \sum_{k=1}^{\infty} \frac{|a|}{r^{k-1}} z^k = \frac{|a|z}{1 - \frac{z}{r}}$$

However, in that case we can use the functional equation (2.7) with the solution (2.8). Thus we have the result

$$\tilde{F}(z) = \frac{(|a|-1)rz}{(|a|-1)r - z} .$$

This function is holomorphic at $z = 0$ and this fact already guarantees that (2.10) has an analytic solution which is holomorphic at 0. In fact, we have the inequality

$$|c_{k+1}| \leq \frac{1}{(|a|-1)^k r^k} .$$

By virtue of the functional equation (2.10) any domain of regularity can be enlarged by a factor $|a|$. Therefore if $\phi(z)$ is free from singularities it can be enlarged indefinitely. In other words, if $\phi(z)$ is entire then (2.10) has an entire solution which is unique with $F'(0) = 1$. Thus we have proved the following property.

<u>THEOREM 2.1.</u> *The functional equation*

$$F(az) = \phi(F(z))$$

where $\phi(z)$ *is holomorphic at* $z = 0$ *with*

$$\phi(0) = 0 \quad , \quad \phi'(0) = a \quad , \quad |a| > 1$$

has a solution $F(z)$ *holomorphic at* $z = 0$ *with* $F(0) = 0$. *With the additional condition* $F'(0) = 1$ *the solution is unique. If* $\phi(z)$ *is an entire function then also* $F(z)$ *is entire.*

It may be of interest to list a few cases of (2.10) where the solution is a well-known special function:

$$(2.14) \qquad F(2z) = \frac{2F(z)}{1-F^2(z)}$$

has the solution

$$(2.15) \qquad F(z) = \tan z.$$

$$(2.16) \qquad F(4z) = \frac{4F(z)(1-F(z))(1-k^2F(z))}{(1-k^2F^2(z))^2}$$

has the solution

(2.17) $F(z) = sn^2\sqrt{z}.$

(2.18) $F(-2z) = F^2(z) - 2F(z)$

has the solution

(2.19) $F(z) = 1 - 2\cos(\frac{z}{\sqrt{3}} + \frac{\pi}{3}).$

3. POWER SERIES EXPANSION

We have seen in the introduction that the logistic iterated map

(3.1) $x_{n+1} = ax_n(1-x_n)$

is elementary for $a = 2$ and $a = 4$. We now give some more details. For $a = 2$ we have the explicit solution (1.8). This formula also shows what can be expected in the complex case. If θ is complex with Re $\theta > 0$ then x_n still converges to the fixed point 1/2, but if Re $\theta < 0$ the sequence x_n diverges to infinity. This means that the basin of attraction for the fixed point 1/2 is the circular region

$$|x_0 - 1/2| < 1/2.$$

On the boundary there are a number of unstable fixed points of higher order corresponding to $\theta = ir$ where r is a rational number. E.g. for $r = 2/3$ we find the period 2 fixed points $\frac{3}{4} \pm \frac{1}{4} i\sqrt{3}$.

For $a = 4$ we have the explicit solution (1.4). We see that for a start with rational θ a periodic orbit is obtained. E.g. for $\theta = 1/7$ a period 3 orbit is obtained, but for $\theta = -1/7$ we get another period 3 orbit. Of course all periodic orbits are unstable. It is helpful to consider θ as an infinite binary fraction $0{\cdot}b_1b_2b_3b_4\ldots$. The effect of an iteration step is merely a shift of one binary position to the left with removal of the integer part

$$0{\cdot}b_1b_2b_3b_4 \ldots \to 0{\cdot}b_2b_3b_4b_5 \ldots$$

However, when working on a computer, θ will be given by a restricted number of binary digits so that after some 40 steps, say, further iteration has lost its meaning. Much depends on the manner the computer fills empty positions with ones or zeros.

The map (1.1) has also the fixed point $x = \frac{3}{4}$. Substitution of

$$x_n = \tfrac{3}{4} - \tfrac{1}{4} y_n \tag{3.2}$$

gives the new map

$$y_{n+1} = y_n^2 - 2y_n. \tag{3.3}$$

According to (2.19) we obtain the parametrization

$$y_n = 1 - 2 \cos(1+(-2)^n\theta) \frac{\pi}{3} \tag{3.4}$$

so that

$$x_n = \cos^2(1+(-2)^n\theta) \frac{\pi}{6} . \tag{3.5}$$

It is of interest to compare this with the parametrization (1.4). However, for (3.5) no elementary continuous extension is possible.

For the general logistic map (3.1) we consider the functional equation

$$F(az) = a\, F(z)(1-F(z)) \quad , \quad a > 1. \tag{3.6}$$

In view of the procedure proposed in the previous section we write

$$F(z) = z - c_2 z^2 + c_3 z^3 - c_4 z^4 + \ldots \tag{3.7}$$

Substitution in (3.6) gives without difficulty

$$(a^k-1)\, c_{k+1} = \sum_{j=1}^{k} c_j c_{k-j+1} \quad , \quad k \geq 1 \quad \text{with } c_1 = 1. \tag{3.8}$$

The first few coefficients are

$$c_1 = 1 \quad , \quad c_2 = \frac{1}{a-1} \quad , \quad c_3 = \frac{2}{(a-1)(a^2-1)} \quad ,$$

$$c_4 = \frac{a+5}{(a-1)(a^2-1)(a^3-1)} \quad , \quad c_5 = \frac{2(2a^2+3a+7)}{(a-1)(a^2-1)(a^3-1)(a^4-1)} \quad \ldots$$

For a = 2,3,4 we have in particular

	a = 2	a = 3	a = 4
c_2	1	1/2	1/3
c_3	2/3	1/8	2/45
c_4	1/3	1/52	1/315
c_5	2/15	17/8320	2/14175

That F(z) is entire follows at once from theorem 1. An independent way of proving the analiticity of F(z) consists in showing that for all k

$$c_{k+1} \leq (a-1)^{-k}$$

by means of (3.8).

Since all coefficients are positive F(z) is of maximal growth on the negative x-axis. If in the usual notation M(r) is the maximum of |F(z)| on the circle |z| = r then

$$M(r) = - F(-r)$$

and

$$M(ar) = a M^2(r) + a M(r). \tag{3.9}$$

This relation can be used to prove the following statement.

THEOREM 3.1. F(z) *is an entire function of order*

$$\mu = \frac{\log 2}{\log a} , \quad a > 1.$$

PROOF. From the inequality

$$(aM(r))^2 \le a\,M(ar) \le (aM(r)+b)^2 - b,$$

where

$$b \ge \max(\tfrac{1}{2}a, 1),$$

we obtain for positive integer n by repeated application

$$(aM(r_0))^{2^n} \le a\,M(a^n r_0) \le (aM(r_0)+b)^{2^n}.$$

Setting $a^n = \frac{r}{r_0}$, $2^n = \left(\frac{r}{r_0}\right)^{\mu}$

we obtain

$$b_i \left(\frac{r}{r_0}\right) \le \log\,(aM(r)) \le b_2 \left(\frac{r}{r_0}\right)^{\mu},$$

where $b_1 = \log\,(aM(r_0))$, $b_2 = \log\,(aM(r_0)+b)$.

This inequality shows that F(z) has the order μ.

Perhaps the simplest entire function of fractional order μ that presents itself is

(3.10) $$\Phi(z,\mu) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{z^k}{(k\mu)!}.$$

Only in a few cases $\Phi(z,\mu)$ is an elementary function. We have e.g.

(3.11a) $$\Phi(z,2) = e^{z^2}(1+\operatorname{erf} z).$$

(3.11b) $$\Phi(z,1) = e^z.$$

(3.11c) $$\Phi(z,\tfrac{1}{2}) = \cosh\sqrt{z}.$$

(3.11d) $$\Phi(z,\tfrac{1}{3}) = \tfrac{1}{3}\left(\exp(z^{1/3}) + \exp(e^{\frac{2}{3}\pi i} z^{1/3}) + \exp(e^{-\frac{2}{3}\pi i} z^{1/3})\right).$$

Integral representation can easily be derived from the Laplace pair

$$(3.12)\qquad \Phi(r^{1/\mu},\mu) \doteqdot \frac{1}{s(1-s^{-1/\mu})}$$

so that e.g.

$$(3.13)\qquad \Phi(z,\mu) = \frac{1}{2\pi i}\int \frac{e^{w}}{w(1-zw)^{-1/\mu}}\,dw\,,$$

where the contour encloses the negative real axis and the poles on the circle $|w|=|z^{\mu}|$. Other representations are

$$(3.14)\qquad \Phi(z,\mu) = \mu\, e^{z^{\mu}} - \frac{\mu}{\pi}\sin\frac{\pi}{\mu}\int_0^{\infty}\frac{\exp-(tz)^{\mu}}{1-2t\cos\frac{\pi}{\mu}+t^2}\,dt$$

valid for $\mu > \frac{1}{2}$, $|\arg z| < \frac{\pi}{2\mu}$,

and

$$\Phi(-z,\mu) = \mu(\exp(e^{\mu\pi i}z^{\mu}) + \exp(e^{-\mu\pi i}z^{\mu})) +$$

$$(3.15)\qquad -\frac{\mu}{\pi}\sin\frac{\pi}{\mu}\int_0^{\infty}\frac{\exp-(tz)^{\mu}}{1+2t\cos\frac{\pi}{\mu}+t^2}\,dt$$

valid for $\frac{1}{2} < \mu < 1$, $|\arg z| < \frac{\pi}{2\mu}$.

The entire functions $F(z)$ and $\Phi(z,\mu)$ seem to be closely related. In fact we have the rather sharp inequality for $2 \le a \le 4$, i.e. for $\frac{1}{2} \le \mu \le 1$,

<u>THEOREM 3.2.</u>

$$(3.16)\qquad 1 - 2F(-r) \le \Phi\,(ar,\mu)\quad ,\quad 2 \le a \le 4,$$

or in terms of the coefficients of (3.7)

$$(3.17)\qquad c_k \le \frac{a^k}{2(k/\mu)!}\,,\quad k = 1,2,\ldots\ .$$

<u>PROOF</u>. The inequalities (3.16) and (3.17) are equalities for $a = 2$ ($\mu=1$) and $a = 4$ ($\mu=1/2$). The proof is somewhat technical but not difficult. At first we show that (3.17) holds for $k = 1$. This is done by using a convexity argument for the logarithm of the gammafunction. For $k \geq 2$ we proceed by induction using (3.8). It turns out that it is sufficient to consider only the lowest value of k, i.e. $k = 2$. Again using a suitable integral representation this can be done in a similar way as the case $k = 1$. The proof uses the following integral representations

$$\text{(3.18)} \qquad \frac{\alpha!}{\left(\frac{\alpha+\beta}{2}\right)!\left(\frac{\alpha-\beta}{2}\right)!} = \frac{2^{\alpha+1}}{\pi}\int_0^{\frac{1}{2}\pi} (\cos t)^{\alpha} \cos \beta t \, dt ,$$

$$\text{(3.19)} \qquad \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} (\cos t)^{\alpha} \frac{\sin(\alpha-\beta)t}{\sin \beta t} \, dt = \frac{1}{2\beta} - \frac{1}{2^{\alpha}} - \frac{\sin\beta\pi}{2^{\alpha}\beta\pi}\int_0^1 \frac{(1-t^{1/\beta})^{\alpha}}{1-2t\cos\beta\pi+t^2} \, dt.$$

The representation (3.18) is well-known but (3.19) will be proved in the Appendix.

The first step in the proof of (3.17) is as follows. The inequality (3.17) for $k = 1$ is

$$\Gamma(\beta+1) \leq 2^{\beta-1} , \qquad 1 \leq \beta \leq 2,$$

where for typographical reasons we have written

$$\beta = \frac{1}{\mu} = \frac{\log a}{\log 2} .$$

We consider the function

$$\phi(\beta) = \log \Gamma(\beta+1) - (\beta-1) \log 2$$

and note that $\phi(1) = \phi(2) = 0$. Since $\phi''(\beta) > 0$ for all β the inequality does hold. Next it is assumed that the inequality is fulfilled up to index k.

Then using (3.8) we should have

$$c_{k+1} \le \frac{a^{k+1}}{4(a^k-1)} \sum_{j=1}^{k} \frac{1}{(j\beta)!((k+1-j)\beta)!} \le \frac{a^{k+1}}{2((k+1)\beta)!} .$$

This means we have to show that

$$(3.20) \qquad \sum_{j=1}^{k} \frac{((k+1)\beta)!}{(j\beta)!((k+1-j)\beta)!} \le 2\left(2^{k\beta}-1\right)$$

for $k \ge 1$ with $1 \le \beta \le 2$.

After a few elementary steps using (3.18) this inequality can be written as

$$(3.21) \qquad \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} (\cos t)^{(k+1)\beta} \frac{\sin k\beta t}{\sin \beta t}\, dt \le 2^{-\beta}(1-2^{-k\beta}).$$

The left-hand side can be replaced by an expression as shown in (3.19). Then the inequality takes the following form

$$\frac{-\sin \beta\pi}{\beta\pi} \int_0^1 \left(\frac{1-t^{1/\beta}}{2}\right)^{(k+1)\beta} \frac{dt}{1-2t\cos\beta\pi+t^2} \le 2^{-\beta} - \frac{1}{2\beta} .$$

The left-hand side takes it largest value for $k = 1$ but instead of proving the last inequality for $k = 1$ we return to (3.21). For $k = 1$ we get

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} (\cos t)^{2\beta}\, dt \le 2^{-\beta}(1-2^{-\beta})$$

or equivalently

$$\frac{1}{2\pi}\int_0^{\pi} (1+\cos t)^{\beta}\, dt \le 1 - 2^{-\beta}.$$

Consider the function

$$\phi(\beta) = \frac{1}{2\pi}\int_0^{\pi} (1+\cos t)^{\beta}\, dt - 1 + 2^{-\beta}$$

and note that $\phi(1) = \phi(2) = 0$. Clearly $\phi''(\beta) > 0$ so that finally $\phi(\beta) < 0$ for $1 < \beta < 2$. This completes the proof.

4. ZEROS

In this section the notation $F(z)$ will be replaced by the more detailed notation $F(z,a)$ if necessary. We consider here the zeros of $F(z)-c$ for a few special values of c such as $c=0$, 1 and 1/2. The zeros of $F(z,a)$ are arranged on rays radiating from the origin into the right-hand half of the complex plane. Obviously if ζ is a zero then also $a\zeta$ is a zero but ζ/a can be a zero of $F(z)-1$. On each ray there is a first zero ζ such that $a^j\zeta$ are zeros for $j=0,1,2,\ldots$ but that $F(\zeta/a) = 1$. Therefore it suffices to consider the zeros of

$$F(z,a) = 1. \tag{4.1}$$

Such a zero can be labelled by an integer index n and is then written as $z_n(a)$. We have the elementary cases

$$\begin{cases} z_n(2) = \pm\,(n+\tfrac{1}{2})\,\pi i\,, \\ z_n(4) = (n+\tfrac{1}{2})^2\,\pi^2\,, \quad n = 0,1,2,\ldots \end{cases} \tag{4.2}$$

Since for $a \neq 4$ the zeros occur in conjugate pairs we need to consider only those with $\operatorname{Im} z_n(a) \geq 0$. If a runs from 2 to 4 we may expect that for each n $z_n(a)$ describes a certain continuous path. We might think that the path of $z_n^{\mu}(a)$ is more or less circular since $|z_n^{\mu}(a)| = (n+\tfrac{1}{2})\pi$ for $a=2$ and $a=4$. However, computer plots obtained from a HP85 computer and a 7225A plotter reveal some unexpected features. There seem to be concentrations of zeros at $n=4,8,16,\ldots$ already for a-values slightly below 4. A perturbation procedure, the details of which will be given later on, shows that this phenomenon has to do with resonances of

$$\tan(2^{-k}(n+\tfrac{1}{2})\pi) \qquad k=1,2,3,\ldots$$

in particular when $\quad 2n+1 \approx 2^k$.

Stated as a theorem

THEOREM 4.1. *For* $a = 4 - \varepsilon^2$, $0 \le \varepsilon \ll 1$, *we have the following expansions*

$$(4.3)\qquad \begin{cases} \operatorname{Re} z_n^{1/2}(a) = (n+\tfrac{1}{2})\pi + A\varepsilon^2 + O(\varepsilon^4) , \\ \operatorname{Im} z_n^{1/2}(a) = \tfrac{1}{2}\varepsilon + B\varepsilon^3 + O(\varepsilon^5), \end{cases}$$

where

$$(4.4)\qquad A = -\frac{1}{8} M((n+\tfrac{1}{2})\pi) ,$$

and

$$(4.5)\qquad B = \frac{1}{24} - \frac{1}{16} M'((n+\tfrac{1}{2})\pi) ,$$

with $$M(x) \overset{\text{def}}{=} x - \sum_{k=1}^{\infty} (2^k \tan(2^{-k}x) - x) =$$

$$(4.6)\qquad = x - \frac{1}{9} x^3 - \frac{2}{225} x^5 - \dots .$$

Of related interest are the zeros of $F'(z,a)$.
Since $F'(az) = F'(z)(1-2F(z))$ they are also arranged along rays. If on a ray ζ is a first zero we must have $F(\zeta/a) = 1/2$. Let us now consider the possibility of a double root of $F(z) = c$ where c is a real or complex constant. Then there exists a non-negative integer m such that

$$F(z_0/a) = 1/2 , \quad F'(a^m z_0) = 0 , \quad F(a^m z_0) = c.$$

However, this means that c can be reached from $1/2$ by forward iteration. Then c is real and belongs to the sequence starting from $1/2$. Thus we have proved

THEOREM 4.2. *The equation* $F(z,a) = c$ *has no double roots unless* c *is real with* $0 < c < \frac{1}{4}a$ *and belonging to the sequence starting from* $1/2$.

We now turn to the analysis of the behaviour of the zeros $z_n(a)$ when a is close to 4. At first sight we could try an expansion such as

$$F(z,a) = F(z,4) + \varepsilon F_1(z) + O(\varepsilon^2),$$

where $a = 4 - \varepsilon$. However, this does not work since $z_n(4)$ is a double zero of $F_n(4)$ which for $\varepsilon > 0$ splits into a conjugate complex pair. Before presenting the right procedure we bring the functional equation (1.11) into a slightly simpler form

$$(4.7) \qquad G(az,a) = G^2(z,a) - \lambda ,$$

where

$$(4.8) \qquad G(z,a) = \tfrac{1}{2}a - a\,F(z,a)$$

and

$$(4.9) \qquad \lambda = \tfrac{1}{4}\,(a^2-2a).$$

The condition $F(z,a) = 1$ is equivalent to $G(z,a) = -\tfrac{1}{2}a$ and next to

$$(4.10) \qquad G(\tfrac{z}{a},a) = \pm\,\tfrac{1}{2}i\,\sqrt{4a-a^2}.$$

This prompts us to set

$$a = 4 - \varepsilon^2 , \quad \varepsilon \geq 0$$

and to solve

$$(4.11) \qquad G(z_n/a,a) = \pm\,i\varepsilon\,(1-\tfrac{1}{8}\varepsilon^2 + O(\varepsilon^4)).$$

In order to do this we need the expansion

$$(4.12) \qquad G(z/a,a) = G(z/4,4) - \varepsilon^2\,H(\sqrt{z}) + O(\varepsilon^4).$$

We have

$$(4.13) \qquad G(z/4,4) = 2\cos\sqrt{z}$$

and

$$(4.14)\qquad H(\sqrt{z}) = \frac{\partial}{\partial a} G(z/a,a) \quad \text{for } a=4.$$

The first few forms of the power series expansion of $H(\sqrt{z})$ can be obtained in a straightforward way from (3.7) and (4.8) as

$$(4.15)\qquad H(\sqrt{z}) = \tfrac{1}{2} - \frac{7}{144} z^2 + \frac{41}{10800} z^3 - \dots$$

We also need information on H for large values of $|z|$ but this will be postponed a bit. We try an expansion of the kind

$$(4.16)\qquad \sqrt{z_n(a)} = (n+\tfrac{1}{2})\pi + iC\varepsilon + A\varepsilon^2 + iB\varepsilon^3 + O(\varepsilon^4),$$

which should be substituted into

$$(4.17)\qquad 2\cos\sqrt{z_n(a)} - \varepsilon^2 H(\sqrt{z_n(a)}) = i\varepsilon(1-\tfrac{1}{8}\varepsilon^2) + O(\varepsilon^4).$$

After some elementary steps we obtain for the upper zeros

$$(4.18)\qquad \begin{cases} C = \tfrac{1}{2}, \quad A = \tfrac{1}{2}(-1)^{n+1} H(\sqrt{z_n(4)}), \\ B = -\frac{1}{12} + \frac{1}{4}(-1)^{n+1} H'(\sqrt{z_n(4)}). \end{cases}$$

The problem of determining $H(\sqrt{z})$ for large values of $|z|$ can be solved by differentiating the functional equation (4.7) with respect to a for $a=4$. A simple calculation gives the following functional equation for H(z)

$$(4.19)\qquad H(2z) = 4\cos z\, H(z) - \frac{3}{2} + \frac{1}{2} z \sin 2z.$$

Substitution of

$$(4.20)\qquad H(z) = \frac{1}{2}\cos z + \frac{1}{4}\sin z\, M(z)$$

changes this into a simpler equation

$$(4.21)\qquad \frac{1}{2} M(2z) - M(z) = z - \tan z.$$

From (4.15) we obtain the first few terms of the power series expansion

$$(4.22) \qquad M(z) = z - \frac{1}{9} z^3 - \frac{2}{225} z^5 - \dots .$$

Noting that $M'(0) = 1$ the solution of (4.21) can be obtained in a straightforward way as

$$(4.23) \qquad M(z) = z - \sum_{k=1}^{\infty} (2^k \tan(2^{-k} z) - z).$$

From the well-known expansion

$$(4.24) \qquad \tan z - z = 2 \sum_{j=2}^{\infty} \frac{2^{2j}-1}{\pi^{2j}} \zeta(2j) z^{2j-1}, \quad |z| < \tfrac{1}{2}\pi,$$

we obtain the general form of the expansion (4.22) as

$$(4.25) \qquad M(z) = z - 2 \sum_{j=2}^{\infty} \frac{2^{2j}-1}{2^{2j-2}-1} \frac{\zeta(2j)}{\pi^{2j}} z^{2j-1}, \quad |z| < \tfrac{1}{2}\pi.$$

If desired the zeta function values can be expressed in Bernoulli coefficients

$$(4.26) \qquad \zeta(2j) = \frac{2^{2j-1}\pi^{2j}}{(2j)!} |B_{2j}| ,$$

with

$$B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \dots .$$

The derivative of $M(z)$ follows from (4.23) as

$$(4.27) \qquad M'(z) = 1 - \sum_{k=1}^{\infty} \tan^2(2^{-k} z) ,$$

and

$$(4.28) \qquad M'(z) = 1 - \frac{1}{3} z^2 - \frac{2}{45} z^4 - \dots .$$

Finally from (4.18) and (4.20) we obtain the expressions (4.4) and (4.5) of theorem 4.1. A number of values of M, M', A and B for $z = (n+\frac{1}{2})\pi$, $n = 0\ (1)\ 40$ is given in tables 2 and 3. However, without consulting numerical data it is obvious that possible extreme values of A and B are to be expected where the right-hand sides of (4.23) and (4.27) contain a large term when

$$2^{-k}\,(n+\tfrac{1}{2})\pi \;\approx\; \tfrac{1}{2}\pi. \tag{4.29}$$

This happens in particular when n is some power of 2. A rough estimate for

$$z = (2^{m}+\tfrac{1}{2})\pi, \qquad m \gg 1 \tag{4.30}$$

gives

$$\begin{cases} M(z) \approx \dfrac{1}{\pi}\, 2^{2m+3} \\ M'(z) \approx -\dfrac{1}{\pi^2}\, 2^{2m+4} \end{cases}. \tag{4.31}$$

Then theorem 4.1 gives at once

$$z_n^{1/2}(a) \;\approx\; (2^{m}+\tfrac{1}{2})\pi + \tfrac{1}{2}\,\varepsilon i + \frac{1}{\pi}\, 2^{2m}\varepsilon^2 + \frac{1}{\pi^2}\, 2^{2m}\,\varepsilon^3 i + \dots, \tag{4.32}$$

with $n = 2^{m}$.

APPENDIX

We consider the integral

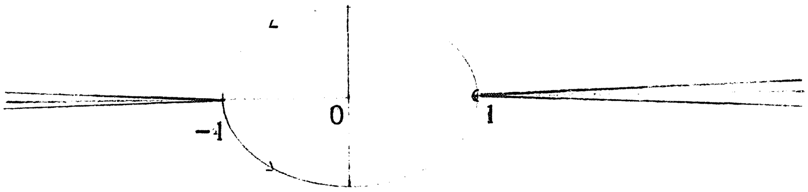
$$I = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} (\cos t)^\alpha \, \frac{\sin(\alpha-\beta)t}{\sin \beta t} \, dt \,,$$

where $\alpha > -1$ and $0 < \beta < 2$.

We write I as a Cauchy integral with $w = \exp it$

$$I = \frac{2^{-\alpha}}{\pi i} \fint_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (w+w^{-1})^\alpha \, \frac{w^{\alpha-\beta}}{w^\beta - w^{-\beta}} \, \frac{dw}{w} .$$

We make the substitution $w^2 \to w$ and note that the new variable describes a path in the complex w-plane as shown in the sketch below.



If the semi-pole at $w = 1$ is split off the path of integration can be closed at $w = 1$ so that it might be contracted. We obtain

$$I = \frac{2^{-\alpha}}{2\pi i} \int \frac{(w+1)^\alpha w^{-1}}{w^\beta - 1} \, dw + \frac{1}{2\beta} .$$

In order to avoid trouble at the pole $w = 0$ we use the trick $1 = (1-w^\beta) + w^\beta$ and write

$$I = \frac{1}{2\beta} - 2^{-\alpha} + \frac{2^{-\alpha}}{2\pi i} \int \frac{(w+1)^\alpha w^{\beta-1}}{w^\beta - 1} \, dw.$$

Finally contraction of the path of integration to lower and upper side of the interval (-1,0) of the negative real axis gives the desired result (3.19).

a		a		a	
2	2	2.68	1.82800	3.36	1.90038
2.02	1.98297	2.70	1.82878	3.38	1.90331
2.04	1.96740	2.72	1.82969	3.40	1.90625
2.06	1.95315	2.74	1.83073	3.42	1.90921
2.08	1.94011	2.76	1.83189	3.44	1.91220
2.10	1.92818	2.78	1.83317	3.46	1.91520
2.12	1.91726	2.80	1.83455	3.48	1.91821
2.14	1.90729	2.82	1.83603	3.50	1.92125
2.16	1.89817	2.84	1.83761	3.52	1.92429
2.18	1.88986	2.86	1.83929	3.54	1.92736
2.20	1.88227	2.88	1.84105	3.56	1.93043
2.22	1.87537	2.90	1.84289	3.58	1.93352
2.24	1.86909	2.92	1.84481	3.60	1.93662
2.26	1.86340	2.94	1.84680	3.62	1.93972
2.28	1.85826	2.96	1.84886	3.64	1.94284
2.30	1.85362	2.98	1.85099	3.66	1.94597
2.32	1.84945	3.00	1.85319	3.63	1.94911
2.34	1.84571	3.02	1.85544	3.70	1.95225
2.36	1.84239	3.04	1.85775	3.72	1.95540
2.38	1.83945	3.06	1.86011	3.74	1.95856
2.40	1.83687	3.08	1.86252	3.76	1.96173
2.42	1.83462	3.10	1.86498	3.78	1.96489
2.44	1.83268	3.12	1.86749	3.80	1.96807
2.46	1.83104	3.14	1.87004	3.82	1.97125
2.48	1.82968	3.16	1.87263	3.84	1.97443
2.50	1.82857	3.18	1.87526	3.86	1.97762
2.52	1.82770	3.20	1.87793	3.88	1.98081
2.54	1.82707	3.22	1.88063	3.90	1.98400
2.56	1.82665	3.24	1.88337	3.92	1.98720
2.58	1.82643	3.26	1.88613	3.94	1.99040
2.60	1.82641	3.28	1.88893	3.96	1.99360
2.62	1.82656	3.30	1.89176	3.98	1.99680
2.64	1.82688	3.32	1.89461	4.00	2
2.66	1.82737	3.34	1.9748		

Table 1. Type of $F(z,a)$ for $2 \leq a \leq 4$.

N	M	M'	A	B
0	1.03	-.22	-.13	.06
1	5.66	-6.40	-.71	.44
2	26.19	-8.44	-3.27	.57
3	4.69	-26.29	-.59	1.68
4	86.44	-27.22	-10.81	1.74
5	58.30	-12.03	-7.29	.79
6	58.09	-17.84	-7.26	1.16
7	-46.32	-104.31	5.79	6.56
8	279.93	-104.77	-34.99	6.59
9	172.57	-19.29	-21.57	1.25
10	165.89	-14.76	-20.74	.96
11	126.34	-31.91	-15.79	2.04
12	188.39	-34.56	-23.55	2.20
13	129.91	-25.06	-16.24	1.61
14	63.12	-52.74	-7.89	3.34
15	-367.90	-415.68	45.99	26.02
16	938.20	-415.91	-117.02	26.04
17	503.73	-53.43	-62.97	3.38
18	434.01	-26.24	-54.25	1.68
19	370.96	-36.29	-46.37	2.31
20	426.59	-34.28	-53.32	2.18
21	378.42	-17.90	-47.30	1.16
22	360.43	-23.39	-45.05	1.50
23	238.28	-110.14	-29.78	6.93
24	545.05	-111.44	-68.13	7.01

Table 2. COEFFICIENTS PERTURBATION EXPANSION ZEROS F-1

N	M	M'	A	B
25	414.52	-27.51	-51.82	1.76
26	378.11	-25.71	-47.26	1.65
27	297.23	-47.79	-37.15	3.03
28	295.82	-60.34	-36.98	3.81
29	124.59	-75.12	-15.57	4.74
30	-203.62	-191.24	25.45	11.99
31	-1938.79	-1660.77	242.35	103.84
32	3276.71	-1660.88	-409.59	103.85
33	1540.83	-191.58	-192.60	12.02
34	1211.18	-75.70	-151.40	4.74
35	1037.77	-61.15	-129.72	3.86
36	1033.42	-48.85	-129.18	3.09
37	948.82	-27.03	-118.60	1.73
38	907.85	-29.11	-113.48	1.86
39	771.86	-113.32	-96.48	7.12
40	1072.23	-112.35	-134.03	7.06
41	942.62	-25.94	-117.83	1.66
42	916.03	-20.84	-114.50	1.34
43	857.99	-37.64	-107.25	2.39
44	902.32	-40.13	-112.79	2.55
45	826.37	-30.63	-103.30	1.96
46	741.90	-58.45	-92.74	3.69
47	292.54	-421.66	-36.57	26.40
48	1577.23	-422.31	-197.15	26.44
49	1123.80	-60.41	-140.48	3.82

Table 3. COEFFICIENTS PERTURBATION EXPANSION ZEROS OF F-1

Description of the figures

Figure 1 gives the positions of the first eleven zeros of F(z,a) for a = 2 to 4 step 0.1. The positions of z^c are given with c = log 2/log a. We note that that for a = 4 $z^c = (n+\frac{1}{2})\pi$, n = 0(1)10 and that for a = 2 $z^c = (n+\frac{1}{2})\pi i$.
Figure 2 gives the positions of the next ten zeros of F(z,a) for a = 3 to 4 step 0.05.
Figure 3 gives the positions of the zeros n = 13(1)18 for a = 3.8 to 4 step 0.01.
Figure 4 gives the graph of $F(t^c,a)$ with a = 3.9 , c = log a/log 2 = 1.963 for 0 < t < 20. Note that for a = 4 we would have $F(t^2,2) = \sin^2 t$.
Figure 5 gives the graph of $F(a^t,a)$ with a = 3.5 for 0 < t < 10 , i.e. $1 < a^t < 275.9$. For this value of a the iterated map has the stable 4-cycle 0.8750 , 0.3828 , 0.8269 , 0.5009. Obviously for $t \to \infty$ the graph of F consists of horizontal parts at the positions of the 4-cycle separated by vertical lines.

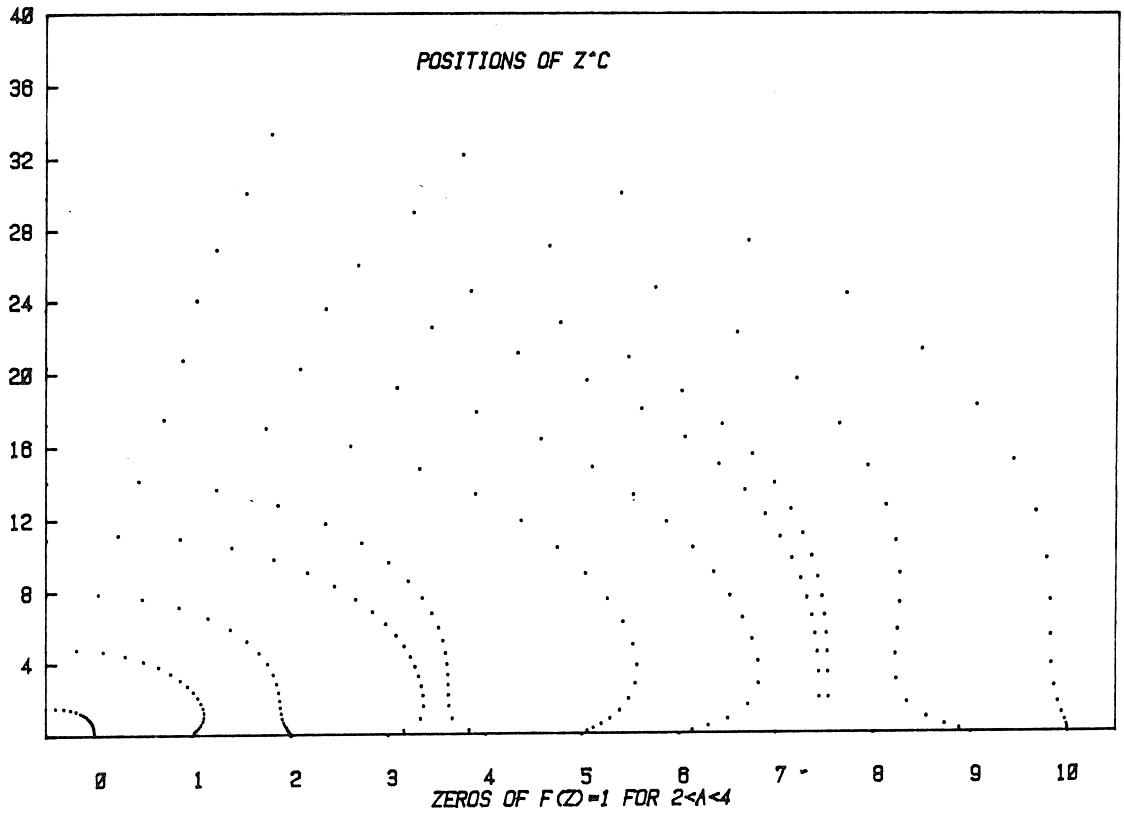
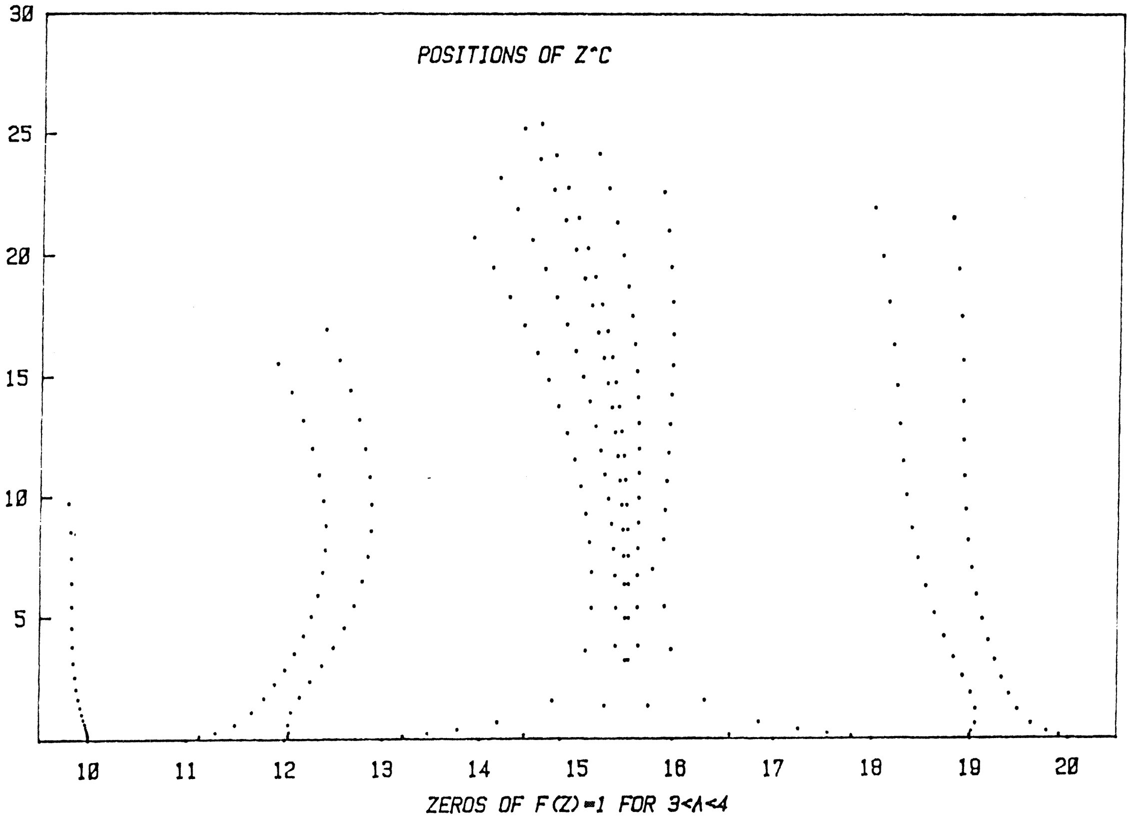
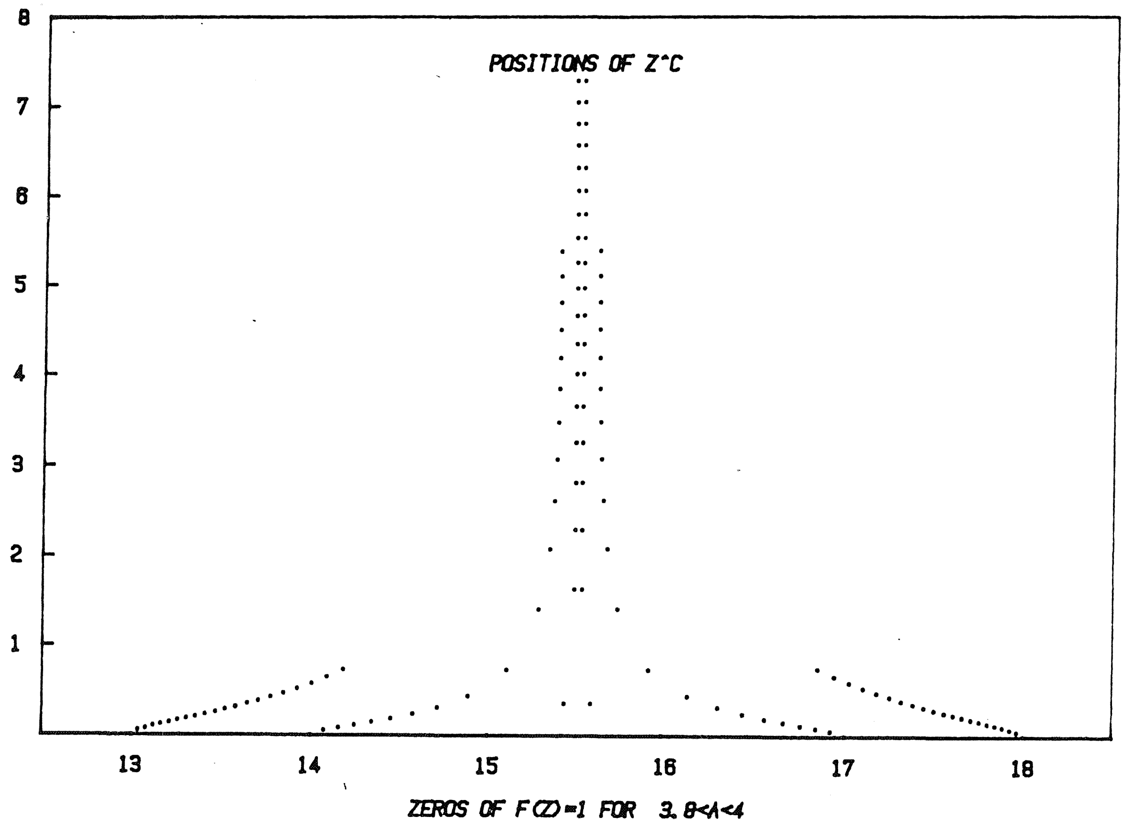


Fig. 1



POSITIONS OF Z*C
30
25
20
15
10
5
10
11
12
13
14
15
16
17
18
19
20
ZEROS OF F(Z)=1 FOR 3<A<4

Fig. 2



POSITIONS OF Z*C
8
7
6
5
4
3
2
1
13
14
15
16
17
18
ZEROS OF F(Z)=1 FOR 3.8<A<4

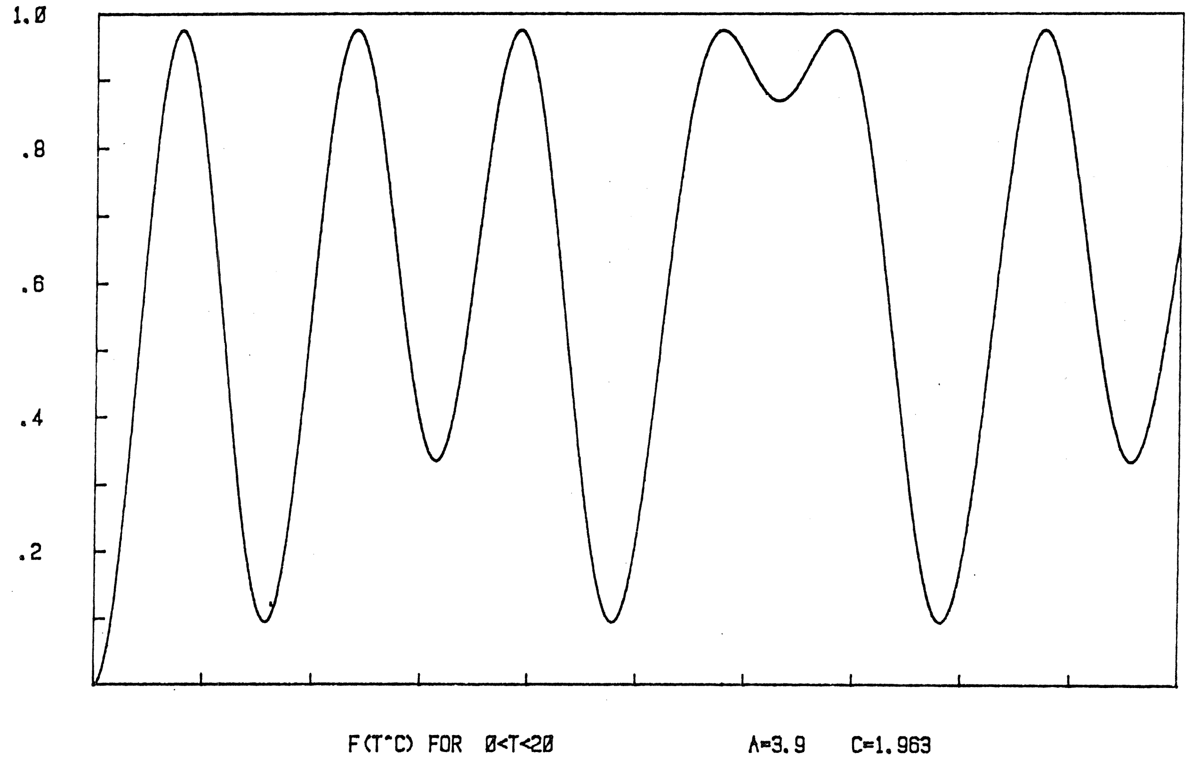


Fig. 4

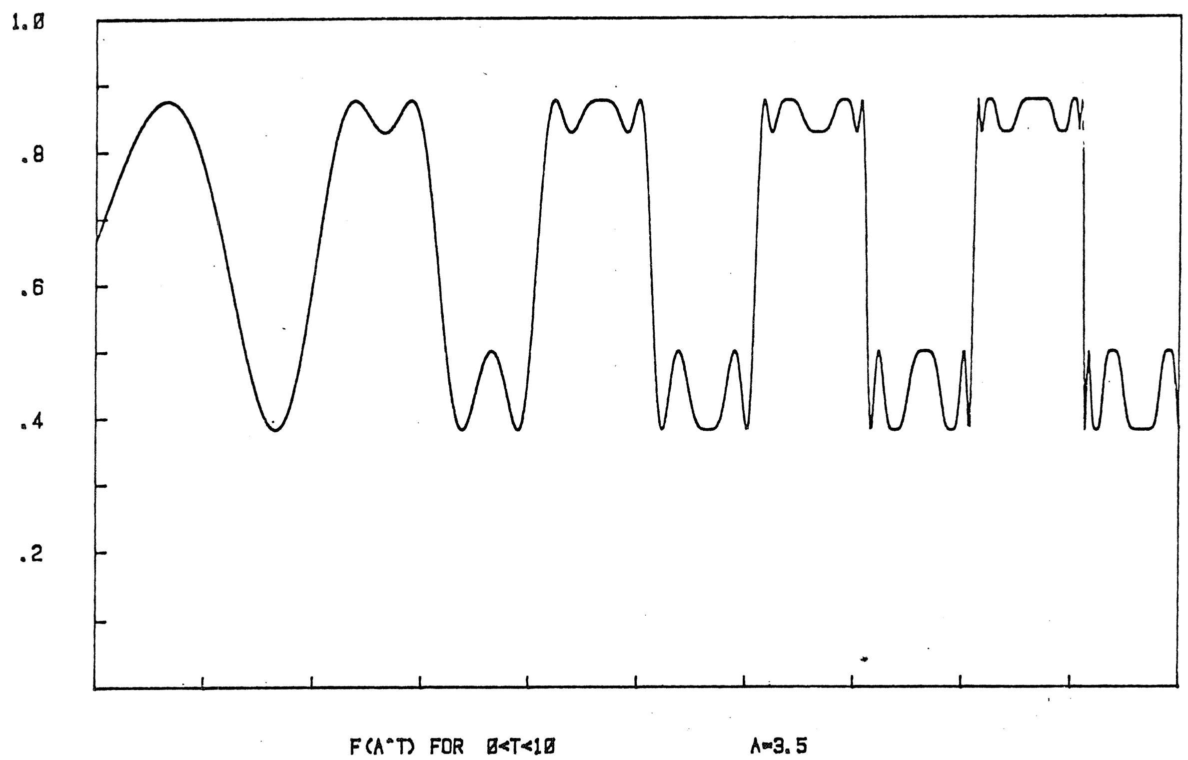


Fig. 5

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